# Lecture 18

- Recall: The Ising model on $\sqrt{n} \times \sqrt{n}$ lattice

**Gibbs Distn'** $\pi(\sigma) = \frac{1}{Z} \exp\Big( \sum_{i \sim j} \beta \sigma_i \sigma_j \Big)$

partition function

inverse temperature

$\exists \beta_c$ st: • $\beta < \beta_c$ (i.e. high enough temperature) no long-range correlations

• $\beta > \beta_c$ (i.e. low enough temp.) long-range correlations

$$\boxed{\beta_c = \frac{1}{2} \ln(1+\sqrt{2})}$$

**Heat-Bath Markov Chain** (Glauber Dynamics)

- start at arbitrary configuration $\sigma$;
- at every step of chain:
  - choose $i$ u.a.r.
  - ignore $i$'s spin in $\sigma$, and sample a new spin $\sigma_i$ from the conditional distn' of $\sigma_i$ ender $\pi$ conditioning on $\sigma_{N(i)}$

easy to check: MC reversible wrt $\pi$

- Remarkable connection between spatial & temporal mixing.

**Theorem [Martinelli-Olivieri '94]**: The mixing time of the Glauber Dynamics for the Ising model on the $\sqrt{n} \times \sqrt{n}$ box of the 2-dimensional lattice satisfies:

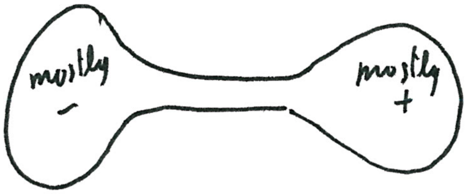
$$\begin{cases} O(n \cdot \log n), & \text{if } \beta < \beta_c; \\ e^{\Omega(\sqrt{n})}, & \text{if } \beta > \beta_c. \end{cases}$$

- last time showed first part of Theorem for $\beta < \frac{1}{2} \ln(5/3) < \beta_c$.

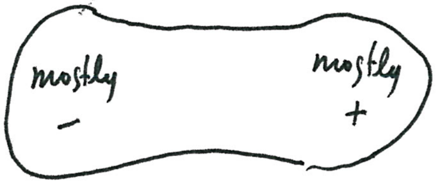
Today we show slow mixing for sufficiently large (but still constant) $\beta$.

**Proof (slow mixing for $\beta > \beta_0$):**

- intuition : low vs high temperatures

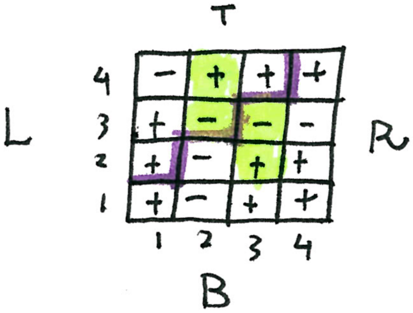


low temperatures
[configurations that are mostly '+' or mostly '−' have a lot of probability under the Gibbs distn' & intermediate states get low probability, thus creating a bottleneck]



high temperatures
[no bottleneck at intermediate configurations]

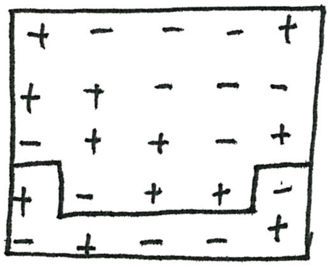
- for convenience, we think of spins as occupying squares of the $[0,\sqrt{n}]^2$ box as follows:



- L, R, T, B : the left, right, top, bottom edges of $[0,\sqrt{n}]^2$ respectively
- **path**: refers to a sequence of boxes that are pairwise adjacent
- **line**: refers to a sequence of segments that are pairwise adjacent and every segment is edge of a subsquare of $[0,\sqrt{n}]^2$

**Def (fault line) of configuration $\sigma$**: A line all of whose segments have squares with different spins under $\sigma$ on their two sides.

