Intro to Bayesian Mechanism Design

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Mechanism Design

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General Theme: resource allocation.

Overview

Optimal Mechanism Design:

- single-item auction.
- objectives: social welfare vs. seller profit.
- characterization of Bayes-Nash equilibrium.
- consequences: solving for and optimizing over BNE.

Single-item Auction

Mechanism Design Problem: *Single-item Auction*

Given:

- one item for sale.
- n bidders (with unknown private values for item, $v_1, \ldots, v_n$)
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- Auction to solicit bids and choose winner and payments.

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Possible Auction Objectives:

- Maximize *social surplus*, i.e., the value of the winner.
- Maximize *seller profit*, i.e., the payment of the winner.

Objective 1: maximize social surplus

Example Auctions

First-price Auction

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Questions:

- what are equilibrium strategies?
- what is equilibrium outcome?
- which has higher surplus in equilibrium?
- which has higher profit in equilibrium?

Second-price Auction Equilibrium Analysis

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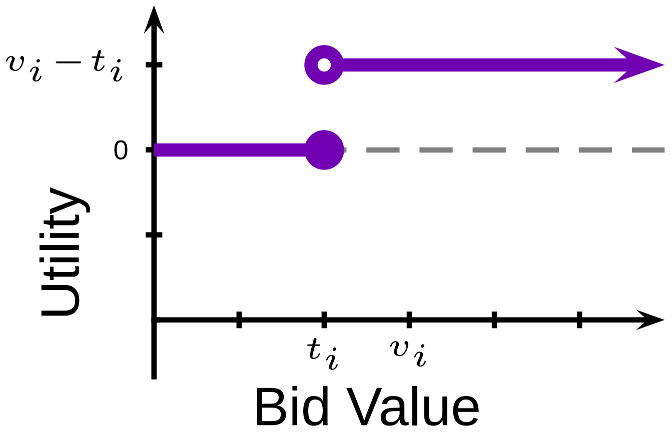

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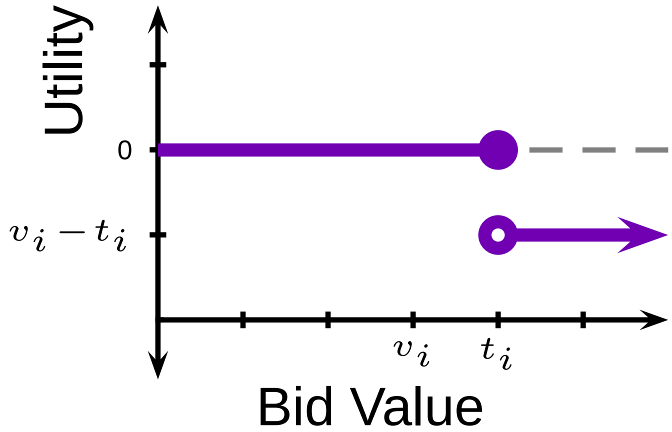

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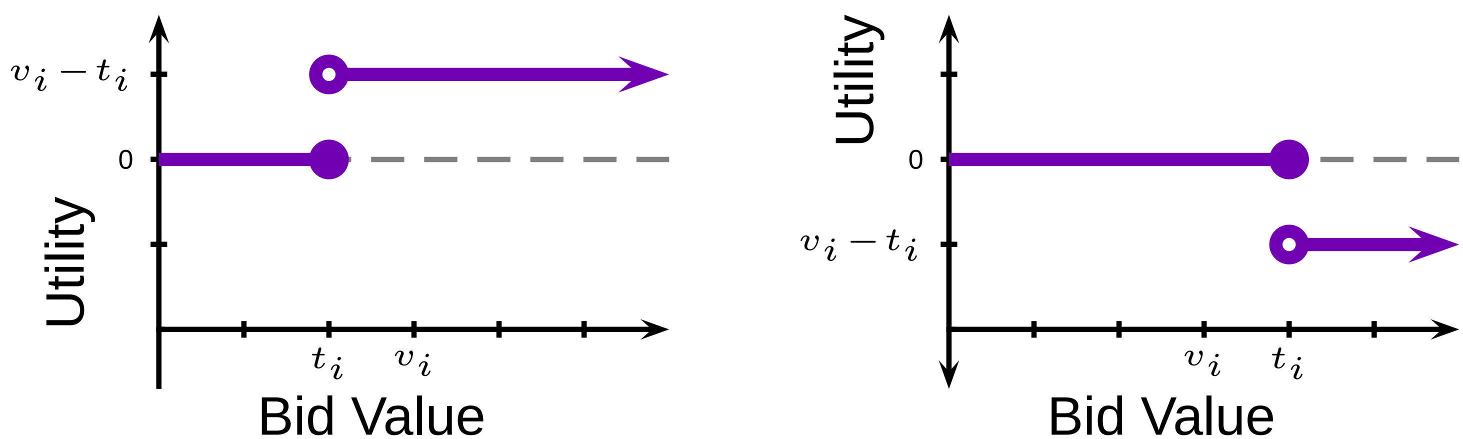

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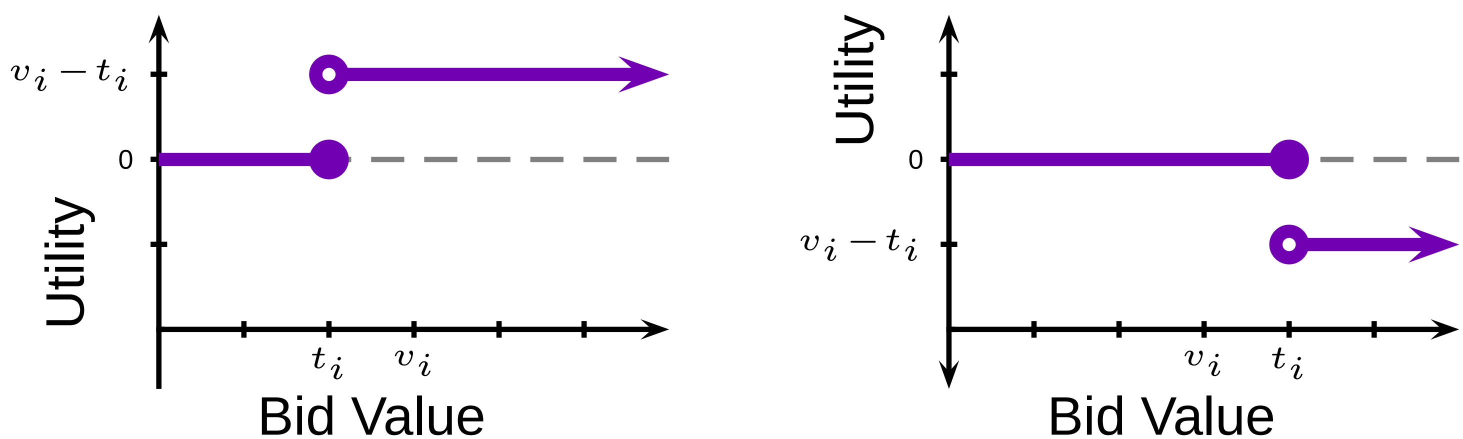

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What about first-price auction?

Recall First-price Auction

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How would you bid?

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Note: first-price auction has no DSE.

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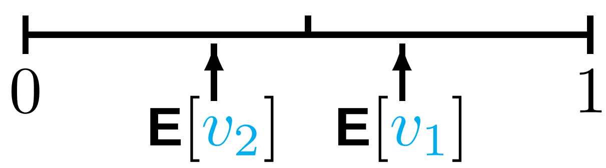

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Conclusion 2: bidder with highest value wins

Conclusion 3: first-price auction maximizes social surplus!

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Notation:

- $F_i(z) = \mathbf{Pr}[v_i \leq z]$ is *cumulative distribution function*, (e.g., $F_i(z) = z$ for uniform distribution)
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Definition: a *strategy profile* is in *Bayes-Nash Equilibrium (BNE)* if for all i, $s_i(v_i)$ is best response when others play $s_j(v_j)$ and $v_j \sim F_j$.

