Lecture 6

(1)

Overview:
- Sampling Graph Colorings
- Path Coupling

Graph Colorings:
- Input: $G=(V,E)$ graph; $Q=\{1,\dots,q\}$ set of colors
- Goal: Sample a u.a.r. legal coloring of G
  ↳ assign a color to each vertex so that no two adjacent nodes get same color
- Denote by Δ the maximum degree of G

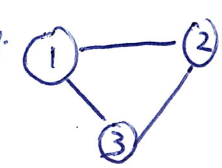
Decision Problem:
- $q \geq \Delta+1$: a legal coloring guaranteed to exist
- $q=\Delta$: Brook's theorem:
  - if $\Delta > 2$, the graph has a Δ-coloring iff it does not contain a $(\Delta+1)$-clique
  - if $\Delta = 2$, Δ-coloring exists $\Leftrightarrow$ no $(\Delta+1)$-clique & no odd cycle
- $q < \Delta$: NP-hard to decide

Working Regime: $q \geq \Delta+1$
→ since $q<\Delta$, NP-hard
→ $q=\Delta$, non-trivial

Natural MC:
- start at arbitrary legal coloring
- pick vertex v u.a.r. and color c u.a.r.
- recolor v with c if this gives legal coloring, o.w. do nothing

Clearly symmetric, aperiodic;

Not necessarily irreducible if $q \leq \Delta+1$; e.g.



$\Delta=2$
$q=3$
no moves

- **Exercise (1 pts)** Chain is irreducible if $q \geq \Delta + 2$.

- **Conjectures:**

  (1). (10 pts) Random sampling of legal colorings can be done in polynomial time if $q \geq \Delta + 1$.

  (2). (10 pts) The MC given above has mixing time $O(n \log n)$ if $q \geq \Delta + 2$

- **This lecture:** Show (2) for $q \geq 2\Delta + 1$.

- **Warming Up:** **Thm:** The mixing time of MC given earlier is $O(n \log n)$ if $q \geq 4\Delta + 1$.

Proof: → Use coupling; X_0, Y_0 are arbitrary; couple $(X_t)_t, (Y_t)_t$ as follows: pick same v, c at all times t

→ Denote: $d_t(X_t, Y_t)$: number of colors where X_t, Y_t differ

→ Accounting: for an arbitrary pair X_t, Y_t, out of all $n \cdot q$ choices of vertex, color:

good moves $(d_{t+1} = d_t - 1)$: choose disagreeing vertex v and choose color c that does not appear in neighborhood of v in neither of the colorings X_t, Y_t and hence will be accepted

at least $(q - 2\Delta) \times d_t$ good moves

bad moves $(d_{t+1} = d_t + 1)$: choose agreeing vertex v in the neighborhood of disagreeing vertex v_0 and color c that is acceptable in X_t & not in Y_t or vice versa

at most $\underbrace{2}_{\text{per disagreeing vertex } v_0}$ $\times$ $\underbrace{\Delta d_t}_{\text{vertices in neighborhood of disagreeing vertices}}$ bad moves

per disagreeing vertex v_0 ~~colors~~

~~are~~ 2 colors $C^{X_t}(v_0)$ & $C^{Y_t}(v_0)$

neutral moves ($d_{t+1} = d_t$): the rest

good moves - bad moves $\geq d_t \times (q - 4\Delta)$

$$\mathbb{E}\left[d_{t+1} \mid X_t, Y_t\right] \leq d_t - 1 \times \frac{d_t(q-2\Delta)}{qn} + 1 \times \frac{d_t \, 2\Delta}{qn} = d_t\left(1 - \frac{q-4\Delta}{qn}\right)$$

hence

$$\mathbb{E}\left[d_t \mid X_0, Y_0\right] \leq d_0 \left(1 - \frac{q-4\Delta}{qn}\right)^t$$

$$\leq n \cdot e^{-t \cdot \frac{q-4\Delta}{qn}} \leq \varepsilon$$

$\hookrightarrow t \geq \frac{q}{q-4\Delta} \cdot n \cdot \left(\log n + \log \frac{1}{\varepsilon}\right)$

$$\Rightarrow \Pr\left[X_t \neq Y_t \mid X_0, Y_0\right] = \Pr\left[d_t \geq 1 \mid X_0, Y_0\right] \leq \mathbb{E}\left[d_t \mid X_0, Y_0\right] \leq \varepsilon$$

$\hookrightarrow t \geq \frac{q}{q-4\Delta} n \left(\log n + \log \frac{1}{\varepsilon}\right)$

hence: $\tau(\varepsilon) = \frac{q}{q-4\Delta} n \left(\log n + \log \frac{1}{\varepsilon}\right)$

$\tau_{mix} = O\left(\frac{q}{q-4\Delta} n \log n\right)$

[sanity check: the larger q is the faster it mixes]

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Part II: Path Coupling

Bubley & Dyer '97

- **Def**: A pre-metric on Ω is a connected, undirected graph with positive weights s.t. every edge is a shortest path
  - ↳ i.e. if $x, y \in \Omega$ are adjacent in pre-metric, the weight of edge (x,y) should be = to shortest path weight
- Can extend pre-metric to metric by considering shortest path distances
- **Idea of Path Coupling**: define coupling only for pairs of states that are adjacent in pre-metric
  (Recall coupling of a MC needs to specify for all pairs of states z_1, z_2: $\Pr[X_{t+1}, Y_{t+1} \mid X_t = z_1, Y_t = z_2]$; path coupling requires that we only specify such distn's for z_1, z_2 that are adjacent in premetric.)

