Revenue Maximization and Ex-Post Budget Constraints

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We consider the problem of a revenue-maximizing seller with m items for sale to n additive bidders with hard budget constraints, assuming that the seller has some prior distribution over bidder values and budgets. The prior may be correlated across items and budgets of the same bidder, but is assumed independent across bidders. We target mechanisms that are Bayesian incentive compatible, but that are *ex-post* individually rational and *ex-post* budget respecting. Virtually no such mechanisms are known that satisfy all these conditions and guarantee any revenue approximation, even with just a single item. We provide a computationally efficient mechanism that is a 3-approximation with respect to all BIC, ex-post IR, and ex-post budget respecting mechanisms. Note that the problem is NP-hard to approximate better than a factor of 16/15, even in the case where the prior is a point mass. We further characterize the optimal mechanism in this setting, showing that it can be interpreted as a *distribution over virtual welfare maximizers.*

We prove our results by making use of a black-box reduction from mechanism to algorithm design developed by Cai et al. Our main technical contribution is a computationally efficient 3-approximation algorithm for the algorithmic problem that results from an application of their framework to this problem. The algorithmic problem has a mixed-sign objective and is NP-hard to optimize exactly, so it is surprising that a computationally efficient approximation is possible at all. In the case of a single item ($m = 1$), the algorithmic problem can be solved exactly via exhaustive search, leading to a computationally efficient exact algorithm and a stronger characterization of the optimal mechanism as a distribution over virtual value maximizers.

CCS Concepts: • **Theory of computation → Algorithmic game theory and mechanism design**; **Algorithmic mechanism design**; **Computational pricing and auctions**;

Additional Key Words and Phrases: Revenue optimization, budget constraints, virtual welfare, generalized assignment problem

ACM Reference format:
Constantinos Daskalakis, Nikhil R. Devanur, and S. Matthew Weinberg. 2018. Revenue Maximization and Ex-Post Budget Constraints. *ACM Trans. Econ. Comput.* 6, 3–4, Article 20 (October 2018), 19 pages.
https://doi.org/10.1145/3274647

C. Daskalakis was supported by a Microsoft Research Faculty Fellowship and NSF Awards CCF-0953960 (CAREER) and CCF-1101491.

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2167-8375/2018/10-ART20 $15.00
https://doi.org/10.1145/3274647

1 INTRODUCTION

Most of auction theory crucially depends on the assumption of quasi-linear utilities, that the utility is equal to valuation minus payments. This assumption fails when bidders are budget constrained.[1] Auctions with budget-constrained bidders are commonplace, and prominent examples of this are ad auctions and auctions for government licensing such as the FCC spectrum auction. An interesting example of budget constraint occurs in the auction for professional cricket players in the Indian Premier League: the league imposes a budget constraint on all the teams as a means of ensuring well balanced teams.[2] Another source of budget constraints is what Che and Gale (1998) call the *moral hazard problem*: procurement is often delegated and budget constraints are imposed as a means of controlling spending. A budget represents the bidder's *ability to pay*, in contrast to the valuation, which represents his *willingness to pay*. For this reason, budgets may be more tangible and easier to estimate than valuations. It is therefore important to understand how budget constraints impact the design of auctions; this has been well established by now (Che and Gale 1998; Pai and Vohra 2014; Benoit and Krishna 2001; Laffont and Robert 1996; Maskin 2000; Malakhov and Vohra 2005; Che and Gale 2000; Bhattacharya et al. 2010).

The theory of auctions in the presence of budget constraints on bidders lags far behind the theory of auctions without budgets. For instance, consider the design of *optimal* (revenue maximizing) auctions that are Bayesian incentive compatible (BIC) and ex-post individually rational (IR). While Myerson (1981) gives a beautiful theory characterizing the optimal auction for any single parameter domain, no such characterization is known in the presence of private budgets (that could be correlated with the valuation). As a way to deal with this difficulty, previous papers have considered special cases and auctions with a subset of the desired properties. (See Section 1.2 for details.) We adopt the computer science approach of approximation, while incorporating all the desired properties. *The main result of this article is a 3-approximation to the optimal auction in the class of auctions that are*

- *BIC,*
- *ex-post IR, and*
- *ex-post budget respecting, with private budgets that could be correlated with the valuations,*

for multiple heterogenous items and additive valuations. This is the first constant factor approximation for this class of auctions. Moreover, the computational problem, even without any incentive constraints is already NP-hard to approximate within a ratio of 16/15 (Chakrabarty and Goel 2010). This too suggests that an approximation is necessary. We provide a computationally efficient algorithm to find such an approximately optimal auction, which itself can be implemented computationally efficiently as well.

1.1 Overview of Techniques

We prove our main result by making use of an algorithmic framework developed in Cai et al. (2013). The computational aspect of their framework provides a *black-box reduction* from a wide class of Bayesian mechanism design problems to problems of purely algorithm design. More specifically, they show that any α-approximation algorithm for a certain incentive-free algorithmic problem (induced by the mechanism design problem at hand) can be leveraged to find a BIC, IR mechanism that is also an α-approximation (to the optimal BIC, IR mechanism) in polynomial time. Significant further details on their reduction and how to employ it can be found in Section 3.1. After

[1]The terms financially constrained bidders or bidders with liquidity constraints are used synonymously.

[2]Such salary caps in fact exist in many professional sports, although in most leagues teams are built indirectly through negotiations rather than directly via an auction.

applying their framework to our problem, there is still the issue of solving the algorithmic problem that pops out of the reduction. This turns out to be essentially a (virtual) welfare maximization problem (without budgets), but where the bidder types are somewhat involved. The optimization involves a mixed sign objective (i.e., the objective is a sum of several terms, each of which can be positive or negative). Such optimization problems are typically solvable *exactly* in polynomial time or computationally hard to approximate within any finite factor, but rarely in between (due to the mixed signs in the objective). Interestingly, we obtain a 3-approximation for our mixed-sign objective problem despite the fact that it is NP-hard to optimize exactly. The design and analysis of our algorithm can be found in Section 4.

Cai et al.'s framework also contains a structural result. We use it to show that the optimal auction in our setting is a *distribution* over virtual welfare maximizers. By this, we mean that the optimal mechanism maintains a distribution over n mappings, one mapping per bidder that maps types to virtual types, and, given a vector of reported types, it samples n mappings from this distribution, uses them to map the reported types to virtual types, and proceeds to choose an allocation that optimizes virtual welfare. Note that by virtual types in the previous sentence, we do *not* mean the specific virtual types as computed by Myerson's virtual transformation, which aren't even defined for multi-dimensional types, but just *some* virtual types that may or may not be the same as the true types. In particular, each mapping in the support of the mechanism's distribution will take as input a type (which is an additive function with non-negative item values, together with a non-negative budget), and output a virtual type without a budget constraint and whose valuation function is the sum of a budgeted-additive function[3] with non-negative item values (which depends on the input type in a very structured way) plus an additive function with possibly negative item values (which may be unstructured with respect to the input type). We provide a formal statement of this structural claim in Section 4 as well. Note that for the special case of a single item auction, this gives a particularly simple structure: the virtual types are now just a single (possibly negative) real number, which could be interpreted as a virtual value. The optimal auction simply maps reported types to virtual *values* and assigns the item to the bidder with the highest virtual value.

Theorem 4.6 describes the format of the optimal mechanism more formally, which can be derived from Cai et al.'s framework. Theorem 4.8 describes the format of an approximately optimal mechanism, which we show can be found computationally efficiently (also using Cai et al.'s framework).

1.2 Related Work

The result that comes closest to characterizing the optimal auction for budgeted bidders is that of Pai and Vohra (2014): they characterize the optimal ex-post budget respecting, BIC, *interim* IR auction for a single item. They make the assumptions that the budgets are drawn independently of the values, and that the marginal distribution over the values satisfies the monotone hazard rate and has weakly decreasing density. They show that the optimal auction takes on a form similar to Myerson's, but with additional *pooling* to enforce that no bidder is asked to pay more than her budget, while also maintaining that no bidder has incentive to underreport her budget. Their auction is implemented as an all-pay auction and is therefore not ex-post IR. Earlier, Laffont and Robert (1996) and Maskin (2000) considered the case where valuations are private information but budgets are common knowledge and identical, for the objectives of revenue and social welfare respectively. Malakhov and Vohra (2005) study the setting where there are two bidders, one has a

[3]A function $v(\cdot)$ is budgeted-additive if there exists a b such that $v(S) = \min\{b, \sum_{i \in S} v(\{i\})\}$ for all S. Note that a buyer with a budgeted-additive valuation behaves differently than an additive buyer with a budget, and that a budgeted-additive buyer indeed has quasi-linear utilities.

known budget constraint while the other does not. Che and Gale (2000) characterize the optimal pricing scheme for a single item with a single bidder, with a private valuation and a budget which may be correlated with each other. The limited special cases considered by these article point to the difficulty of characterizing the optimal auction, which motivates the search for efficient approximations.