Surplus Maximization Conclusions

Conclusions:

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Questions?

Objective 2: maximize seller profit

(other objectives are similar)

An example

Example Scenario: two bidders, uniform values

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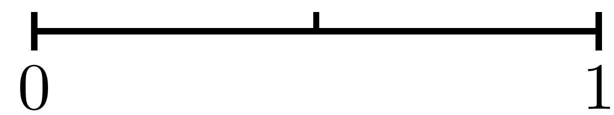

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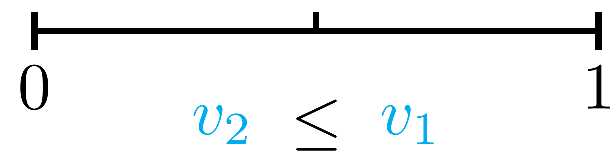

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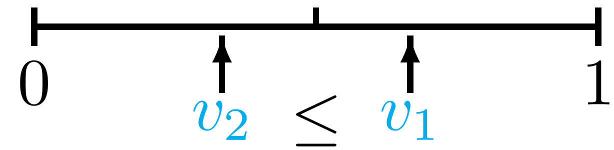

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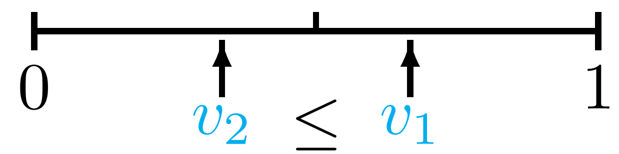

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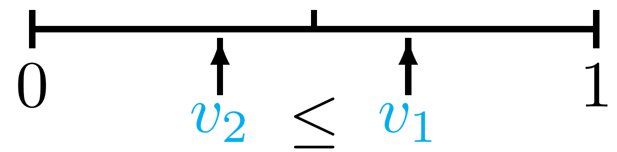

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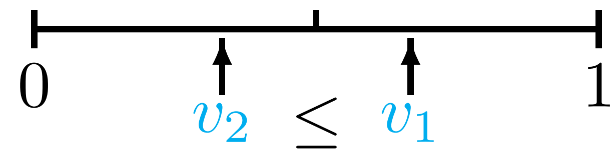

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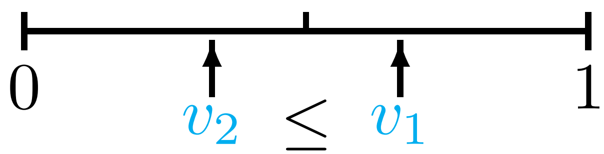

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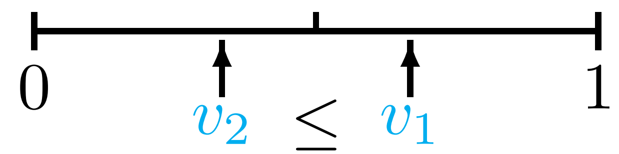

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Surprising Result: second-price and first-price auctions have same expected profit.

Can we get more profit?

Second-price with reserve price

Second-price Auction with reserve r

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Case 2: $v_1 \geq v_2 \geq \frac{1}{2}$	$1/4$	**E**[v_2 | Case 2]
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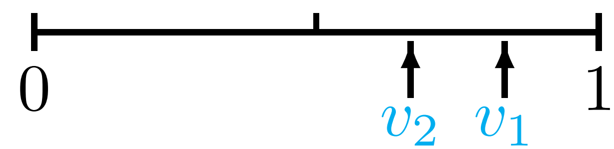

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Case 2: $v_1 \geq v_2 \geq \frac{1}{2}$	$1/4$	**E**[v_2 | Case 2] $= \frac{2}{3}$
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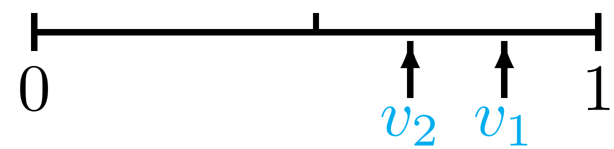

- Sort values, $v_1 \geq v_2$

Case Analysis:	**Pr**[Case i]	**E**[Profit]
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Case 2: $v_1 \geq v_2 \geq \frac{1}{2}$	$1/4$	**E**$[v_2 \mid \text{Case } 2] = \frac{2}{3}$
Case 3: $v_1 \geq \frac{1}{2} > v_2$	$1/2$	$\frac{1}{2}$

E[profit of 2nd-price with reserve] $= \frac{1}{4} \cdot 0 + \frac{1}{4} \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{2} = \frac{5}{12}$

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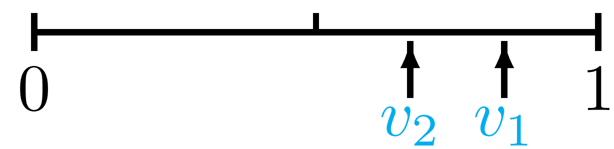

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Observations:

- pretending to value the good increases seller profit.
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Bayes-Nash Equilibrium Characterization and Consequences

- solving for BNE
- optimizing over BNE

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- $\mathbf{x}$ is an allocation, x_i the allocation for i.
- $\mathbf{x}(\mathbf{v})$ is BNE allocation of mech. on valuations $\mathbf{v}$.
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Analogously, define $\mathbf{p}$, $\mathbf{p}(\mathbf{v})$, and $p_i(v_i)$ for payments.

Characterization of BNE

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
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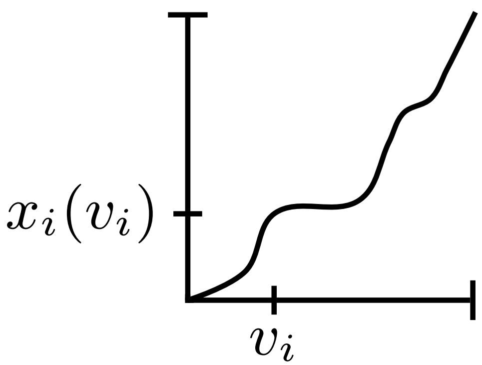

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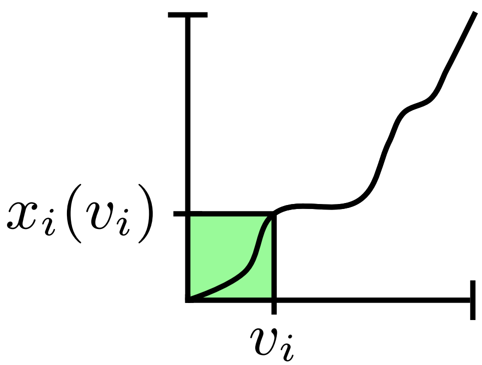

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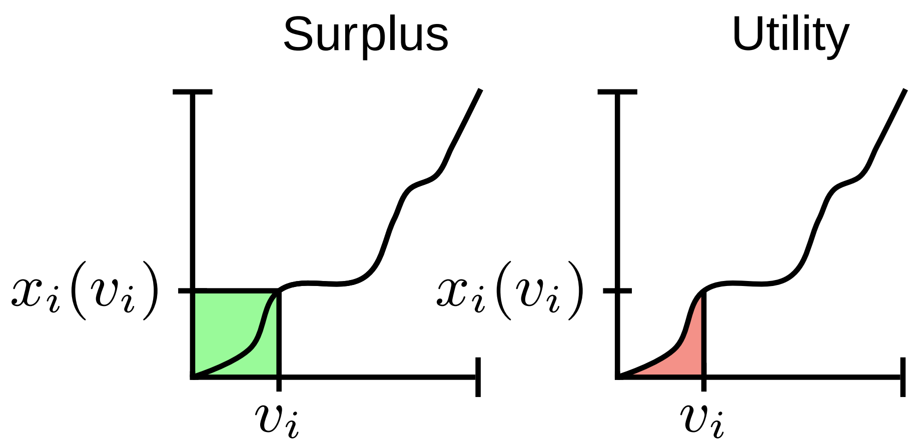

