Top-in-at-Random Shuffle {Lecture 5 Part I: Mixing by coupling} ①

- take top card & insert it at random position;
- repeat.

- Inverse Shuffle: - pick card c from deck u.a.r.
  - move card c to the top.

- Mixing time of inverse Shuffle: - Couple $(X_t)_t$, $(Y_t)_t$ as follows: they choose the same card c, at all times t
  - Observation: if card c is chosen at time t, then c is going to be at same location for all $t' > t$
  - Hence $T_{xy}$ is dominated by the coupon collector time for n coupons

$$\Pr[T_{xy} > n\log n + cn] \leq e^{-c}$$

Hence $T_{mix}^{inv} = O(n\log n)$

- How is the mixing time of a chain related to the mixing time of inverse chain?

more general framework: - R.W. on a group G specified by a set of generators $\{g_1, \dots, g_k\}$ and some probability distn' over the generators P
  - R.W.: at each step pick random generator g according to P and apply generator to current state.
- Inverse R.W.: apply $g^{-1}$ (instead of g) [i.e. $\{g_1^{-1}, \dots, g_k^{-1}\}$ with same P i.e. $P(g_i^{-1}) = P(g_i)$

Claim: $\Delta(t) = \Delta^{inv}(t)$

variation distance from stationarity for inverse R.W

**Proof of Claim:**

- Define 1-to-1 mapping between paths starting at $x$ in R.W. & inverse R.W.

$$f(x \circ \sigma_1 \circ \sigma_2 \dots \circ \sigma_t) = x \circ \sigma_t^{-1} \circ \sigma_{t-1}^{-1} \circ \dots \circ \sigma_1^{-1}$$

- Notice $f$ preserves probabilities of paths

- Moreover:

  if
$$x \circ \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_t = x \circ \tau_1 \circ \tau_2 \circ \dots \circ \tau_t$$

$$\Rightarrow \sigma_1 \circ \sigma_2 \circ \dots \circ \sigma_t = \tau_1 \circ \dots \circ \tau_t$$

$$\Rightarrow \sigma_t^{-1} \circ \dots \circ \sigma_1^{-1} = \tau_t^{-1} \circ \dots \circ \tau_1^{-1}$$

$$\Rightarrow x \circ \sigma_t^{-1} \circ \dots \circ \sigma_1^{-1} = x \circ \tau_t^{-1} \circ \dots \circ \tau_1^{-1}$$

- So $f$ induces bijection $\hat{f}$ between set of states reachable from $x$ in R.W. and those reachable from $x$ in inverse R.W. & preserves probabilities between those states

  i.e.
$$P_x^{(t)}(y) = P_x^{inv(t)}(\hat{f}(y)), \forall y$$

- Observation: walks are doubly stochastic

  $\Rightarrow$ stationary distn's for both are uniform distn $\pi$

- Hence: $\| P_x^{(t)} - \pi \|_{TV} = \| P_x^{inv(t)} - \pi \|_{TV}, \forall t$

$\boxtimes$

## Back to top-in-at-random shuffle

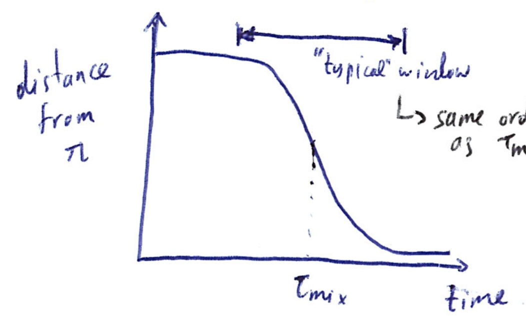
- Argued that $\forall x,y$: $\Pr[T_{xy} > n\cdot\log n + c\cdot n] \le e^{-c}$ $(*)$