Another line of work ranks different auction formats by the revenue generated in the presence of budgets. Che and Gale (1998) compare first price, second price, and all-pay auctions, while Benoit and Krishna (2001) compare sequential and simultaneous auctions.

In the computer science tradition, Bhattacharya et al. (2010) give a 4-approximation for multiple items with additive valuations, but they assume that the budgets are publicly known, and the auction is not ex-post IR. Chawla et al. (2011) give a 2-approximation in a single parameter domain, but assume that the budgets are public. They also consider private budgets, where budgets and values are independently distributed, in single parameter matroid domains, and MHR Distributions, and give a $3(1 + e)$-approximation. Finally, Cai et al. (2012), provide exactly optimal mechanisms for multiple items, additive valuations and private budgets, but their auctions are interim-IR. Once again, all these auctions make additional assumptions when compared to us.

Cai et al. (2013) give a general reduction from mechanism design to algorithm design, which we use for our results. Without concern for computation, we use their framework to obtain a structural characterization of the optimal mechanism for our setting in Theorem 4.8. With concern for computation, we show that for the special case of a single item, the algorithmic problem obtained through this reduction is quite easy to solve optimally, resulting in computationally efficient and exactly optimal single-item auctions with budgets. However, when there are multiple items the resulting algorithmic problem becomes NP-hard (Chakrabarty and Goel 2010). We give a (computationally efficient) 3-approximation to this algorithmic problem, which through the reduction yields a (computationally efficient) 3-approximately optimal multi-item auction with budgets. Recently, Bhalgat et al. (2013) showed that (a weaker form of) the reduction of Cai et al. (2013) could be obtained using the simpler multiplicative weight update method instead of the ellipsoid algorithm used originally, and consider the variant of our setting where the items are *divisible*. The algorithmic problem in this case is once again easy. Daskalakis and Weinberg (2015) also use the reduction in Cai et al. (2013) to design an auction for a non-linear objective, namely the *makespan* of an assigment of jobs to machines.

The auction design problem has also been considered in a *worst-case* model, as opposed to a Bayesian model. A standard framework is that of *competitive* auctions, where a bound is shown on the ratio of the revenue of an optimal auction to the revenue of the given auction on any instantiation of valuations and budgets. Borgs et al. (2005) and Abrams (2006) give constant competitive auctions for multi-unit auctions, under an assumption of *bidder dominance*, that the contribution of a single bidder to the total revenue is sufficiently small. (Devanur et al. 2013) give constant competitive auctions for single parameter downward-closed domains with a public, common budget constraint. Since the worst-case setting is decidedly more difficult than the Bayesian setting, these results are not comparable to ours. Another line of work considers the design of *Pareto-optimal* auctions: Dobzinski et al. (2008) characterize single item auctions that are Pareto-optimal, with public budgets and show an impossibility of a similar auction for private budgets. Goel et al. (2012) extend this auction to a more general poly-matroidal setting.

1.3 Contributions

The goals of revenue-optimality, ex-post individual rationality, and ex-post budget feasibility seem to be at odds with one another. This is highlighted by the fact that, prior to our work, no known auctions even approximately satisfied all three conditions, even with just a single item and private budgets that are independent of values. We provide a computationally efficient 3-approximation for the significantly more general case of auctions for multiple heterogeneous goods and additive bidders with private budgets that can be correlated with their values.

2 AN INTRODUCTORY EXAMPLE

Before getting into the full details of our model and results, we analyze a quick example. The purpose of this example is just to show what the input might possibly look like, and what format optimal mechanisms might possibly take. It is not meant to illustrate any computational aspects of our result.

Consider a setting with two bidders and two items. Each bidder's (value, value, budget) triplet is drawn independently from the same distribution D, where D samples the triplets $(2, 0, 1)$, $(0, 2, 1)$, $(2, 2, 2)$ each with probability $1/3$ (the first number refers to the bidder's value for item one, the second for item two, and the third is their private budget).

If there were no budget constraints (i.e., the third number were ∞ with probability one), then the optimal mechanism would simply sell each item separately using a second-price auction with reserve two. This auction in fact generates expected revenue equal to expected welfare and is therefore optimal. However, the budget constraints would often be violated, so it is infeasible when taking those into consideration. The expected revenue of this mechanism is $32/9$.

If we were interested in the optimal *interim IR*, budget-respecting mechanism, we would instead do the following. We could offer to each bidder the option to receive any number of items, each independently with probability $1/2$, and pay 1 per item. That is, each bidder could pay 1 to receive item 1 w.p. $1/2$, pay 1 to receive item 2 w.p. $1/2$, or pay 2 to receive both items (independently) w.p. $1/2$ (or nothing to receive no items). Clearly, this allocation rule is feasible, as it never promises any item to any bidder with probability $> 1/2$. Moreover, each bidder will elect to pay their full budget no matter their type: if the bidder has type $(2, 0, 1)$ or $(0, 2, 1)$, they will elect to purchase a single item. If the bidder has type $(2, 2, 2)$, they will elect to purchase both. As clearly the expected revenue cannot exceed the expected sum of budgets, this mechanism is optimal. The expected revenue of this mechanism is $8/3 < 32/9$.

In this article, we are interested in the optimal *ex-post IR*, budget-respecting mechanism. Because the previous mechanism sometimes charges a bidder a non-zero amount and gives them no items, it is not ex-post IR. The optimal mechanism here is more complex, and doesn't have a short proof of optimality. The optimal mechanism is the following:

- If one bidder reports $(2, 0, 1)$ and the other reports $(0, 2, 1)$, the bidders each pay 1 and receive the item they value. It is clear that no ex-post IR mechanism can generate more revenue on these profiles, as each bidder pays their budget.
- If both bidders report $(2, 0, 1)$, they each receive item one with probability $1/2$, and pay 1 only in the event that they receive it. The $(0, 2, 1)$ case is symmetric. It is clear that no ex-post IR mechanism can generate more revenue on these profiles, as only one bidder can get non-zero value, and this bidder pays their budget.
- If both bidders report $(2, 2, 2)$, they each receive a uniformly random item and pay 2. It is again clear that no ex-post IR mechanism can generate more revenue on these profiles, as each bidder pays their budget.
- If one bidder reports $(2, 0, 1)$ and the other reports $(2, 2, 2)$, the $(2, 2, 2)$ bidder receives item two with probability one, item one with probability $1/2$ and pays 2. The $(2, 0, 1)$ bidder receives item one with probability $1/2$ and pays 1 in the event that they receive item one. The $(0, 2, 1)$ vs. $(2, 2, 2)$ case is symmetric. It's not obvious that no ex-post IR mechanism can generate more revenue on these profiles, as it's *a priori* not clear why we can't instead give

item one to the $(2, 0, 1)$ bidder and charge her her budget, while still charging the $(2, 2, 2)$ bidder her budget to receive only item two. Indeed, however, this is the case as implied by BIC constraints (but this is not at all obvious without writing down the full optimization problem and verifying).

This mechanism generates expected revenue $20/9 < 8/3$. Note that when the bidder has type $(2, 0, 1)$, they receive item one with probability $1/3 + 1/6 + 1/6 = 2/3$ and pay $2/3$ in expectation (they never receive item two), enjoying an expected utility of $2/3$. Similarly, when the bidder has type $(0, 2, 1)$, they receive item two with probability $2/3$ and pay $2/3$ in expectation (never receiving item one), again enjoying an expected utility of $2/3$. When the bidder has type $(2, 2, 2)$, they receive each item with probability $2/3$ and pay 2 deterministically, again enjoying an expected utility of $2/3$. Note that the type $(2, 2, 2)$ would also enjoy expected utility $2/3$ for reporting $(2, 0, 1)$ or $(0, 2, 1)$ instead.