## Example (fault line)



fault line from L to R

**Claim 1:** Let F be the set of configurations containing a fault line of length $\geq \sqrt{n}$. Then $\pi(F) \leq e^{-c\sqrt{n}}$, for some $c > 0$, if $\beta$ is large enough.

**Proof:**
- fix a fault line L w/ length $\ell \geq \sqrt{n}$
- Let $F(L)$ be configurations containing $\ell$
- take $\sigma \in F(L)$ and flip spins on one side of fault line (chosen according to some fixed rule)
- the weight of $\sigma$ goes up by a factor of $e^{2\beta\ell}$

  moreover, the mapping is one-to-one

  hence $\pi(F(L)) < e^{-2\beta\ell}$.

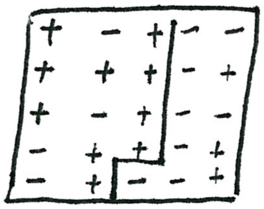
- $$\pi(F) \leq \underbrace{2\sqrt{n}}_{\text{starting point of fault line (left or bottom edge of } [0,\sqrt{n}]^2)} \sum_{\ell \geq \sqrt{n}} \underbrace{3}_{\text{choices at every step.}}^{\ell} e^{-2\beta\ell} \leq e^{-c\sqrt{n}}, \quad \beta > \frac{1}{2}\ln 3.$$

⊠

**Claim 2:** If (in some configuration $\sigma$) there is no monochromatic (i.e. all '+' or all '−') path crossing from left side to right side, then there is a fault line from top to bottom (in $\sigma$).

**Proof:** • Let

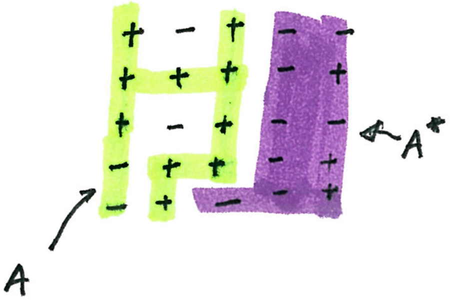
e.g.



$A = \{ \text{squares } x \mid \text{there is a path from left side to } x \text{ using squares of spin } \sigma(x) \}$

• Now let $A^* = \{ x \in \bar{A} \mid x \text{ is reachable from right side using squares in } \bar{A} \}$

• In above example:



• The boundary of $A^*$ consists of part of the boundary of $[0, \sqrt{n}]^2$ and a fault line. Indeed the only impediment to the boundary of $A^*$ from advancing is either reaching the boundary of $[0, \sqrt{n}]^2$ or a square of different spin that belongs to $A$.

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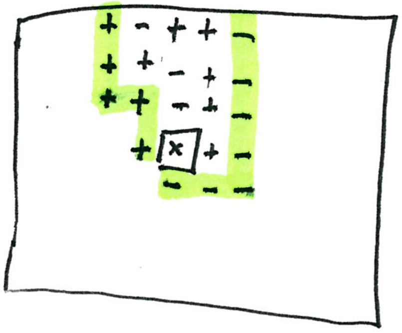
**Claim 3**: Suppose there is a square x for which [ignoring $\sigma(x)$] there is a path of squares that are all + from x to the top and there is a path of squares that are all − from x to the top. Then there is a line from x to the top s.t. every segment of the line has different spins on its two sides.

e.g.