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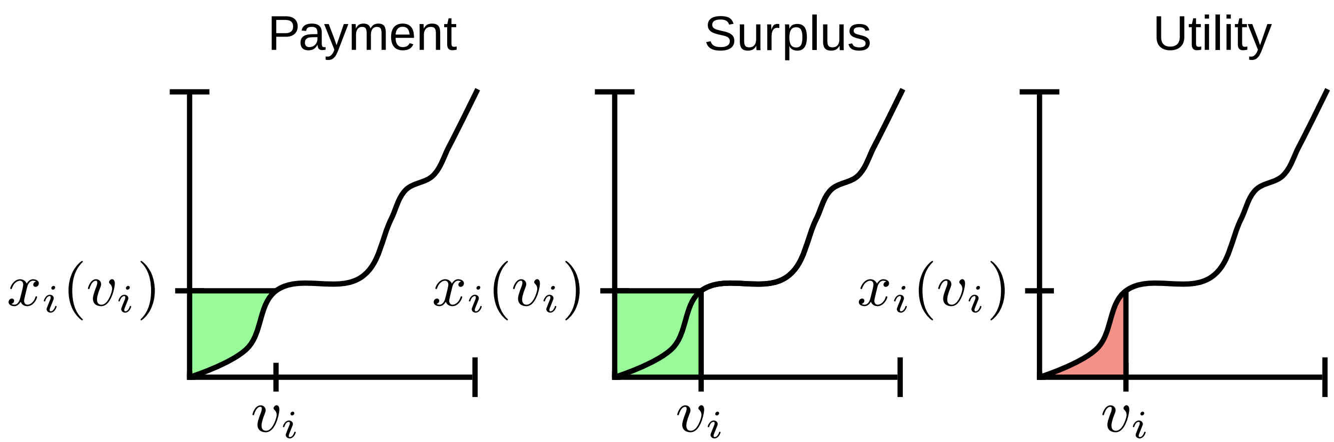

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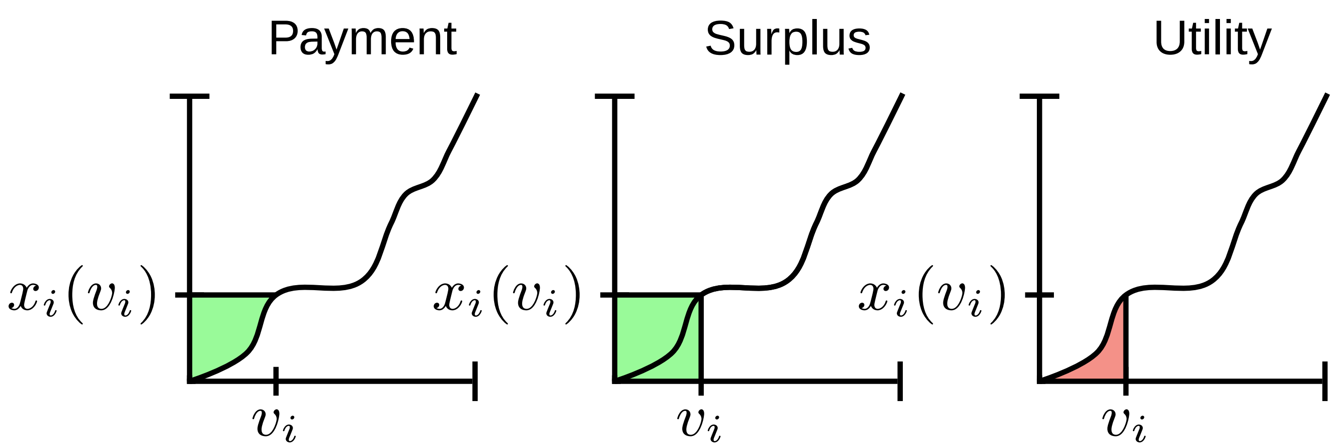

Consequence: *(revenue equivalence)* in BNE, auctions with same outcome have same revenue (e.g., first and second-price auctions)

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Solving for equilbrium:

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3. Verify guess and BNE: $b(v)$ continuous, strictly increasing, symmetric.

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Proof: expected virtual valuation of winner $=$ expected payment.

Proof of Lemma

Recall Lemma: In BNE, $\mathbf{E}[p_i(v_i)] = \mathbf{E}\left[\left(v_i - \frac{1-F_i(v_i)}{f_i(v_i)}\right) x_i(v_i)\right]$.

Proof Sketch:

- Use characterization: $p_i(v_i) = v_i x_i(v_i) - \int_0^{v_i} x_i(v)dv$.
- Use definition of expectation (integrate payment $\times$ density).
- Swap order of integration.
- Simplify.

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What is optimal single-item auction for $U[0,1]$?

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Optimal auction for $U[0,1]$:

- $F(v_i) = v_i$.
- $f(v_i) = 1$.
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- So, optimal auction is Second-price Auction with reserve 1/2!

Conclusions:

- expected virtual value $=$ expected revenue
- optimal mechanism maximizes virtual surplus.
- optimal auction depends on distribution.
- i.i.d., regular distributions: second-price with reserve is optimal.
- theory is “descriptive”.

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Bayes-Nash Equilibrium Characterization Proof

Proof Overview

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Proof Overview:

$\Longrightarrow$ 1. BNE $\Leftarrow$ M & PI

2. BNE $\Rightarrow$ M

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BNE $\Leftarrow$ M & PI

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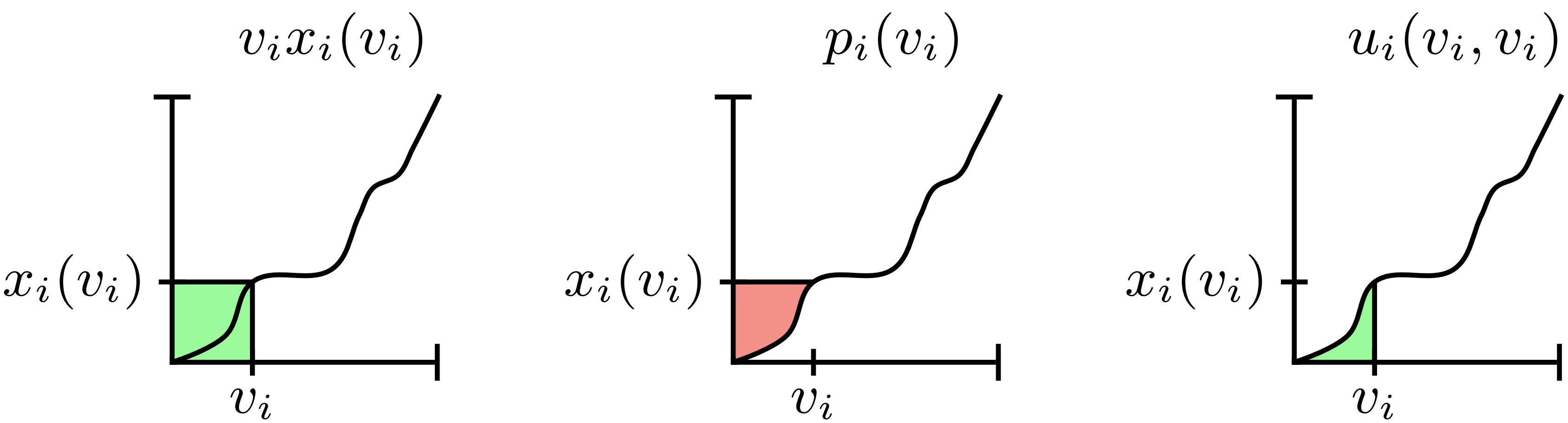