Proving the optimality of the above scheme is outside the scope of this article, but for the reader familiar with techniques of Cai et al. (2016), one can start from the LP formulation in the following section and put a Lagrangian multiplier of $1/5$ on each of the two BIC constraints, guaranteeing that the $(2, 2, 2)$ type prefers to tell the truth rather than report $(2, 0, 1)$ or $(0, 2, 1)$. We won't go into further detail why the above proposed mechanism satisfies complementary slackness with this proposed "partial dual," but one key factor is that the choice to use probability $1/2$ in the last bullet above makes the $(2, 2, 2)$ type enjoy the same utility regardless of the type they report.

Again, this example is just meant to give a sample of what format the problem input and optimal mechanism might take. It also illustrates some complexities of the problem we consider (as compared to budget-less or interim IR) and motivates taking an algorithmic rather than analytic approach.

3 PRELIMINARIES

We begin with formal definitions of the mechanism design problem we study. We then outline the reduction of Cai et al. (2013) (Section 3.1) and its implications (Section 3.2) for our problem. Finally,we state a related problem (Section 3.3), the generalized assignment problem, which we use in the design of our algorithm.

Bidders. There are n bidders, each with additive valuations over m items and a hard budget constraint. Specifically, bidder i has value v_{ij} for item j, value $\sum_{j \in S} v_{ij}$ for set S, and hard budget b_i. We denote by $\vec{v}_i$ the vector of bidder i's values for all m items. We denote by $\mathcal{D}_i$ the joint distribution of $(\vec{v}_i, b_i)$. We denote by $\mathcal{D} = \times_i \mathcal{D}_i$ the joint distribution of all bidders' valuations and budgets.

Types and Virtual Types. We will use the following notation when referring to a bidder's "type" and "virtual type."

- Valuation function: A bidder's valuation function takes as input a set of items and outputs their value for that set (maps $2^{[m]}$ to $\mathbb{R}_+$).
- Value: We will use the term value to refer to a bidder's valuation function evaluated on a singleton set.
- Type: A bidder's type is a function that determines their utility for every possible outcome. For instance, if bidders are additive and quasi-linear, then their type can be represented as a vector $\vec{v}$, with the convention that the bidder's utility for receiving set S and paying p is $\sum_{i \in S} v_i - p$. Similarly, if bidders are additive with a hard budget constraint, then their type can be represented as $(\vec{v}, b)$, with the convention that the bidder's utility for receiving set S and paying p is $-\infty$ if $p > b$, or $\sum_{i \in S} v_i - p$ otherwise.

—Welfare: Sums over all bidders their value for items received.
—Virtual valuation function: A bidder's virtual valuation function is a function that maps $2^{[m]}$ to $\mathbb{R}$, which may or may not be their actual valuation function.
—Virtual value: We will use the term virtual value to refer to a bidder's virtual valuation function evaluated on a singleton set.
—Virtual type: A bidder's virtual type is a utility function associated to that bidder, which may or may not be their actual type.
—Virtual welfare: Sums over all bidders their virtual value for items received.
—Virtual transformation: A (possibly randomized) mapping from valuation functions/types to virtual valuation functions/types. We will use ϕ (often with super- and subscripts) to denote this mapping. Note that $\phi(t)(X)$ first maps valuation function/type t to virtual valuation function/type $\phi(t)$, and then evaluates outcome X.

Mechanisms. Our goal is to design BIC mechanisms that are IR and that respect budgets ex-post. Formally, for a (randomized) mechanism M, we can denote by $x_{ij}^M(\vec{v}, \vec{b}, r)$ to be 1 if bidder i receives item j when the profile of values/budgets reported to M is $(\vec{v}, \vec{b})$ and the random seed used by M is r, or 0 otherwise. Similarly, we denote the price paid by bidder i as $q_i^M(\vec{v}, \vec{b}, r)$ under the same conditions. We can then define the interim allocation probability $\pi_{ij}^M(\vec{v}_i, b_i)$ to be the probability that bidder i receives item j when reporting $(\vec{v}_i, b_i)$ over the randomness of other agent's valuations and budgets, $(\vec{v}_{-i}, \vec{b}_{-i})$, as drawn from $\mathcal{D}_{-i}$, and the randomness in M captured by r. We can similarly define the interim price $p_i^M(\vec{v}_i, b_i)$ to be the expected payment made by bidder i over the same randomness. Formally, $\pi_{ij}^M(\vec{v}_i, b_i) = \mathbb{E}_{(\vec{v}_{-i}, \vec{b}_{-i}) \leftarrow \mathcal{D}_{-i}, r}[x_{ij}^M(\vec{v}_i; \vec{v}_{-i}, b_i; \vec{b}_{-i}, r)]$ and $p_{ij}^M(\vec{v}_i, b_i) = \mathbb{E}_{(\vec{v}_{-i}, \vec{b}_{-i}) \leftarrow \mathcal{D}_{-i}, r}[q_{ij}^M(\vec{v}_i; \vec{v}_{-i}, b_i; \vec{b}_{-i}, r)]$. Formal definitions of BIC, IR, and ex-post budgets are below.

Definition 3.1 (Bayesian Incentive Compatible). A mechanism M is BIC if for all bidders i, and types $(\vec{v}_i, b_i), (\vec{v}_i', b_i')$ the following holds:[4]

$$\vec{v}_i \cdot \vec{\pi}_i^M(\vec{v}_i, b_i) - p_i^M(\vec{v}_i, b_i) \geq \vec{v}_i \cdot \vec{\pi}_i^M(\vec{v}_i', b_i') - p_i^M(\vec{v}_i', b_i').$$

A mechanism is said to be ϵ-BIC if for all bidders i, and types $(\vec{v}_i, b_i), (\vec{v}_i', b_i')$ the following holds:

$$\vec{v}_i \cdot \vec{\pi}_i^M(\vec{v}_i, b_i) - p_i^M(\vec{v}_i, b_i) \geq \vec{v}_i \cdot \vec{\pi}_i^M(\vec{v}_i', b_i') - p_i^M(\vec{v}_i', b_i') - \epsilon.$$

Definition 3.2 (Interim/Ex-Post Individually Rational). A mechanism M is interim IR if for all bidders i, and types $(\vec{v}_i, b_i)$ the following holds:

$$\vec{v}_i \cdot \vec{\pi}_i^M(\vec{v}_i, b_i) \geq p_i^M(\vec{v}_i, b_i).$$

Further, it is ex-post IR if for all bidders i, all profiles $(\vec{v}, \vec{b})$, and random seeds r, we have

$$\vec{v}_i \cdot \vec{x}_i^M(\vec{v}, \vec{b}, r) \geq q_i^M(\vec{v}, \vec{b}, r).$$

Definition 3.3 (Ex-Post Budget Respecting). A mechanism M respects budgets ex-post if for all type profiles $(\vec{v}, \vec{b})$, all random seeds r, and all bidders i we have

$$q_i^M(\vec{v}, \vec{b}, r) \leq b_i.$$

[4]Note that bidders with budgets are often modeled as being able only to under-report their budget (but not over-report). This could be enforced, for instance, by asking each bidder to front their budget as a refundable deposit. For a mechanism to be BIC in such a setting, we would only require that the following holds for all types $(\vec{v}_i, b_i)$, $(\vec{v}', b_i')$ with $b_i' \leq b_i$. The Cai et al. framework (and therefore our results) apply in such settings, as well as when bidders can both over- and under-report their budget (for which BIC mechanisms require the inequality to hold for all $(\vec{v}_i, b_i)$, $(\vec{v}_i', b_i')$).

Definition 3.4 (No Positive Transfers). A mechanism M has no positive transfers if for all type profiles $(\vec{v}, \vec{b})$, all random seeds r, and all bidders i we have

$$q_i^M(\vec{v}, \vec{b}, r) \geq 0.$$

3.1 Reduction from Mechanism to Algorithm Design

In recent work, (Cai et al. 2013) provide an algorithmic framework for mechanism design, showing how to design mechanisms by solving purely algorithmic problems. We use this construction to reduce our mechanism design problem to an algorithm design problem and show a 3-approximation to this algorithmic problem. In the rest of this section, we state the general formulations of the mechanism design and the corresponding algorithm design problems considered by Cai et al. (2013). Then we give the precise statement of their reduction, and a structural characterization of the optimal mechanism obtained as a byproduct of their reduction. Finally, we instantiate these to state the corresponding problems in our setting, and massage the resulting problems to simplify them. We will provide some very high-level intuition for their reduction when appropriate, and refer the reader to Cai et al. (2013) for further detail.