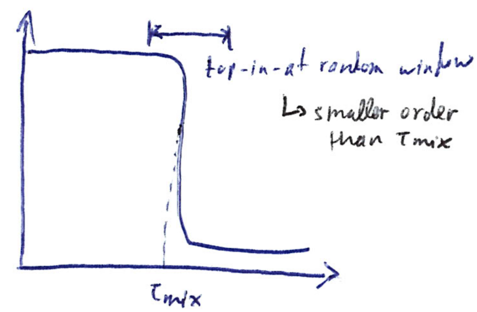
  hence $\Delta(n\cdot\log n + (\log 2+1)n) \le \frac{1}{2e}$ $\qquad \tau_{mix} = \lceil n\log n + (1+\log 2)n\rceil$

  $\Delta(n\log n + n\log\frac{1}{\varepsilon}) \le \varepsilon$ $\qquad \tau(\varepsilon) = n\log n + n\cdot\log\frac{1}{\varepsilon}$

- Notice that $\tau(\varepsilon) \ll \tau_{mix}\cdot\log\frac{1}{\varepsilon}$ (as implied by the bound $\Delta(t) \le e^{-\lfloor \frac{t}{\tau_{mix}} \rfloor}$)

- this MC has a sharp cutoff




- more precisely; we show that

  i. $\Delta(n\cdot\log n + n\cdot\alpha(n)) \xrightarrow{n\to+\infty} 0$

  ii. $\Delta(n\cdot\log n - n\cdot\alpha(n)) \xrightarrow{n\to+\infty} 1$

  for any function $\alpha(n) \xrightarrow{n\to+\infty} +\infty$

- Proof: i. follows from $(*)$

  ii. • define event that has very different probability under $P_x^{(t)}$ and $\pi$, if $t \le n\cdot\log n - n\cdot\alpha(n)$

  • look at $k$ bottom cards in configuration $x$; say $C_1, C_2, \ldots, C_k$

  • Define: $A =$ "$C_1 < C_2 < \ldots < C_k$", i.e. cards $C_1, C_2, \ldots, C_k$ are in the same order as in $x$

  • $\Pr[A \text{ holds at time } t] \ge \Pr[C_k \text{ has not been reinserted yet}]$

  $= \Pr[T_k + T_{k+1} + \ldots + T_{n-1} + 1 > t]$

  $T_k$: time for $C_k$ to go from position $k \to k+1$

  $T_{k+1}$: time $k+1 \to k+2$

  $T_{n-1}$: time $(n-1) \to n$

  $+1$: reinsertion step

Notice $T_i$ geometric r.v. w. parameter $\frac{i}{n}$ (same as time to collect $(n-i+1)$-th coupon in coupon collector problem)

so "$1 + T_{n-1} + \dots + T_{k+1} + T_k > t$" $\equiv$ "after $t$ steps $k$ coupons are still uncollected"

(exercise 1 point) for fixed $k$: ✓ $\mathbb{P}r\left[\text{"after } t = n\log n - n\cdot d(n) \text{ steps } k \text{ coupons are still uncollected"}\right] \to 1, \quad n \to +\infty$

Hence $\mathbb{P}r[A \text{ holds at time } n\log n - n\cdot d(n)] \to 1, \quad n \to +\infty$

but $\pi[A \text{ holds}] = \frac{1}{k!}$

Hence $\|P_x^{(t)} - \pi\|_{TV} \geq 1 - \frac{1}{k!} - o(1)$

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## Part II: mixing via strong stationary times

**Def:** • A stopping time is a random variable $T \in \mathbb{N}$ s.t. event $\{T = t\}$ depends only on $X_0, X_1, \dots, X_t$