Theorem: • Suppose exists coupling $(X, Y) \xrightarrow{\Pr} (X', Y')$ defined only for pairs (X, Y) of states that are adjacent in pre-metric, & such that:

$$\mathbb{E}\left[d(X', Y') \mid X, Y\right] \leq (1-\alpha) \cdot d(X, Y) \qquad (*)$$

for some $\alpha \in [0,1]$, where d is the metric extending the pre-metric.

• Then can extend coupling to full coupling so that $(*)$ is satisfied for all $X, Y \in \Omega \times \Omega$.

Proof: • Let x, y be arbitrary states that are non-adjacent in premetric.

• fix arbitrary shortest path between these states:

$$x \equiv z_0 \to z_1 \to \dots \to z_{k-1} \to z_k \equiv y$$

• Need to specify: $\Pr[Z_0', Z_k' \mid z_0, z_k]$ that is a valid MC coupling from (z_0, z_k).

• We construct this from $\Pr[Z_0', Z_1' \mid z_0, z_1], \dots, \Pr[Z_{k-1}', Z_k' \mid z_{k-1}, z_k]$ which are known and given

- $\mathbb{P}r[Z_0', Z_k' \mid z_0, z_k]$ is sampled via the following randomized procedure:

  (1) sample Z_0', Z_1' from $\mathbb{P}r[Z_0', Z_1' \mid z_0, z_1]$

  (2) conditioning on sampled value of Z_1' sample Z_2' from distribution $\mathbb{P}r[Z_1', Z_2' \mid z_1, z_2]$

  ⋮

  (k) conditioning on sampled value of Z_{k-1}' sample Z_k' from distribution $\mathbb{P}r[Z_{k-1}', Z_k' \mid z_{k-1}, z_k]$

- **Claim 1**: The above procedure:
  i. is well-defined
  ii. is a valid coupling

- **Proof**: • We show by induction that $\mathbb{P}r[Z_i' = w] = P(z_i, w)$, $\forall i, w \in \Omega$. (MC matrix)
  i.e. we show that the step $z_i \longrightarrow Z_i'$ is a valid Markov Chain step from z.

  - For Z_0', Z_1' obvious, since $\mathbb{P}[Z_0', Z_1' \mid z_0, z_1]$ defines valid coupling.
  - Suppose claim true for Z_i; will show true for Z_{i+1}

    Since $\mathbb{P}r[Z_i', Z_{i+1}' \mid z_i, z_{i+1}]$ is valid coupling
    & claim is true for Z_i'

    we have
$$\mathbb{P}r[Z_{i+1}' = w] = \sum_{z_i'} P(z_i, z_i') \cdot \frac{\mathbb{P}r[Z_i' = z_i'; Z_{i+1}' = w \mid z_i, z_{i+1}]}{\underbrace{\sum_w \mathbb{P}r[Z_i' = z_i'; Z_{i+1}' = w \mid z_i, z_{i+1}]}_{\equiv\, P(z_i, z_i') \text{ since } \mathbb{P}r[Z_i', Z_{i+1}' \mid z_i, z_{i+1}] \text{ is valid coupling}}} \quad (*)$$

z_i' ↳ value sampled for Z_i' in step i of procedure

$P(z_i, z_i')$: probability that z_i' is sampled in previous step; by induction hypothesis this is equal to $P(z_i, z_i')$

$$\Rightarrow \mathbb{P}r[Z_{i+1}' = w] = \underbrace{\sum_{z_i'} \mathbb{P}r[Z_i' = z_i'; Z_{i+1}' = w \mid z_i, z_{i+1}]}_{\equiv\, P(z_{i+1}, w)}$$

because $\mathbb{P}r[Z_i', Z_{i+1}' \mid z_i, z_{i+1}]$ is valid coupling

$$\Rightarrow \Pr[Z'_{i+1} = w] \equiv P(z_{i+1}, w) \quad , \forall i, w \in \Omega$$

→ this readily establishes (ii)

→ Regarding (i), we have already implicitly shown it:

- notice that the highlighted ratio in (*) is the conditional probability from which we sample Z'_{i+1} conditioning on the value z'_i that was sampled for Z'_i in previous step;
- the above analysis shows that we only compute this ratio for values z'_i in the support of $P(z_i, z'_i)$ and hence the denominator is never 0.

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Claim 2: The distribution of (Z'_i, Z'_{i+1}) sampled from the procedure is identical to $\Pr[Z'_i, Z'_{i+1} | z_i, z_{i+1}]$, for all i.

Proof: follows easily from claim 1.

Now:

triangle inequality

$$\mathbb{E}\left[d(Z'_0, Z'_k) | z_0, z_k\right] \le \mathbb{E}\left[\sum_{i=0}^{k-1} d(Z'_i, Z'_{i+1}) | z_0, z_k\right]$$

linearity of expectation

$$\le \sum_{i=0}^{k-1} \mathbb{E}\left[d(Z'_i, Z'_{i+1}) | z_0, z_k\right]$$

using claim 2

$$\le \sum_{i=0}^{k-1} (1-\alpha)\, d(z_i, z_{i+1})$$

$$= (1-\alpha) \cdot \boxed{\sum_{i=0}^{k-1} d(z_i, z_{i+1})}$$

$$\| \quad d(z_0, z_k).$$

because $z_0 \to z_1 \to \dots \to z_k$ was a shortest path.

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