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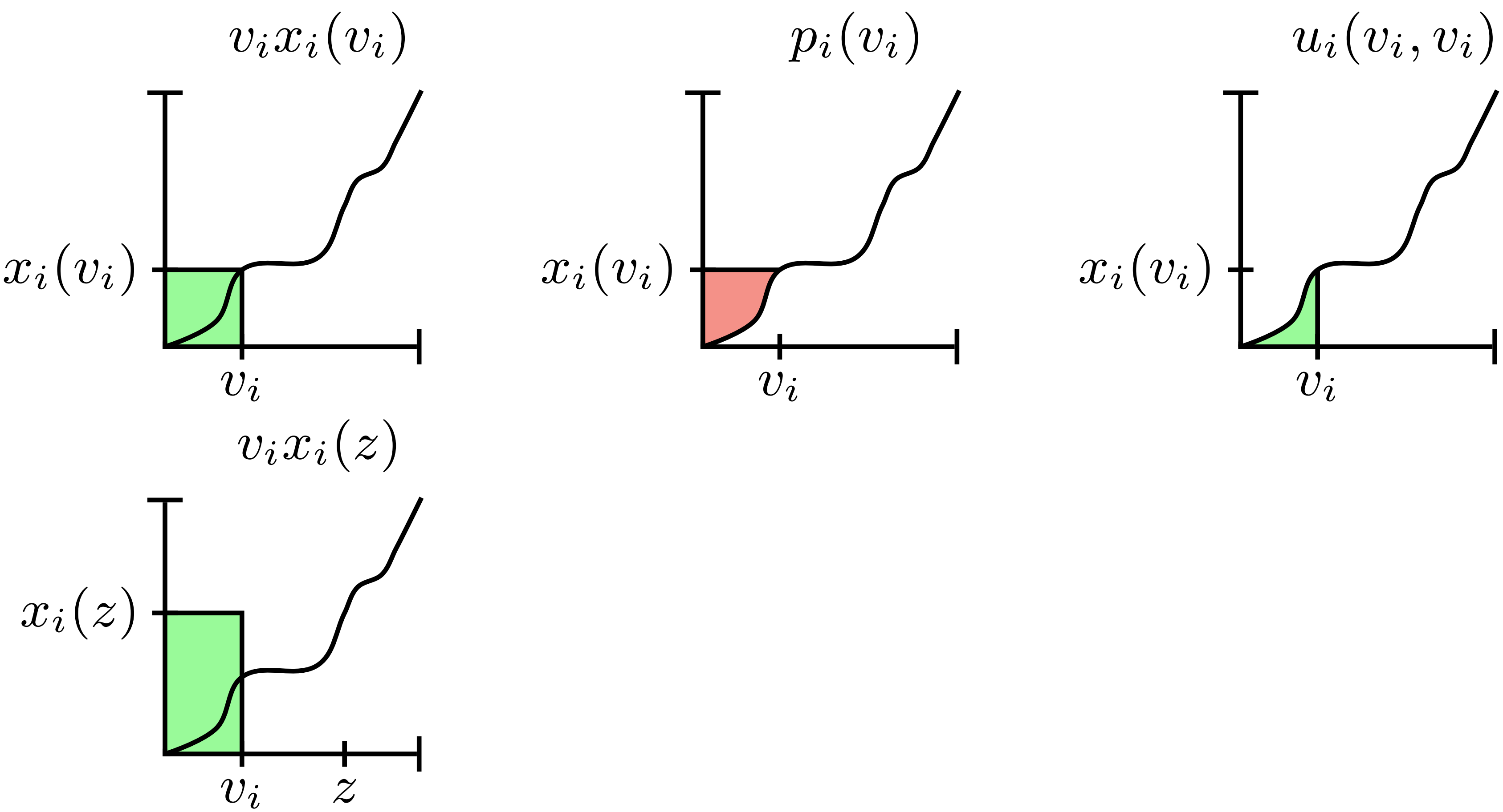

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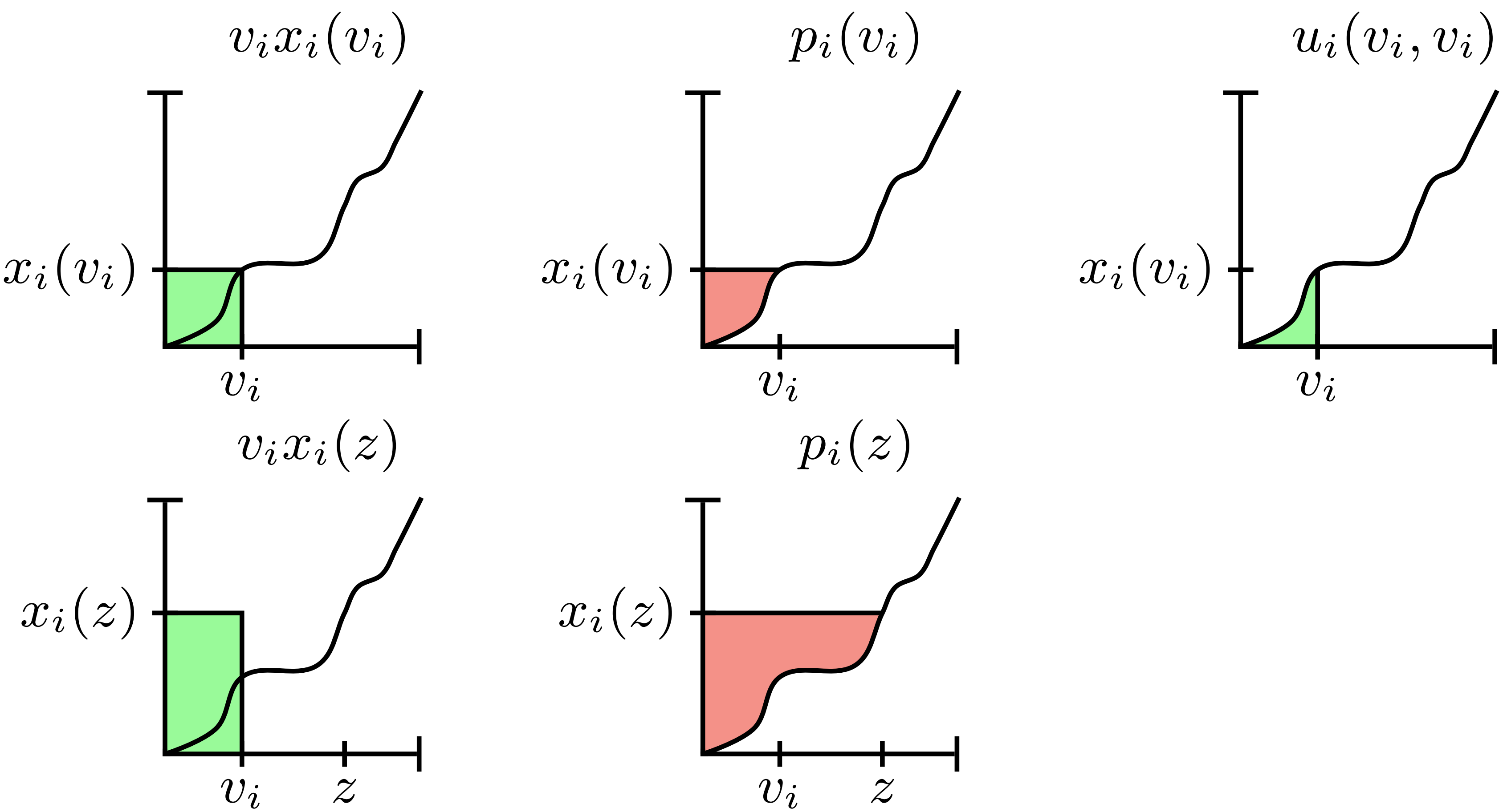