• A stopping time is a strong stationary time (SST), if

$$\forall z, x: \mathbb{P}r[X_t = z \mid T = t, X_0 = x] = \pi(z).$$

**Lemma:** If $T$ is SST, $\Delta_x(t) \leq \mathbb{P}r[T > t \mid X_0 = x]$, $\forall x$

**Proof:** $\mathbb{P}r[X_t = z \mid X_0 = x] = \underbrace{\mathbb{P}r[T \leq t \mid X_0 = x]} \cdot \mathbb{P}r[X_t = z \mid T \leq t, X_0 = x] + \mathbb{P}r[T > t \mid X_0 = x] \cdot \mathbb{P}r[X_t = z \mid T > t, X_0 = x]$

$$\left(1 - \mathbb{P}r[T > t \mid X_0 = x]\right) \cdot \pi(z) + \mathbb{P}r[T > t \mid X_0 = x] \cdot \mathbb{P}r[X_t = z \mid T > t, X_0 = x]$$

$$\frac{1}{2}\sum_z \left|\mathbb{P}r[X_t = z \mid X_0 = x] - \pi(z)\right| \leq \frac{1}{2}\sum_z \mathbb{P}r[T > t \mid X_0 = x] \cdot \left(\pi(z) + \mathbb{P}r[X_t = z \mid T > t, X_0 = x]\right) = \mathbb{P}r[T > t \mid X_0 = x]$$

$\boxtimes$

# Riffle - Shuffle

- Cut deck into stacks ~~of size~~ L, R where $|L| \sim Bin(n, \frac{1}{2})$

$\mathbb{P}r[|L| = k] = \frac{\binom{n}{k}}{2^n}$

- interleave L, R u.a.r.

each intelearing has probability $\frac{1}{\binom{n}{k}}$

(indeed there are $\left[\begin{matrix} |R|+1 \\ |L| \end{matrix}\right] = \binom{|R|+|L|}{|L|} = \binom{n}{k}$ interleavin

- repeat

Hence, a pair of cut & intelearing has probability:

$\mathbb{P}r[\text{-//-}] = \frac{1}{2^n}$

## Inverse Shuffle:

- label cards 0/1 u.a.r.
- keeping order of 0 cards pull them out of the stack and place them on top

Probability of each distinct move: $\frac{1}{2^n}$

Claim: Inverse Shuffle is inverse of Riffle Shuffle.

⇒ analyze inverse shuffle.

- after every repetition od the inverse shuffle, record the bit that was assigned to each card.
- after $t$ steps: each card $t$-digit number.
- Define stopping time: $T = \min\{t : \text{all cards different numbers}\}$
- Claim: $T$ is a SST.

[ If two cards, say 1♥ and 2♣, have different numbers then their relative position is random.]

- Analyze $T$

Relate to birthday problem → k people random birthdays from $c \cdot k^2$ dates then $\mathbb{P}r[\text{some pair same birthday}] \xrightarrow[k\to+\infty]{} 1-e^{-1/2c}$

$$\left( \mathbb{P}r[\text{~~some~~ no pair same birthday}] = \left(1-\frac{1}{ck^2}\right)\left(1-\frac{2}{ck^2}\right)\cdots\left(1-\frac{k-1}{ck^2}\right) \right.$$

$$\left. \leq e^{-\frac{1}{ck^2}-\frac{2}{ck^2}-\cdots-\frac{k-1}{ck^2}} = e^{-\frac{1}{ck^2}\frac{(k-1)\cdot k}{2}} = e^{-\frac{1}{2c}\cdot\left(1-\frac{1}{k}\right)} \xrightarrow[k\to+\infty]{} e^{-1/2c} \right)$$

- Back to inverse Riffle-Shuffle:

| | |
|---|---|
| # people | $n$ |
| # days | $2^t$ |

if $\underbrace{2^t \geq c^* n^2}$ then $\mathbb{P}r[T>t] \approx 1-e^{-1/2c^*} < \frac{1}{2e}$ for appropriate choice of $c^*$

$\downarrow$

$$t \geq 2\log_2 n + \log(c^*)$$

So $\tau_{mix} \leq 2\cdot\log_2 n + \Theta(1)$

Aldous: $\tau_{mix} \sim \frac{3}{2}\log_2 n$

Bayer + Diaconis: numerical computation of mixing time for any $t, n$

| $t$ | $\leq 4$ | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|
| $\Delta(t)$ | 1.00 | 0.92 | 0.61 | 0.33 | 0.17 | 0.09 |