Cai et al. (2013) call the mechanism design problems of study BMeD$(\mathcal{F}, \mathcal{V}, \mathcal{O})$,[5] where feasibility constraints $\mathcal{F}$, possible valuations $\mathcal{V}$, and optimization objective $\mathcal{O}$ parameterize the problem. Formally, this problem is defined as

BMeD$(\mathcal{F}, \mathcal{V}, \mathcal{O})$:
Input: For each bidder $i \in [n]$, a finite set $T_i \subseteq \mathcal{V}$, and a distribution D_i over T_i, presented by explicitly listing all types in T_i and their corresponding probability.
Output: A feasible (selects an allocation in $\mathcal{F}$ with probability 1), BIC, (interim) IR mechanism for bidders drawn from $D = \times_i D_i$.
Goal: Find the mechanism that optimizes $\mathcal{O}$ in expectation, with respect to all BIC, IR mechanisms (when bidders with types drawn from D play truthfully).
Approximation: An algorithm is said to be an (ϵ, α)-approximation if it finds an ϵ-BIC mechanism whose expected value of $\mathcal{O}$ (when bidders drawn from D report truthfully) is at least $\alpha \cdot \text{OPT} - \epsilon$.

In our problem, the feasible allocations are those that award each item to at most one bidder. So we can denote the index set of feasible allocations as $[m+1]^n$ (with the convention that selecting the allocation $\vec{a}$ awards item j to bidder a_j if $a_j > 0$, or no one if $a_j = 0$). The possible bidder types are all additive functions over items (with non-negative multipliers), and non-negative budgets, which we can denote as $\mathbb{R}_+^{m+1}$. Our objective is revenue. To ensure that all feasible mechanisms are ex-post IR (note that their reduction only guarantees interim IR without extra work) and ex-post budget respecting, we will define the objective function Revenue as follows. Revenue takes as input a valuation profile $(\vec{v}, \vec{b})$, an allocation $\vec{x}$ (where $x_{ij} = 1$ iff bidder i is awarded item j), and a price vector $\vec{p}$. We define $\text{Revenue}(\vec{v}, \vec{b}, \vec{x}, \vec{p}) = \sum_i p_i$, if $0 \leq p_i \leq \min\{b_i, \vec{v}_i \cdot \vec{x}_i\}$ for all i, or $\text{Revenue}(\vec{v}, \vec{b}, \vec{x}, \vec{p}) = -\infty$ otherwise. Note that when we refer to the optimal mechanism, we mean the solution to BMeD.

Informally, the main result of Cai et al. (2013) states that, for all $\mathcal{F}, \mathcal{V}, \mathcal{O}$, the problem BMeD$(\mathcal{F}, \mathcal{V}, \mathcal{O})$ can be solved in polynomial time with black-box access to a poly-time algorithm for a purely algorithmic problem that they call GOOP$(\mathcal{F}, \mathcal{V}, \mathcal{O})$.[6] Below, $\mathcal{V}^\times$ denotes the closure of $\mathcal{V}$ under addition and (possibly negative) scalar multiplications (so for instance, $(\mathbb{R}_+^m)^\times = \mathbb{R}^m$).

[5] BMeD stands for Bayesian Mechanism Design.
[6] GOOP stands for Generalized Objective Optimization Problem.

GOOP($\mathcal{F}, \mathcal{V}, \mathcal{O}$):
Input: A type $t_i \in \mathcal{V}$, multiplier $m_i \in \mathbb{R}$, and virtual type $g_i \in \mathcal{V}^{\times}$ for each $i \in [n]$.[7]
Output: An allocation $x \in \mathcal{F}$ and price vector $\vec{p} \in \mathbb{R}_+^n$.
Goal: Find $\arg\max_{x \in \mathcal{F}, \vec{p}} \{\mathcal{O}(\vec{t}, x, \vec{p}) + \sum_i m_i p_i + \sum_i g_i(x)\}$.
Approximation: $(x^*, \vec{p}^*)$ is said to be an α-approximation if $\mathcal{O}(\vec{t}, x^*, \vec{p}^*) + \sum_i m_i p_i^* + \sum_i g_i(x^*) \geq \alpha \cdot \arg\max_{x \in \mathcal{F}, \vec{p}} \{\mathcal{O}(\vec{t}, x, \vec{p}) + \sum_i m_i p_i + \sum_i g_i(x)\}$.

Further below, we provide much more detail on the structure of the algorithmic focus of this article, GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue), but we first conclude our discussion of the reduction we employ. The main result of Cai et al. (2013) states that for all $\epsilon > 0$, an (ϵ, α)-approximation for BMeD($\mathcal{F}, \mathcal{V}, \mathcal{O}$) can be obtained from a poly-time α-approximation for GOOP($\mathcal{F}, \mathcal{V}, \mathcal{O}$). The additive error (and failure probability in the theorem statement) is due to a sampling procedure in the execution of the reduction. We provide a full statement of their main result below.[8]

Theorem 3.5 (Theorem 4 of (Cai et al. 2013)). *For all $\mathcal{F}, \mathcal{V}, \mathcal{O}$, and $\epsilon > 0$, if there is a poly-time α-approximation algorithm, G, for GOOP($\mathcal{F}, \mathcal{V}, \mathcal{O}$), there is a poly-time (ϵ, α)-approximation algorithm for BMeD($\mathcal{F}, \mathcal{V}, \mathcal{O}$) as well. Specifically, if ℓ denotes the input length of a BMeD($\mathcal{F}, \mathcal{V}, \mathcal{O}$) instance, the algorithm runs in time poly$(\ell, 1/\epsilon)$, makes poly$(\ell, 1/\epsilon)$ black box calls to G on inputs of size poly$(\ell, 1/\epsilon)$, and succeeds with probability $1 - exp(-poly(\ell, 1/\epsilon))$.*

Cai et al. (2013) prove Theorem 3.5 above by considering a linear program that optimizes over the space of interim forms that are both truthful (that satisfy the linear constraints in Definitions 3.1 and 3.2), and feasible (those that correspond to an actual mechanism that selects an outcome $x \in \mathcal{F}$ on every profile with probability 1).[9] Linear constraints enforcing that an interim form is BIC and interim IR can be written explicitly, but a computationally efficient separation oracle for the space of feasible interim forms is still required to solve the linear program. They show how to obtain such a separation oracle with black-box access to an algorithm that solves GOOP, and that this entire process preserves approximation as well. So, an execution of the ellipsoid algorithm to solve this LP will make many queries to such a separation oracle. The separation oracle will make many queries to an algorithm that solves GOOP. The input on which the GOOP algorithm is queried loosely corresponds to dual variables/Lagrangian multipliers for the incentive constraints.

Cai et al. (2013) further provide a structural characterization of the space of all feasible mechanisms (truthful or not), leading to a structured implementation of whatever interim form is output by the LP. Specifically, they show that the extreme points of the space of feasible interim forms correspond to mechanisms that associate a virtual type $g_i(t_i)(\cdot)$ and price multiplier $m_i(t_i)$ to each type $t_i \in T_i$, and then selects on profile $(t_1, \ldots, t_n)$ the allocation and price vector that solves GOOP on input $t_1, \ldots, t_n, m_1(t_1), \ldots, m_n(t_n), \sum_i g_i(t_i)(\cdot)$. They show further that solving the linear program explicitly finds a list of virtual types and multipliers whose resulting interim forms contain the optimal (truthful) interim form in their convex hull. Theorem 3.6 below captures the structural aspect of their result.

Theorem 3.6 (Implicit in Cai et al. (2013)). *For all BMeD instances, the optimal mechanism can be implemented as a distribution over generalized objective optimizers. Specifically, there exists a distribution Δ over mappings $(f_1^\delta, \ldots, f_n^\delta)$. Each mapping f_i^δ takes types t_i in T_i to price multipliers*

[7]For other applications, the inputs $g_i(\cdot)$ are instead sometimes called cost functions.
[8]The theorem statement is identical in content, but reworded for clarity and cleanliness.
[9]In fact, they need to work with a generalization of interim forms, called *implicit forms*, to accommodate non-additive valuations. But we describe their proof for additive valuations for clarity of exposition, and because it is relevant for our setting.

$m_i^\delta(t_i) \in \mathbb{R}$ and virtual types $g_i^\delta(t_i)(\cdot) \in \mathcal{V}^\times$. The optimal mechanism first samples $(f_1^\delta, \ldots, f_n^\delta)$ from Δ, and on profile $\vec{t}$, selects the outcome and price vector $\arg\max_{x \in \mathcal{F}, \vec{p}} \{O(\vec{t}, x, \vec{p}) + \sum_i m_i^\delta(t_i) \cdot p_i + \sum_i g_i^\delta(t_i)(x)\}$.