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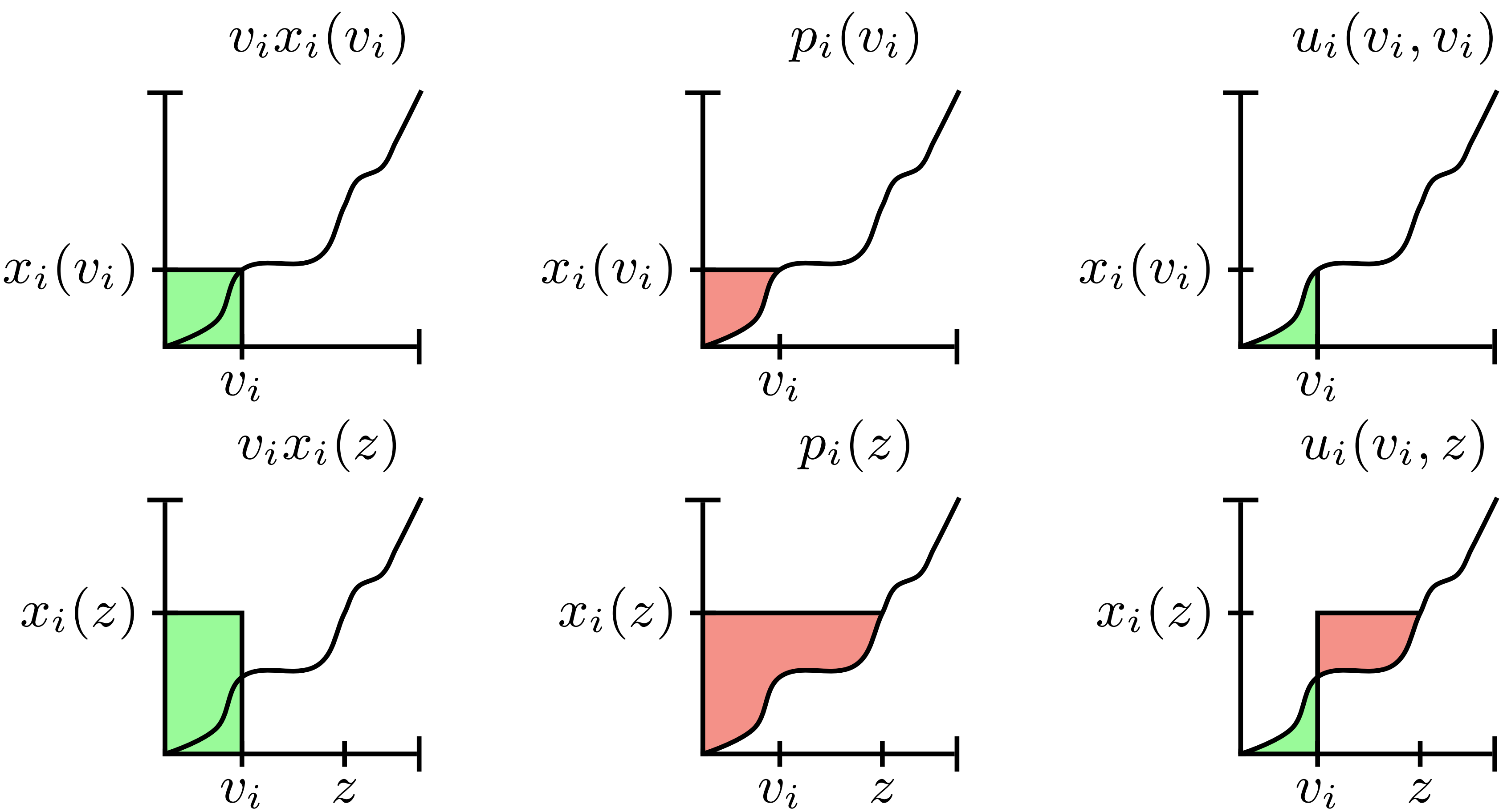

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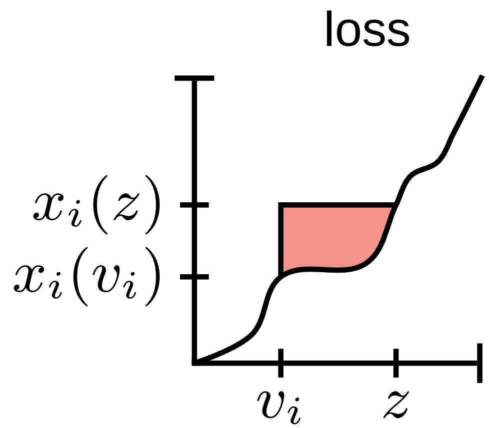

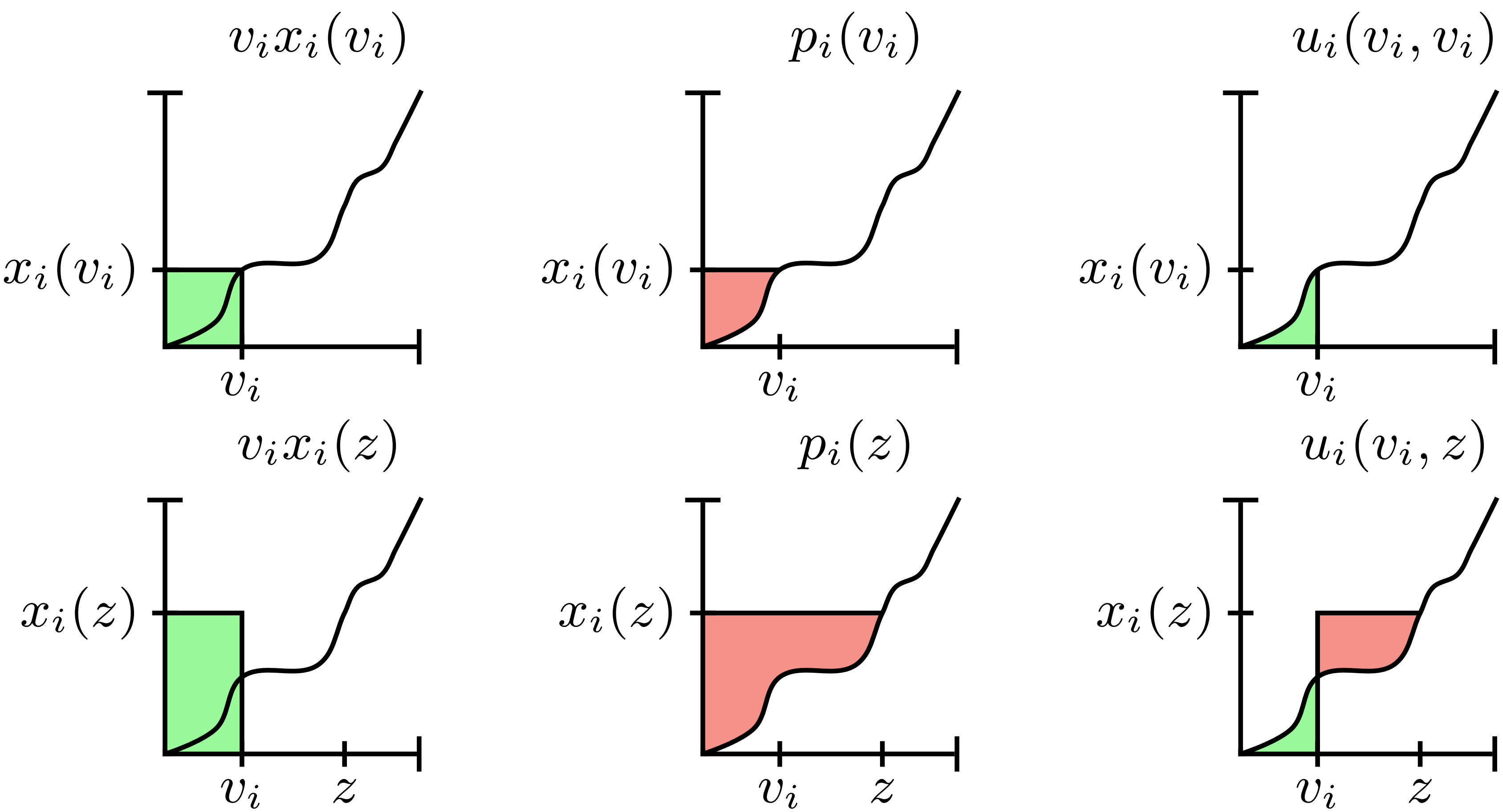

BNE $\Leftarrow$ M & PI (cont)

Claim: BNE $\Leftarrow$ M & PI

Case 2: mimicking $z < v_i$

BNE $\Leftarrow$ M & PI (cont)

Claim: BNE $\Leftarrow$ M & PI

Case 2: mimicking $z < v_i$

Recall: loss $= u_i(v_i, v_i) - u_i(v_i, z)$.