Proof of claim: ex. 1pt

- Now let $S_+$ : configurations with both a left-right '+' crossing and a top-bottom '+' crossing

  $S_-$ : similarly for '−'

then clearly: if $\sigma \in \overline{S_+ \cup S_-}$, then $\sigma$ either has no monochromatic left-right crossing or no monocromatic top-bottom crossing; hence by fact 2 it has a L-R or T-B fault line

then using Fact 1, and symmetry: $\pi(S_-) = \pi(S_+) \to \frac{1}{2}$, as $n \to \infty$

(in fact $\pi(S_-) = \pi(S_+) = \frac{1}{2} - O(e^{-c\sqrt{n}})$)

- Now let $\partial S_+$ be the exterior boundary of $S_+$ (i.e. configurations that would be in $S_+$ if we flipped one spin)

**Claim:** $\pi(\partial S_+) \le e^{-c'\sqrt{n}}$.

**Proof:** ∘ suppose $\sigma \in \partial S_+ \cap \overline{S_+ \cup S_-}$ ; then $\sigma$ has a T-B or L-R fault line.

so by Fact 1:

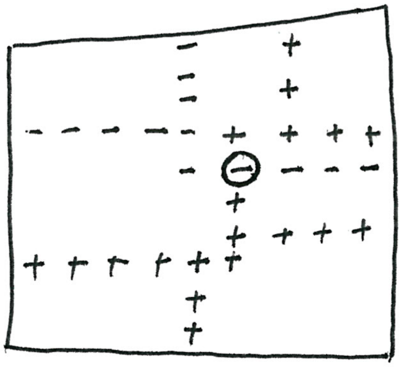
$$\pi(\partial S_+ \cap \overline{S_+ \cup S_-}) \le e^{-c\sqrt{n}}.$$

∘ consider now $\sigma \in \partial S_+ \cap S_-$

- there exists a square $^{x}$ s.t. flipping x's spin from − to + removes $\sigma$ from $S_-$ and places $\sigma$ in $S_+$
- it is not hard to see (using the simple observation that there cannot be a + L-R path if there is a − TB path and vice versa) that: square x has '+' paths to top, bottom, left and right

  x has '−' paths to —〃— —〃— —〃— —〃—
- using claim 3, there is a fault line from L to x and another fault line from x to R.

  e.g.



- we can connect these lines to form a line from L to R that is a fault line w/ at most a constant number of defects around square x
- But the proof of claim 1 is robust enough to handle a constant number of defects

⟶

- It follows that $\pi(\partial S_+ \cap S_-) \leq e^{-c'\sqrt{n}}$.

- Hence overall
$$\pi(\partial S_+) \leq e^{-c''\sqrt{n}}.$$
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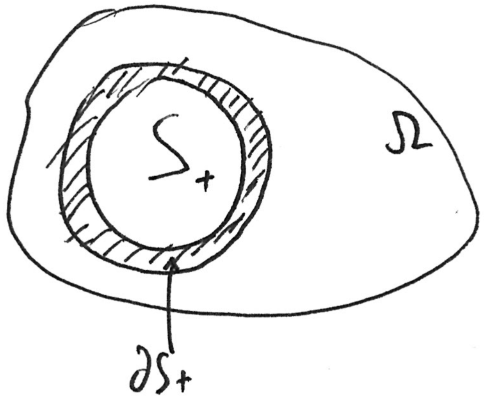
- So



$$\pi(\partial S_+) \leq e^{-c''\sqrt{n}}$$
$$\pi(S_+) = \frac{1}{2} - O(e^{-c\sqrt{n}})$$

- Recall from last time: for any MC and any $S \subseteq \Omega$ w/ $\pi(S) \leq 1/2$

  $\exists$ starting distn $x$ s.t.: $\tau_x(1/4) \geq \dfrac{1}{4\Phi(S)}$.

- In our case:

$$\Phi(S_+) = \frac{\sum_{x\in S_+, y\in\partial S_+} \pi(x) P(x,y)}{\pi(S_+)} \overset{\text{reversibility}}{=} \frac{\sum_{x\in S_+, y\in\partial S_+} \pi(y) P(y,x)}{\pi(S_+)}$$
$$\leq \frac{\pi(\partial S_+)}{\pi(S_+)} \leq e^{-c'''\sqrt{n}}$$

Hence $\tau_{mix} \geq e^{+c'''\sqrt{n}}$

$\boxtimes$