In the section below, we provide further details surrounding instantiations of Theorems 3.5 and 3.6 as they pertain to the problem at hand.

3.2 Instantiations

The goal of this section is to provide more details of the instantiation of Theorems 3.5 and 3.6 to our setting, but not to provide proofs (for which we refer the reader to Cai et al. (2013)). We begin by describing the linear program that the reduction of Cai et al. (2013) would solve for our setting. Below, $F([n+1]^m, \mathbb{R}_+^{m+1}, \textsc{Revenue})$ denotes the space of interim forms of all feasible (not necessarily truthful) mechanisms. Specifically, $(O, \vec{\pi}, \vec{p}) \in F([n+1]^m, \mathbb{R}_+^{m+1}, \textsc{Revenue})$ if and only if there is a mechanism M that awards each item at most once on every profile, is ex-post IR and ex-post budget respecting, awards bidder i item j when she reports type t_i with probability exactly $\pi_{ij}(t_i)$ (w.r.t. all other bidders' types and the randomness in the mechanism) and chargers bidder i price $p_i(t_i)$ in expectation (over all other bidders' types and the randomness in the mechanism), and whose expected revenue is exactly O. With this definition in mind, the linear program they solve is stated below.

Variables:

- O, denoting the expected revenue of the interim form found.
- $\pi_{ij}(t_i)$ for all bidders i, items j, types t_i, denoting the probability that bidder i receives item j when reporting type t_i.
- $p_i(t_i)$ for all bidders i and types t_i, denoting the expected price paid by bidder i when reporting type t_i.

Constraints:

(1) $\sum_j \pi_{ij}(t_i) \cdot v_{ij}(t_i) - p_i(t_i) \geq \sum_j \pi_{ij}(t_i') \cdot v_{ij}(t_i) - p_i(t_i')$, for all bidders i and types t_i, t_i', guaranteeing that the interim form corresponds to a BIC mechanism.
(2) $\sum_j \pi_{ij}(t_i) \cdot v_{ij}(t_i) - p_i(t_i) \geq 0$, for all bidders i and types t_i, guaranteeing that the interim form corresponds to an interim IR mechanism.[10]
(3) $(O, \vec{\pi}, \vec{p}) \in F([n+1]^m, \mathbb{R}_+^{m+1}, \textsc{Revenue})$, guaranteeing that the interim form corresponds to a feasible mechanism.

Maximizing:

- O, the expected revenue.

The solution to this LP is the interim form of the optimal mechanism. The LP can be solved in polynomial time, so long as we have a poly-time separation oracle for the space $F([n+1]^m, \mathbb{R}_+^{m+1}, \textsc{Revenue})$. Cai et al. (2013) shows that this can be obtained via an algorithm for the related GOOP problem, which we instantiate in our setting below.

Budgeted-Additive Virtual Welfare Maximization. As discussed above, to find (approximately) optimal mechanisms for our setting, we need to study the purely algorithmic problem GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue), so we instantiate it below. Recall that when we write "virtual value," we mean the evaluation of a virtual valuation function on a singleton set. We use this language below

[10] Actually, this constraint is redundant as we will also enforce that the mechanism be ex-post IR to be considered feasible.

to emphasize the specific format of the input to GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue), but also to make the connection to the input format posed earlier.

GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue):
Input: Values $v_{ij} \geq 0$ and virtual values $w_{ij} \in \mathbb{R}$ for all i, j. Budget $b_i \in \mathbb{R}_+$ and price multiplier $m_i \in \mathbb{R}$ for all i.
Output: An allocation $\vec{x} \in \{0,1\}^{mn}$ and prices $\vec{p}$ such that $\sum_i x_{ij} \leq 1$ for all j (each item awarded at most once), $\sum_j x_{ij} v_{ij} \geq p_i$ (ex-post IR), $p_i \leq b_i$ (ex-post budget respecting), and $p_i \geq 0$ (no positive transfers).
Goal: Find $\arg\max_{\vec{x},\vec{p}} \{\sum_i (m_i + 1) p_i + \sum_{ij} x_{ij} w_{ij}\}$.

Note that in the above formulation, we have folded cases where Revenue evaluates to $-\infty$ into feasibility constraints on the output. We make two quick further observations about the structure of GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue), and call the reformulation Budgeted-Additive Virtual Welfare Maximization (BAVWM). Also, for cleanliness, we will replace the input price multipliers m_i by $m_i - 1$ so that the term in the objective will be $\sum_i m_i p_i$. This is without loss of generality as each m_i could be any real number.

Observation 1. *If $m_i > 0$, the optimal choice for p_i is always $\min\{b_i, \sum_j x_{ij} v_{ij}\}$. If $m_i \leq 0$, the best choice for p_i is 0.*

Observation 2. *For all possible solutions $(\vec{x}, \vec{p})$, the quality of $(\vec{x}, \vec{p})$ for the input instance $(\vec{v}, \vec{w}, \vec{b}, \vec{m})$ is the same as for the instance $(\vec{v}', \vec{w}, \vec{b}, \vec{m})$ where $v'_{ij} = \min\{v_{ij}, b_i\}$, for all i, j.*

In light of these, we may set all negative m_i to 0, and all v_{ij} to $\min\{v_{ij}, b_i\}$ without changing the problem, leading to the following reformulation.

Budgeted-Additive Virtual Welfare Maximization:
Input: Budget b_i for all agents. Values $v_{ij} \in [0, b_i]$ for all agents and items. Price multiplier $m_i \geq 0$ for all agents, and virtual value $w_{ij} \in \mathbb{R}$ for all agents and items.
Output: An allocation $\vec{x} \in \{0,1\}^{mn}$ such that $\sum_i x_{ij} \leq 1$ for all j (each item awarded at most once).
Goal: Find $\arg\max_{\vec{x}} \{\sum_i (m_i \min\{b_i, \sum_j x_{ij} v_{ij}\} + \sum_j x_{ij} w_{ij})\}$.

Note that in the above formulation, we no longer need to optimize over the price vector, due to Observation 1. The problem can now be interpreted as just a welfare maximization problem, where bidder i's valuation function is the sum of a budgeted-additive function (with non-negative item values) and an additive function (with possibly negative item values). Finally, note that we can re-formulate the above problem to remove the multipliers $(m_i)_i$ from the input and the objective by incorporating them in the b_i's and the v_{ij}'s. We choose to leave them in so that it is more transparent how the inputs to BAVWM are related to the types reported by the bidders of the mechanism output by the Cai et al. (2013) reduction.

3.2.1 A Remark about Welfare Optimization. It is also previously unknown how to design the *welfare*-optimal BIC, ex-post IR mechanism for additive bidders that respects budget constraints ex-post (because VCG might charge payments that exceed budgets). Our same approach for revenue, almost word-for-word, can be used to find (approximately) welfare-optimal mechanisms as well. In this section, we'll formally explain why - this will also give the reader another example instantiating GOOP.

Define the objective function Welfare to take as input a valuation profile $(\vec{v}, \vec{b})$, an allocation $\vec{x}$ (where $x_{ij} = 1$ iff bidder i is awarded item j), and a price vector $\vec{p}$. Define Welfare$(\vec{v}, \vec{b}, \vec{x}, \vec{p}) = \sum_i x_i \cdot v_i$, if $0 \leq p_i \leq \min\{b_i, \vec{v}_i \cdot \vec{x}_i\}$ for all i, or Welfare$(\vec{v}, \vec{b}, \vec{x}, \vec{p}) = -\infty$ otherwise. Then the problem GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Welfare) is defined as follows:

GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Welfare):
Input: Values $v_{ij} \geq 0$ and virtual values $w_{ij} \in \mathbb{R}$ for all i, j. Budget $b_i \in \mathbb{R}_+$ and price multiplier $m_i \in \mathbb{R}$ for all i.
Output: An allocation $\vec{x} \in \{0,1\}^{mn}$ and prices $\vec{p}$ such that $\sum_i x_{ij} \leq 1$ for all j (each item awarded at most once), $\sum_j x_{ij} v_{ij} \geq p_i$ (ex-post IR), $p_i \leq b_i$ (ex-post budget respecting), and $p_i \geq 0$ (no positive transfers).
Goal: Find $\arg\max_{\vec{x},\vec{p}} \{\sum_i m_i \cdot p_i + \sum_{ij} x_{ij}(w_{ij} + v_{ij})\}$.