Recall: $u_i(v_i, z) = v_i x_i(z) - p_i(z)$

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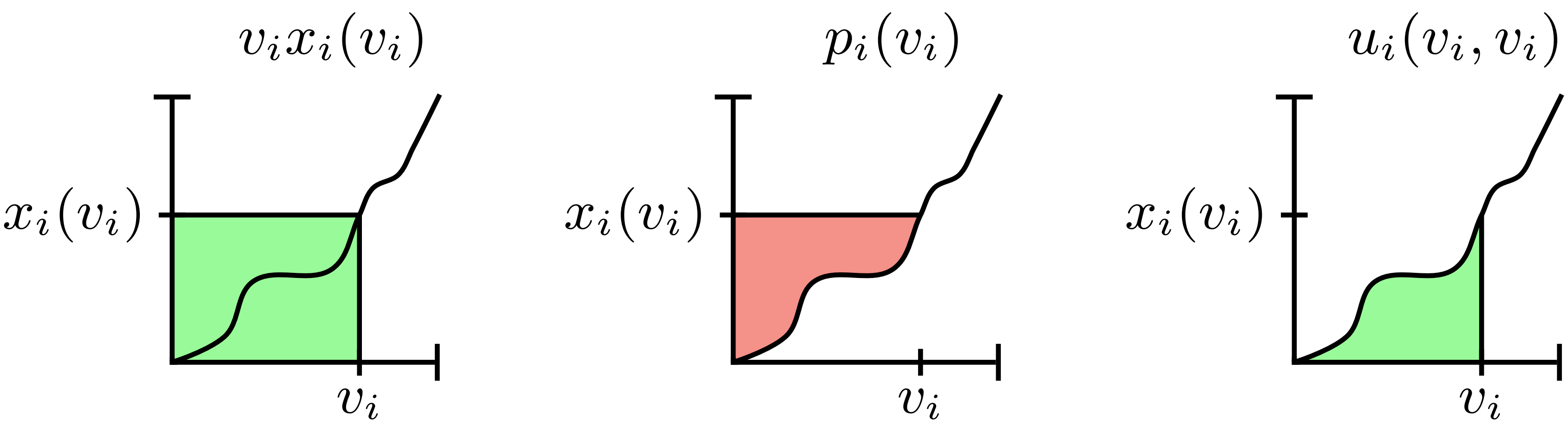

BNE $\Leftarrow$ M & PI (cont)

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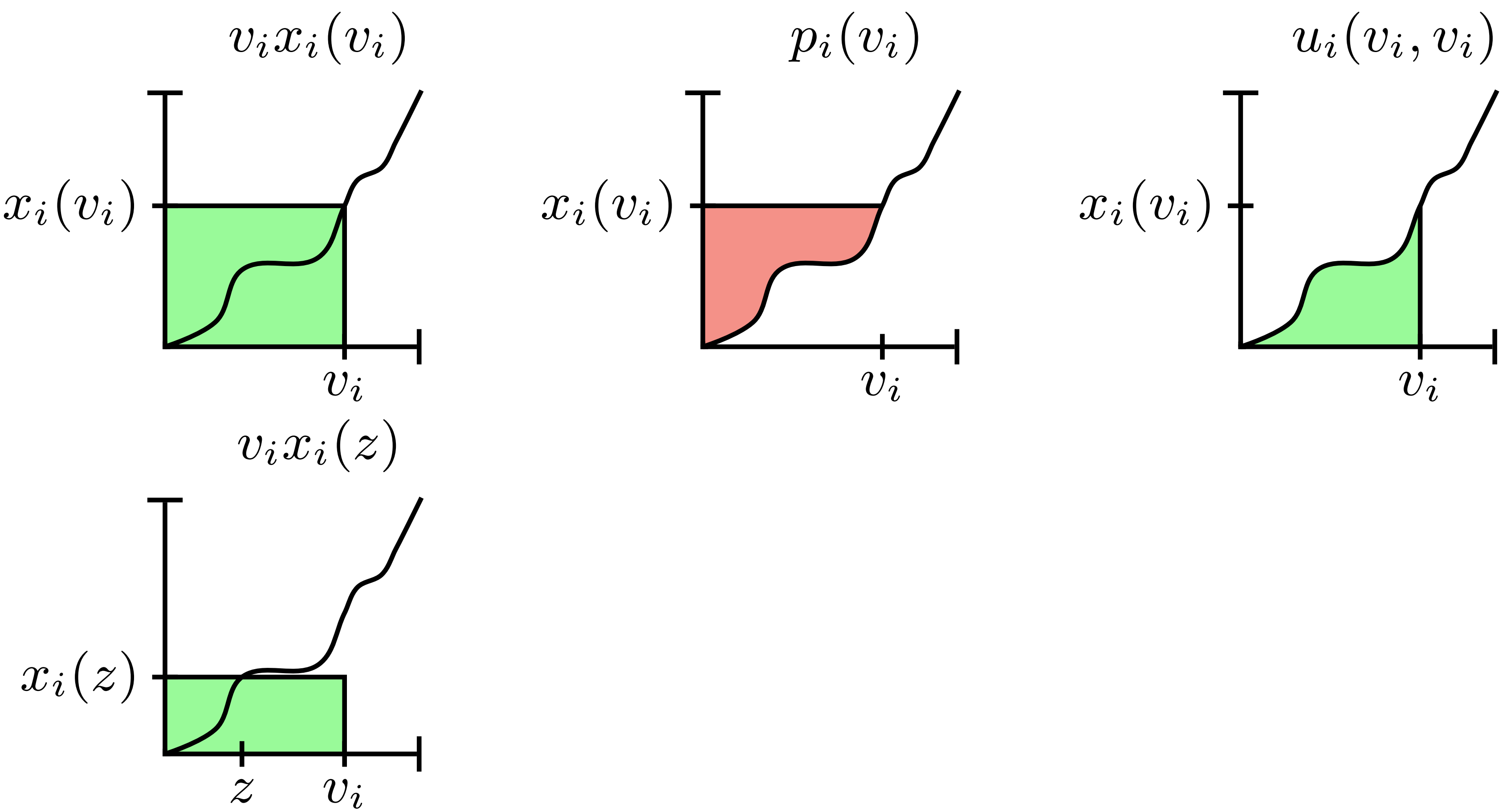

BNE $\Leftarrow$ M & PI (cont)