Again, as each w_{ij} can be arbitrary, we can replace $(w_{ij} + v_{ij})$ in the objective with w_{ij}. From here, Observations 1 and 2 hold again, and we see that GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Welfare) is also equivalent to BAVWM (and therefore GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue) as well). By Theorem 3.5, this means that BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, Welfare) can also be (approximately) solved by designing (approximation) algorithms for BAVWM. As our theorem statements are already somewhat lengthy, we will state our results in terms of Revenue, and only note here that all theorem statements and proofs hold as-is after replacing Revenue with Welfare as well.

3.3 The Generalized Assignment Problem

Our main technical result will make use of a rounding algorithm for the *generalized assignment problem*. We give here a statement of the problem and a rounding theorem due to Shmoys and Tardos (1993).

Generalized Assignment Problem:
Input: Processing times $p_{ij} \in \mathbb{R}_+$ and costs $c_{ij} \in \mathbb{R}$ for all machines i and jobs j, capacities T_i for all machines i.[11]
Output: An allocation $\vec{x} \in \{0,1\}^{mn}$ of jobs to machines such that $\sum_i x_{ij} = 1$ for all j (each job is assigned) and $\sum_j x_{ij} p_{ij} \leq T_i$ (each machine processes at most its capacity).
Goal: Find $\arg\max_{\vec{x}} \{\sum_{i,j} x_{ij} c_{ij}\}$ (total cost).[12]

Now, we provide an LP due to Shmoys and Tardos that outputs a fractional solution at least as good as OPT.

Variables:

— x_{ij}, for all machines i and jobs j, denoting the fraction of job j assigned to machine i.

Constraints:

(1) $\sum_i x_{ij} = 1$, for all j, guaranteeing that every job is processed exactly once.
(2) $\sum_j x_{ij} \leq T_i$, for all i, guaranteeing that no machine's capacity is violated.
(3) $x_{ij} = 0$ if $p_{ij} > T_i$.

Maximizing:

— $\sum_{i,j} x_{ij} c_{ij}$, the total cost.

Theorem 3.7 ((Shmoys and Tardos 1993)). *The optimal fractional solution to the above LP can be rounded in polynomial time to an integral solution such that:*

(1) $\sum_i x_{ij} = 1$*, for all* j*.*
(2) $\sum_j x_{ij} \leq 2T_i$*, for all* i*.*
(3) $\sum_j x_{ij} c_{ij} \geq OPT$*.*

[11]Traditionally, some consider only costs $c_{ij} \in \mathbb{R}_+$, but the result we cite applies for negative costs as well.
[12]Traditionally, it makes sense to minimize total cost. As costs are possibly negative, the use of max or min is irrelevant.

4 MAIN RESULTS

In Section 4.1 below, we provide our main computational result: a poly-time approximation algorithm for BAVWM, which implies a poly-time truthful mechanism for revenue maximization that respects ex-post IR and ex-post budget constraints. In Section 4.2, we detail the structure of the optimal mechanism in this setting, as well as our computationally efficient mechanism from Section 4.1.

4.1 Computational Results

In this section, we provide a poly-time 3-approximation for BAVWM. We begin by writing an LP relaxation, allowing the designer to award fractions of items as long as the total fraction awarded doesn't exceed 1. We split the fraction of item j awarded to bidder i into two parts, $\bar{x}_{ij}$ and $\hat{x}_{ij}$. Let $\bar{x}_{ij}$ denote the fraction of item j assigned to agent i before exceeding b_i. And let $\hat{x}_{ij}$ denote the fraction of item j assigned after. In other words, if x_{ij} is the fraction of item j assigned to agent i, we have $\bar{x}_{ij} + \hat{x}_{ij} = x_{ij}$, $\sum_j \bar{x}_{ij} v_{ij} \leq b_i$, and $\sum_j \bar{x}_{ij} v_{ij} = b_i$ if for any j, $\hat{x}_{ij} > 0$. The idea is that assigning more of item j to agent i before exceeding his budget increases both terms in the "goal" above, but assigning more after exceeding the budget only affects the second term. Note that there may be numerous ways to split a fractional allocation x into "valid" $\hat{x}, \bar{x}$ as defined above. The following LP relaxation provides a specific construction of such $\hat{x}, \bar{x}$.:

Variables:

- $\bar{x}_{ij}$, for all agents i and items j, denoting the fraction of item j assigned to agent i, contributing to both the budgeted-additive and additive terms in bidder i's (virtual) valuation.
- $\hat{x}_{ij}$, for all agents i and items j, denoting the fraction of item j assigned to agent i, contributing to just the additive term in bidder i's (virtual) valuation.

Constraints:

(1) $\sum_i (\bar{x}_{ij} + \hat{x}_{ij}) \leq 1$, for all j, guaranteeing that no item is allocated more than once.
(2) $\sum_j \bar{x}_{ij} v_{ij} \leq b_i$, for all i, guaranteeing that contributions to the budgeted-additive term are not overcounted.

Maximizing:

- $\sum_{ij} m_i \bar{x}_{ij} v_{ij} + \sum_{ij} w_{ij}(\bar{x}_{ij} + \hat{x}_{ij})$, the virtual welfare. Note that as each $m_i \geq 0$ and $v_{ij} \geq 0$, the optimal solution will never have $\hat{x}_{ij} > 0$ unless $\sum_j \bar{x}_{ij} = b_i$.

It is clear that any solution to BAVWM has a corresponding fractional solution to this LP. So the goal is to solve this LP and round the fractional solution to an integral one without too much loss. The idea is that the feasible region now looks pretty similar to that of the generalized assignment problem, asking for an assignment of jobs to machines such that the capacity of machine i is at most b_i. We first prove the following rounding theorem, which is a near-direct application of Theorem 3.7.

THEOREM 4.1. *The optimal fractional solution to the above LP can be rounded in polynomial time to an integral assignment such that:*

(1) $\sum_i (\bar{x}_{ij} + \hat{x}_{ij}) \leq 1$ *for all* j.
(2) $\sum_j \bar{x}_{ij} v_{ij} \leq 2b_i$ *for all* i.
(3) $\sum_{ij} m_i \bar{x}_{ij} v_{ij} + \sum_{ij} w_{ij}(\bar{x}_{ij} + \hat{x}_{ij}) \geq OPT$, *where* OPT *is the value of the LP.*

PROOF. We show how to interpret our LP as an instantiation of a fractional LP for the generalized assignment problem, and then directly apply Theorem 3.7. We use p_{ij} to denote processing times, c_{ij} to denote costs, and T_i to denote capacities in the created generalized assignment problem instance.

- Machines:
  (1) A dummy machine, 0.
  (2) For all bidders i, a hat machine $\hat{i}$ (corresponding to the hat variables in our LP).
  (3) For all bidders i, a bar machine $\bar{i}$ (corresponding to the bar variables in our LP).
- Jobs: A job j for all items j.
- Processing times and costs:
  (1) $p_{0j} = c_{0j} = 0$ for all j. $T_0 = 0$.
  (2) $\hat{p}_{ij} = 0$ for all j. $\hat{c}_{ij} = w_{ij}$ for all j. $\hat{T}_i = 0$.
  (3) $\bar{p}_{ij} = v_{ij}$. $\bar{c}_{ij} = m_i v_{ij} + w_{ij}$. $\bar{T}_i = b_i$.

The fractional LP referenced in Theorem 3.7 on this instance would then be (note that the capacity constraints for machines 0 and all $\hat{i}$ are vacuously satisfied, and that there do not exist any i, j for which $p_{ij} > T_i$ by Observation 2):

Variables:

- x_{0j}, for all jobs j, denoting the fraction of job j assigned to machine 0.
- $\bar{x}_{ij}$, for all machines i and jobs j, denoting the fraction of job j assigned to machine $\bar{i}$.
- $\hat{x}_{ij}$, for all machines i and jobs j, denoting the fraction of job j assigned to machine $\hat{i}$.

Constraints:

(1) $x_{0j} + \sum_i (\bar{x}_{ij} + \hat{x}_{ij}) = 1$, for all j, guaranteeing that every job is allocated exactly once.
(2) $\sum_j \bar{x}_{ij} v_{ij} \leq b_i$, for all i, guaranteeing that the total processing time on machine $\bar{i}$ is at most b_i.

Maximizing:

- $\sum_{ij} m_i \bar{x}_{ij} v_{ij} + \sum_{ij} w_{ij} (\bar{x}_{ij} + \hat{x}_{ij})$, the cost.

It's clear that this LP is exactly the same as our LP, just with an additional dummy bidder 0 who collects all unallocated fractions of items. By Theorem 3.7, the optimal fractional solution to this LP can be rounded in polynomial time to an integral solution whose total cost is at least as large, but where the capacity of machine $\bar{i}$ could be as large as $2b_i$, which is exactly an integral allocation of items to bidders with the desired properties. □

After applying Theorem 4.1, we now have an integral solution that is at least as good as the optimum, except our solution is infeasible. It's infeasible because it's "getting credit" for (virtual) welfare in the budgeted-additive term that is perhaps up to twice the budget (i.e., up to $2b_i$). An "obvious" fix to this problem might be to take this integral solution and only take credit for budgeted-additive values up to b_i, thereby making the solution feasible again. Unfortunately, because the objective is mixed sign, the resulting solution doesn't provide any approximation guarantee.[13] Instead, we provide a simple procedure to select a feasible suballocation of this infeasible one that loses a factor of 3.

[13] Consider, for example, the following instance: there is one buyer and two items. $v_{11} = v_{12} = 3$, $b_1 = 3$, $w_{11} = w_{12} = -2$. Then the allocation that awards both items and "gets credit" for up to $2b_i$ is believed to have virtual welfare 2. However, the correctly computed virtual welfare of this allocation is actually -1, which clearly provides no meaningful approximation. Instead we must develop a procedure that, on this instance, would allocate just one of the items.

THEOREM 4.2. *Given an integral allocation $\vec{x}$ satisfying $\sum_i \bar{x}_{ij} + \hat{x}_{ij} \leq 1$ for all j, $\sum_j \bar{x}_{ij} v_{ij} \leq 2b_i$ for all i, and $\sum_{ij} m_i \bar{x}_{ij} v_{ij} + \sum_{ij} w_{ij}(\bar{x}_{ij} + \hat{x}_{ij}) = C$, one can find in poly-time an integral allocation $\vec{y}$ such that:*

(1) $\sum_i (\bar{y}_{ij} + \hat{y}_{ij}) \leq 1$ for all j.
(2) $\sum_j \bar{y}_{ij} v_{ij} \leq b_i$ for all i.
(3) $\sum_{ij} m_i \bar{y}_{ij} v_{ij} + \sum_{ij} w_{ij}(\bar{y}_{ij} + \hat{y}_{ij}) \geq C/3$.

PROOF. For each i, we wish to partition the set of items assigned to i via $\bar{x}_{ij}$ (of the infeasible integral solution), S, into three disjoint sets S_i^1, S_i^2, S_i^3 such that $\sum_{j \in S_i^k} v_{ij} \leq b_i$ for all k. This is always possible: consider sorting the elements in decreasing order of v_{ij} and greedily adding them one at a time to the S_i^k with minimal weight so far. Assume for contradiction that some item j^*, when added, pushes some S_i^k from below b_i to above b_i. Then, without j^*, each of S_i^1, S_i^2, S_i^3 must have had weight strictly larger than $b_i - v_{ij^*}$. As the total weight in all three (without j^*) is at most $2b_i - v_{ij^*}$, this means that $2b_i - v_{ij^*} > 3(b_i - v_{ij^*}) \Rightarrow v_{ij^*} > b_i/2$. But as we processed elements in decreasing order of v_{ij}, this would imply that j^* was the third (or earlier) item processed, meaning that some set must have been empty, and j^* couldn't have possibly pushed it over the limit (as $v_{ij} \leq b_i$ for all j). Therefore, at termination we must have $\sum_{j \in S_i^k} v_{ij} \leq b_i$ for all k. Now, define $k^* = \arg\max_k \{\sum_{j \in S_i^k} m_i v_{ij} + w_{ij}\}$. Let $\bar{y}_{ij} = 1$ iff $j \in S_i^{k^*}$, and $\hat{y}_{ij} = \hat{x}_{ij}$ for all j.

It's clear that $\sum_j \bar{y}_{ij} v_{ij} \leq b_i$ for all i. As $\bar{y}_{ij} \leq \bar{x}_{ij}$ for all i, j, it's also clear that $\sum_i \bar{y}_{ij} + \hat{y}_{ij} \leq 1$ for all j. Finally, by choice of k^* it's also clear that $\sum_{ij}(m_i v_{ij} + w_{ij})\bar{y}_{ij} \geq \sum_{ij}(m_i v_{ij} + w_{ij})\bar{x}_{ij}/3$, and therefore $\sum_{ij} m_i \bar{y}_{ij} v_{ij} + \sum_{ij} w_{ij}(\bar{y}_{ij} + \hat{y}_{ij}) \geq C/3$, as desired. □

Combining Theorems 4.1 and 4.2 yields a feasible, integral allocation that is a 3-approximation by rounding the fractional solution output by our LP, and it is easy to see that the entire procedure runs in polynomial time.

THEOREM 4.3. *There is a poly-time 3-approximation algorithm for Budgeted-Additive Virtual Welfare Maximization, which is a reformulation of GOOP($[n+1]^m, \mathbb{R}_+^{m+1}$, REVENUE). Therefore, for all $\epsilon > 0$, there is a poly-time $(\epsilon, 3)$-approximation algorithm for BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, REVENUE). Specifically, if ℓ is the input length to an instance of BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, REVENUE), the algorithm terminates in time poly$(\ell, 1/\epsilon)$ and succeeds with probability $1 - \exp(-\text{poly}(\ell, 1/\epsilon))$.*

We conclude this section with a remark about the special case of a single (or small constant) number of items. Notice that BAVWM can be solved exactly by exhaustive search in time poly(n^m). If m is a small constant, exhaustive search may be computationally feasible, resulting in an exact algorithm (instead of a 3-approximation).

Remark 4.4. BAVWM can be solved exactly in time poly(n^m) by exhaustive search. Therefore, for all $\epsilon > 0$, there is an $(\epsilon, 1)$-approximation algorithm for BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, REVENUE). Specifically, if ℓ is the input length to an instance of BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, REVENUE), the algorithm terminates in time poly$(\ell, n^m, 1/\epsilon)$ and succeeds with probability $1 - \exp(-\text{poly}(\ell, 1/\epsilon))$.

Finally, we remark that the single-item case is *especially* simpler than even the two item case. We refer the reader to Cai et al. (2012, 2013) for complete details, but essentially the sampling procedure that results in the ϵ error of Theorem 3.5 can be replaced by an exact computation *only* in the single item case (and not even in the two item case), and ϵ can be set to exactly 0.

Remark 4.5. BAVWM with $m = 1$ can be solved exactly in time poly(n) by exhaustive search: there are only n possible outcomes, corresponding to assigning the item to exactly one of the agents. Therefore, there is a $(0, 1)$-approximation algorithm (i.e., an exact algorithm) for

BMeD($[n+1], \mathbb{R}_+^2$, Revenue) (i.e., the single item case). Specifically, if ℓ is the input length to an instance of BMeD($[n+1], \mathbb{R}_+^2$, Revenue), the algorithm terminates in time poly(ℓ), and succeeds with probability 1.

4.2 Structural Results

In this section, we discuss the structure of the optimal mechanism, and of the computationally efficient mechanism from Section 4.1. We begin by characterizing the optimal mechanism by combining Theorem 3.6 with Observation 1.