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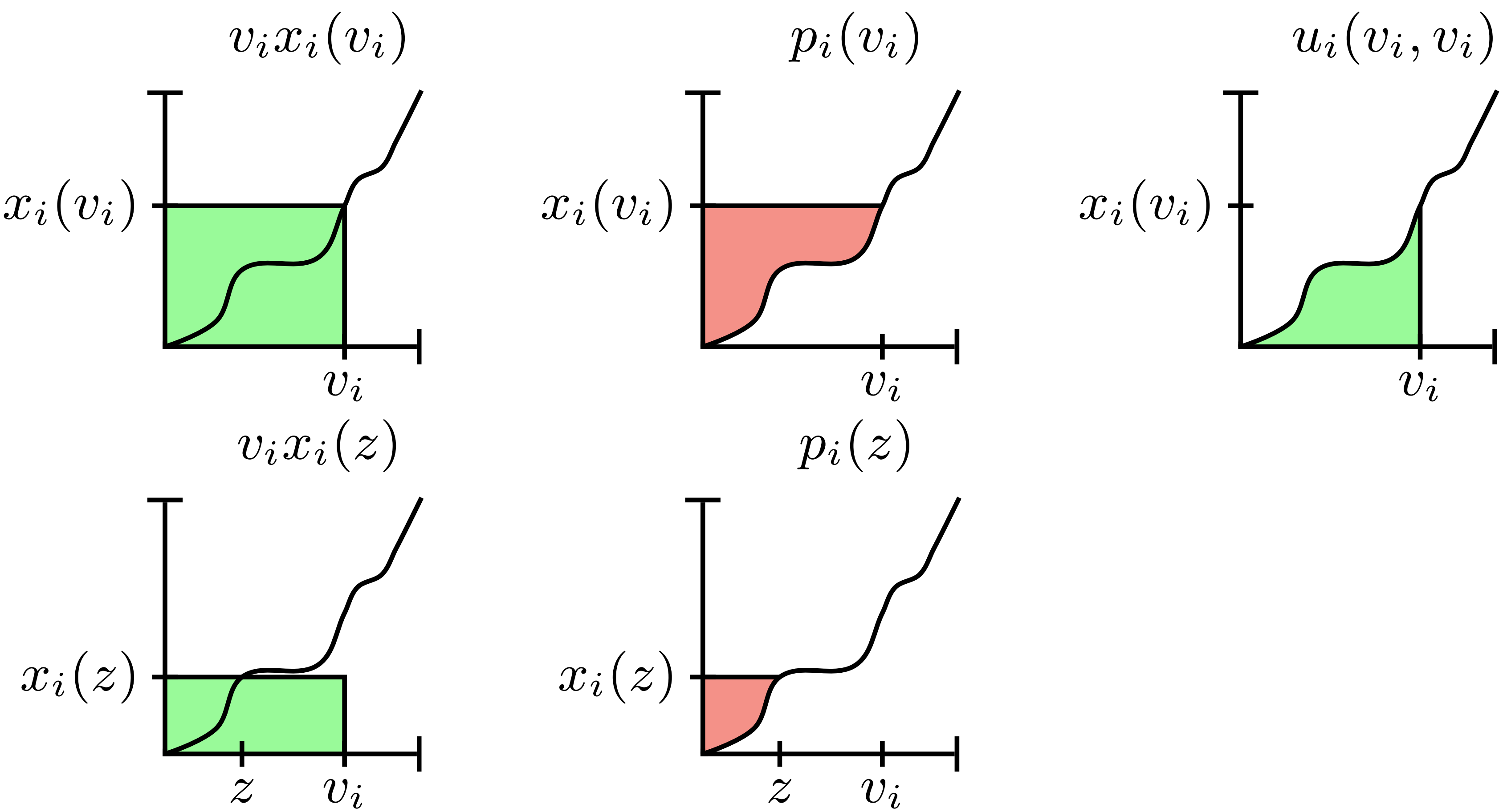

BNE $\Leftarrow$ M & PI (cont)

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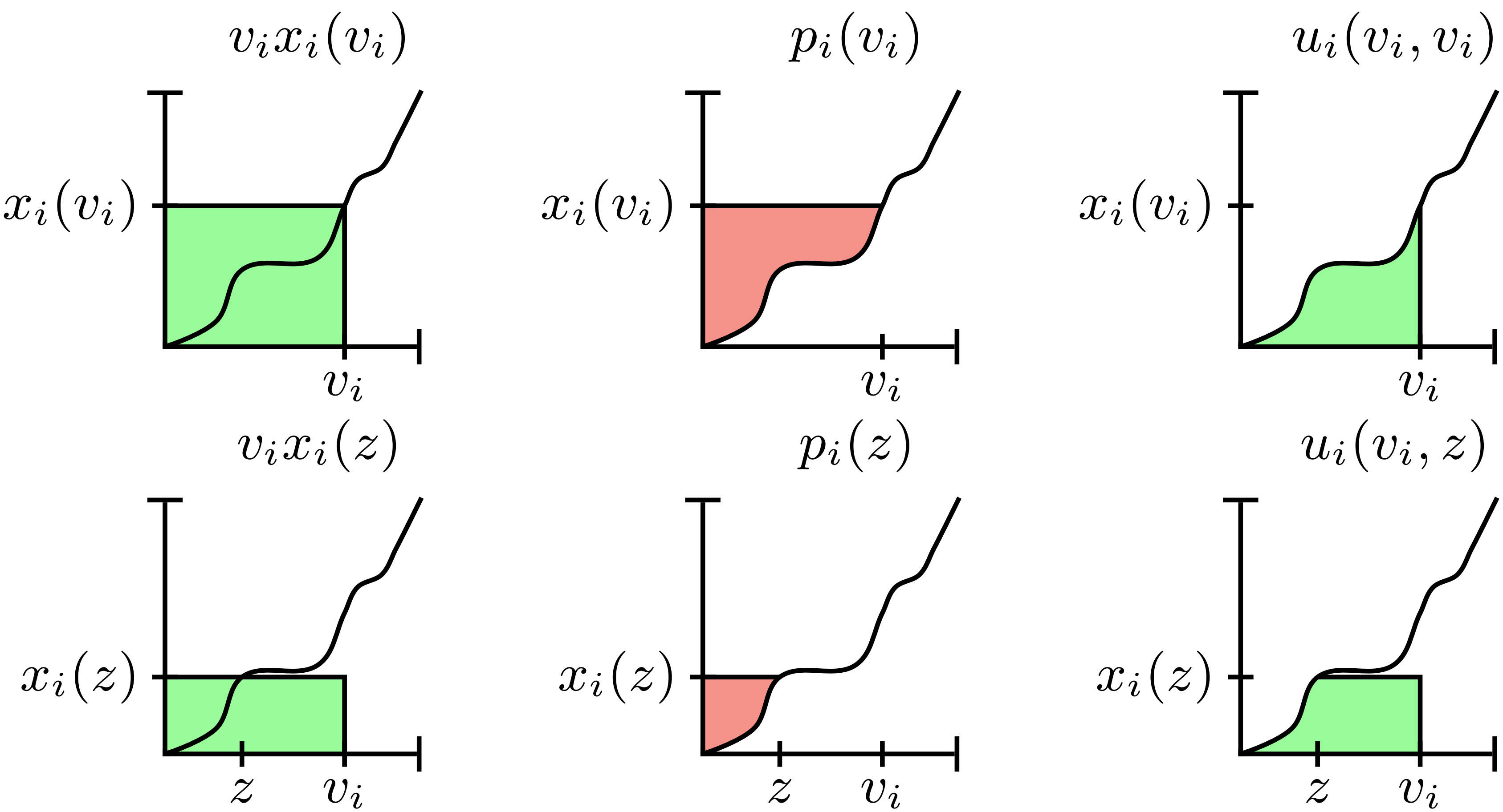

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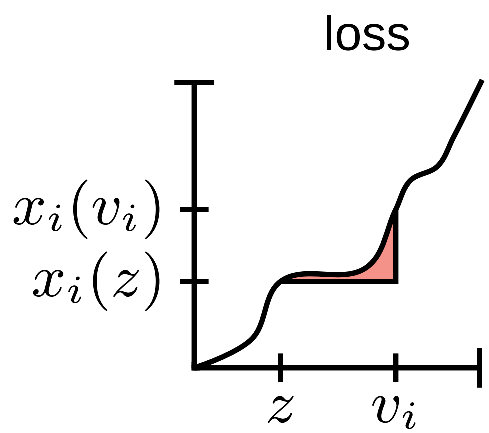

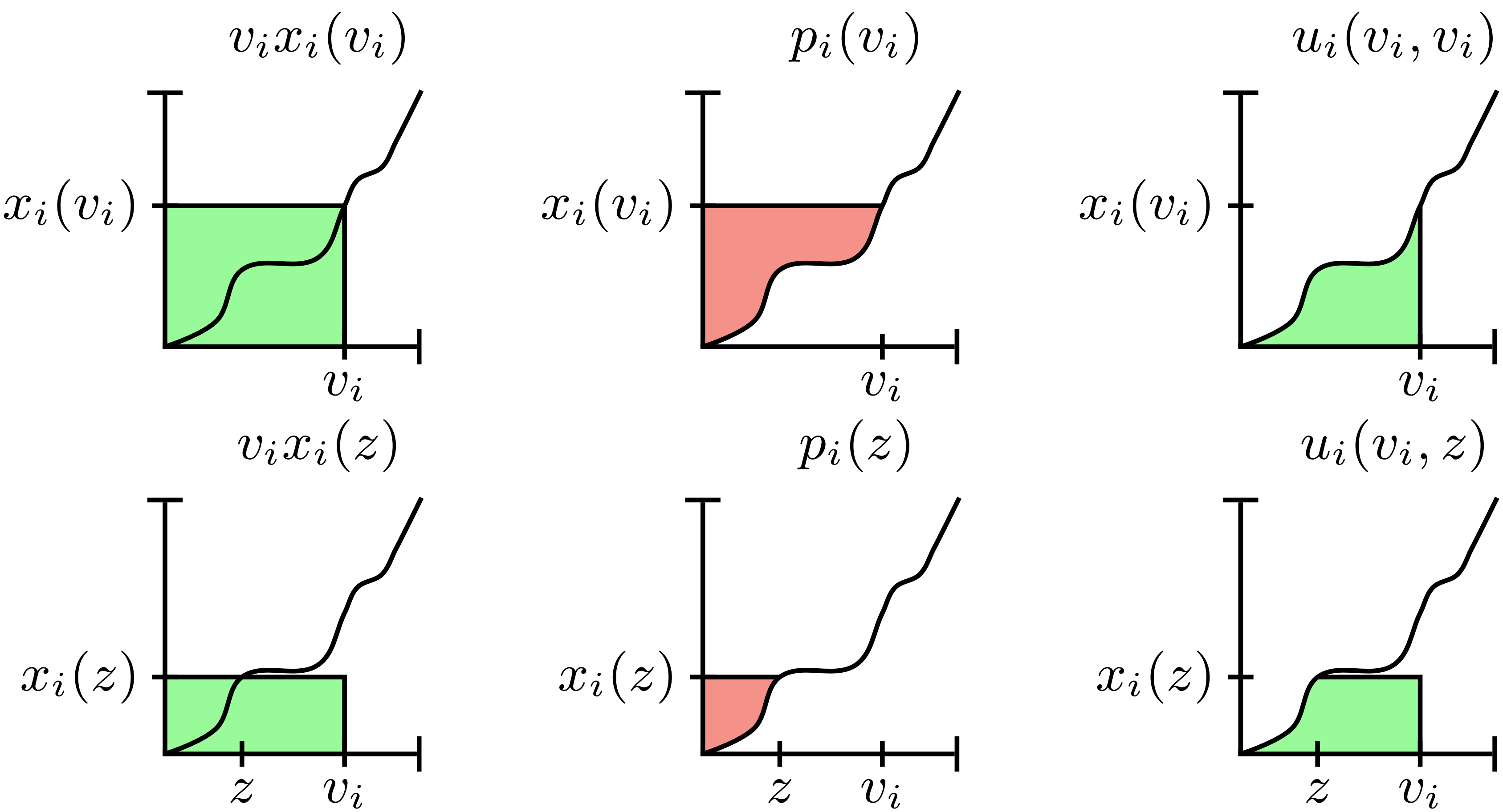

Proof Overview

Thm: a mechanism and strategy profile is in BNE iff

1. *monotonicity (M)*: $x_i(v_i)$ is monotone in v_i.
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and usually $p_i(0) = 0$.

Proof Overview:

1. BNE $\Leftarrow$ M & PI
2. BNE $\Rightarrow$ M
3. BNE $\Rightarrow$ PI

BNE $\Rightarrow$ M

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- Take $v_i = z'$ and $z = z''$ and vice versa:

$$z'' x_i(z'') - p_i(z'') \geq z'' x_i(z') - p_i(z')$$
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$$z''x_i(z'') + z'x_i(z') \geq z''x_i(z') + z'x_i(z'')$$

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- Regroup:

$$(z'' - z')(x_i(z'') - x_i(z')) \geq 0$$

- So $x_i(z)$ is monotone:

$$z'' - z' > 0 \Rightarrow x(z'') \geq x(z')$$

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Thm: a mechanism and strategy profile is in BNE iff

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- Picture:

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

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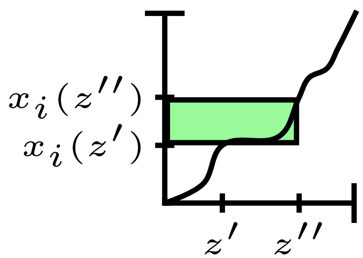

upper bound

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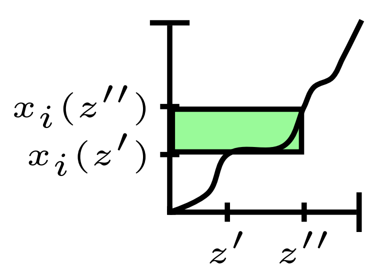
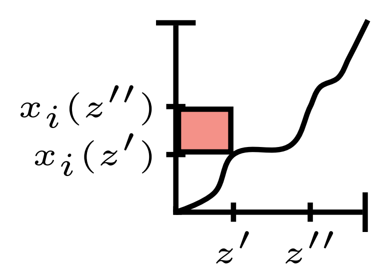


upper bound



lower bound

BNE $\Rightarrow$ PI

Claim: BNE $\Rightarrow$ PI.

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- Take $v_i = z'$ and $z = z''$ and vice versa:

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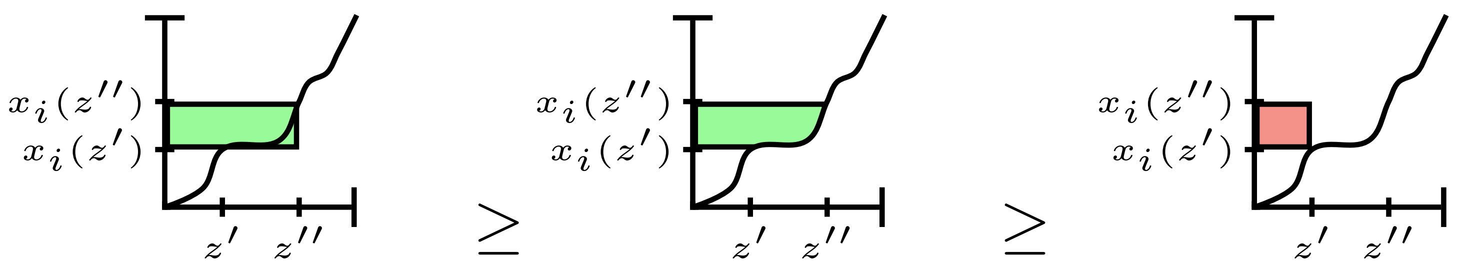
- Picture:



upper bound   only solution   lower bound

Characterization Conclusion

Thm: a mechanism and strategy profile is in BNE iff

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Questions?

Workshop Overview

- Are there simple mechanisms that are approximately optimal? Are there prior-independent mechanisms that are approximately optimal? [Roughgarden 10am & 11am]

- What are optimal auctions for multi-dimensional agent preferences, is it tractable to compute? [Daskalakis 11:30am]

- Are there black-box reductions for converting generic algorithms to mechanisms? [Immorlica 2:30pm]

- Are there good mechanisms for non-linear objectives (e.g., makespan)? [Chawla 3:30pm & 4:30pm]

- Are practical mechanisms good in equilibrium (e.g., “price of anarchy”)? [Tardos 5pm]