Theorem 4.6. *In any BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue) instance, the optimal mechanism can be implemented as a distribution over virtual welfare maximizers. Specifically, there exists a distribution Δ over mappings $(f_1^\delta, \dots, f_n^\delta)$. Each mapping f_i^δ maps types $(\vec{v}_i, b_i) \in \mathbb{R}_+^{m+1}$ to a multiplier $m_i^\delta(\vec{v}_i, b_i) \in \mathbb{R}_+$ and a vector $\vec{w}^\delta(\vec{v}, b_i) \in \mathbb{R}^m$. Define ϕ_i^δ to be the mapping that takes as input types $(\vec{v}_i, b_i) \in \mathbb{R}_+^{m+1}$ and outputs a valuation function $\phi_i^\delta(\vec{v}_i, b_i)(\cdot)$ with $\phi_i^\delta(\vec{v}_i, b_i)(S) = m_i^\delta(\vec{v}_i, b_i) \cdot \min\{b_i, \sum_{j\in S} v_{ij}\} + \sum_{j \in S} w_{ij}^\delta(\vec{v}_i, b_i)$. The allocation rule of the optimal mechanism first samples $(f_1^\delta, \dots, f_n^\delta)$ from Δ, and on profile $(\vec{v}, \vec{b})$, allocates the items according to $\arg\max_{S_1 \sqcup \ldots \sqcup S_n \subseteq [m]} \{\sum_i \phi_i^\delta(\vec{v}_i, b_i)(S_i)\}$. Furthermore, if $m_i^\delta(\vec{v}_i, b_i) > 0$, bidder i is charged $\min\{b_i, \sum_{j \in S_i} v_{ij}\}$. If $m_i^\delta(\vec{v}_i, b_i) = 0$, then bidder i is charged 0.*

Proof. The proof starts with an application of Theorem 3.6 to the problem BMeD($[n+1]^m, \mathbb{R}_+^{m+1}$, Revenue). By Observation 1, the joint optimization over allocations x and price vectors $\vec{p}$ can be accomplished by transforming the optimization into one that depends only on the allocation. Once the allocation is found, optimization of the price vector follows as in Observation 1. □

We remark that the virtual types involved in Theorem 4.6 have valuation functions that are the sum of a budgeted-additive function, and an additive function (the latter may have negative item values). We also note that the budgeted-additive component depends in a very structured way on the input type $(\vec{v}_i, b_i)$. Specifically, b_i is turned into a hard cap on the bidder's maximum valuation instead of a hard budget on her ability to pay, and the additive valuation $\vec{v}_i$ is kept the same, forming a budgeted-additive function that is scaled by a positive multiplier m_i. The multiplier m_i and additional values $\vec{w}_i$ may show little structure with respect to the input types (or perhaps none at all).

We also remark that the structure is especially simple in the case of a single item, because a budgeted-additive function for a single item is just a typical valuation function (where the bidder's value for the item is the minimum of her value and her budget). Specifically, the virtual type parameterized by $m_i^\delta(v_i, b_i)$ and $w_i(v_i, b_i)$ values the item at $m_i \min\{v_i, b_i\} + w_i(v_i, b_i)$. This observation leads to the following simplification:

Remark 4.7. In any BMeD($[n+1], \mathbb{R}_+^2$, Revenue) instance (i.e., the single item case), the optimal mechanism can be implemented as a distribution over virtual value maximizers. Specifically, there exists a distribution Δ over mappings $(f_1^\delta, \dots, f_n^\delta)$. Each mapping f_i^δ maps types $(v_i, b_i) \in \mathbb{R}_+^2$ to an indicator bit $m_i^\delta(v_i, b_i) \in \{0, 1\}$ and a virtual value $\phi_i^\delta(v_i, b_i)$. The allocation rule of the optimal mechanism first samples $(f_1^\delta, \dots, f_n^\delta)$ from Δ, and on profile $(\vec{v}, \vec{b})$, allocates the item to any bidder $i^* \in \arg\max_i\{\phi_i^\delta(v_i, b_i)\}$ if her virtual value is non-negative, and doesn't allocate the item otherwise. Furthermore, if $m_{i^*}^\delta(v_{i^*}, b_{i^*}) = 1$, bidder i^* is charged $\min\{b_{i^*}, v_{i^*}\}$. If $m_{i^*}^\delta(v_{i^*}, b_{i^*}) = 0$, then bidder i^* is charged 0.

We conclude with a statement regarding the format of our computationally efficient mechanisms from Section 4.1 (which Theorem 4.3 states can be found computationally efficiently). This is an instantiation of Algorithm 2 in Cai et al. (2013), which is used to prove Theorem 3.5.

THEOREM 4.8. *The mechanism providing the guarantee of Theorem 4.3 has the following format:*
Phase One, Find the Mechanism:

(1) Write a linear program that optimizes revenue over the space of truthful, feasible interim forms (Section 3.2).
(2) Pick an $\epsilon > 0$. Using the algorithm developed in Section 4.1, and the reduction of (Cai et al. 2013), solve this linear program approximately.
(3) This yields an interim form corresponding to a mechanism that is an $(\epsilon, 3)$-approximation.
(4) The linear program also outputs auxiliary information in the form of a distribution Δ over mappings $(f_1^\delta, \ldots, f_n^\delta)$ of the same format from Theorem 4.6.

Phase Two, Run the Mechanism:

(1) Sample a mapping from Δ (provided in Phase One).
(2) On profile $(\vec{v}, b)$, run the approximation algorithm of Section 4.1 for BAVWM, with input budgets b_i, input values v_{ij}, input price multipliers $m_i^\delta(\vec{v}_i, b_i)$, and input virtual values $w_{ij}^\delta(\vec{v}_i, b_i)$. Select this allocation.
(3) If $m_i^\delta(\vec{v}_i, b_i) > 0$, charge bidder i the minimum of their budget and their value for the items they receive. Otherwise, charge them nothing.

Note that this mechanism has basically the same structure as the optimal mechanism, except that on every profile it only approximately maximizes virtual welfare (and we also first have to find the mechanism, which is completely described by the distribution Δ). In the special case of a single item, the structure can again be simplified.

Remark 4.9. In the special case of a single item, the following algorithm finds the optimal mechanism in polynomial time:
Phase One, Find the Mechanism:

(1) Write a linear program that optimizes revenue over the space of truthful, feasible interim forms (Section 3.2).
(2) Using the reduction of Cai et al. (2013) and the observation in Remark 4.5 that BAVWM with $m = 1$ can be solved exactly, solve this linear program exactly. This yields an interim form corresponding to the optimal mechanism.
(3) The linear program also outputs auxiliary information in the form of a distribution Δ over mappings $(f_1^\delta, \ldots, f_n^\delta)$ of the same format from Remark 4.7.

Phase Two, Run the Mechanism:

(1) Sample a mapping from Δ (provided in Phase One).
(2) On profile $(\vec{v}, b)$, award item j to any bidder $i^* \in \arg\max_i \{\phi_i^\delta(v_i, b_i)\}$ if her virtual value is non-negative. Don't allocate item j otherwise.
(3) If $m_i^\delta(\vec{v}_i, b_i) = 1$, charge bidder i the minimum of their budget and their value for the items they receive. Otherwise, charge them nothing.

5 CONCLUSION

In this article, we consider revenue maximizing auctions in the presence of budgets that could be correlated with values. We want the auctions to be BIC, ex-post IR, and ex-post budget respecting. No nice characterization of the optimal auction is known in this setting, even when there is a single item, and when the budgets and values are drawn independently. Prior to our work, no auction that achieves a constant factor of the optimal revenue was known for this special case either. In this article, we provide a polynomial time algorithm that achieves a 3-approximation for this problem, with multiple items and additive valuations, while allowing the values and budgets to be arbitrarily correlated for a given bidder. The types of different bidders are assumed to be independently drawn. We note that even the algorithmic version of our problem, and, in particular, the case where the prior is a point mass, is NP-hard to approximate within a factor of 16/15. The technical core of the result is to design an approximation algorithm for an allocation problem with a mixed-sign objective. The result then follows from the mechanism design to algorithm design reduction of Cai et al. (2013).

While budget constraints occur fairly often, especially when the sums involved are huge such as in spectrum auctions, much of the recent research into budget constraints in auctions is motivated by *ad auctions*. Our results represent a step toward a principled approach to handling budgets for this application. A natural next step would be to generalize our results to handle the more complicated structure of allocation constraints in these auctions. Here we highlight this and other questions for future research that our work raises.

(1) Generalize our results to the more general allocation constraints involved in ad auctions.
(2) Find similar guarantees for other classes of valuations, such as unit-demand, submodular, weak gross substitutes, XOS, and subadditive valuations. (All of these, except unit-demand are generalizations of additive valuations.)
(3) Extend the recent spate of simple vs. optimal auction results (Hart and Nisan 2012; Babaioff et al. 2014; Rubinstein and Weinberg 2015; Yao 2015; Bateni et al. 2015; Goldner and Karlin 2016; Cai et al. 2016) to incorporate budgets.
(4) In the other direction, for the special case of a single item, the problem of characterizing the optimal BIC and IIR auction, when budgets and values are correlated, has resisted all efforts so far. In fact, even when the budgets and values are independent, characterizing the optimal auction for *all* (i.e., possibly irregular) distributions is open. Very recent work has only just now resolved this question for the case of a *single* bidder (Devanur and Weinberg 2017), using techniques related to the so-called FedEx Problem (Fiat et al. 2016).

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