Combinatorial Rearrangements, Restricted Permutations, and Matrix Permanents

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Abstract

Enumerative Combinatorics is the study of counting problems. Frequently, we can solve such problems by first defining equivalent problems to prove more manageable ones, an important example is the Fibonacci numbers, which can be shown to be the number of distinct paths in a chessboard. As such we begin with some standard recurrences and closed form expressions for the Fibonacci numbers, as well as the Lucas numbers. Our focus will be on using combinatorial arguments to generate closed forms and recurrences for several families of numbers arising from some chess pieces, and generalizing our model to count in more abstract settings.

Introduction

In the early 1990’s, mathematicians from UCM calculated the number of rearrangements of a 3 × 3 rectangular chessboard, with a single game piece on each square, how many rearrangements are possible if piece i must exactly make legal moves? At most one move? Can these problems be solved with recurrence equations? We begin by considering rearrangements of kings, queens, bishops, and rooks. We then extend this idea to Consider the number of rearrangements of certain tiles on a board, and how they can be applied to all varieties of problems.

Generalizations

In general, counting the number of rearrangements of a particular context-dependent structure will be used to identify classes of rearrangements.

Notation

We will use the notation R(n) to represent the number of possible rearrangements of a particular context-dependent structure. Subscripts will be used to identify classes of rearrangements.

Seating Rearrangements

We began our research by constructing combinatorial proofs of the enumerations given in [1–7]. To simplify the 3 × 3 case, we colored the classroom like a chessboard, and considered the movements of the students from the north, south, east, and west. We then extended this to the "extended" seating arrangements and the Fibonacci tilings of a 3 × 3 rectangle to show that R(n) = F(n).

Fibonacci and Lucas Tilings

The second extension of this model dealt with the related question, "How many rearrangements exist in a m × n classroom if the students are not allowed to remain in their seats?" Clasifying the number of a 3 × 3 seating classroom leads to the following homogeneous recurrence

R(n) = 2R(n−1) + 3R(n−2) + 3R(n−3) + 2R(n−4) + R(n−5).

One of the most elegant integer sequences is the Fibonacci sequence and the Lucas sequence. We can model both of these sequences with our model and its generalizations. The seating rearrangements and digraphs are straightforward generalizations, but other theories may come about to count the Fibonacci numbers we let \( F(n) = \left[ \begin{array}{cc} 2 & 2 \\ 3 & 1 \end{array} \right] n + 1, \) and define another \( F'(n) = \left[ \begin{array}{cc} 1 & 1 \\ 1 & 0 \end{array} \right] n + 1. \)

Permanents and Cycle Covers

Given a digraph, \( D \), the adjacency matrix of \( D \) is \( \Omega \) in which entries \( \Omega_{ij} = 1 \) if \( i \) is adjacent to \( j \), and \( \Omega_{ij} = 0 \) otherwise. Several of these sequences were equivalent to ones already in the OEIS, but few were not. The rearrangements for the kings, bishops, and rooks led to a variety of results, while queens and rooks rearrangements do not. However, the number of rearrangements in the Lucas sequence grows geometrically. For example, the king rearrangements satisfy recurrences of \( 2, 3, 5, 8, \) and \( 13 \) with \( 0 \) and \( 1 \), each claim being a more complex result.

Theorem 4. The number of rearrangements on the 3 × 3 square graph, \( \Omega = \left[ \begin{array}{cc} 2 & 2 \\ 3 & 1 \end{array} \right] n + 1 \), is \( 1, 2, 6, 14, 31, 71, 156, 333, 694, 1438, 2981, 6082, 12200, 24608, 49354, 100010, 200474 \), and \( 401036 \).

Graph Families

Finally, we applied our rearrangement model to three well-known families of graphs, in order to analyze perfect matchings (fig. 6). Perfect matchings are often used by computer scientists to analyze network topologies, and the theory of perfect matchings can be used to develop efficient search algorithms for many kinds of combinatorial problems, including graph interaction [2]. These studies led to the following theorems.

Theorem 2. The number of perfect matchings on the \( n \) th \( n \)-dimensional \( n \)-cube \( (\lambda - 1) \), for \( \lambda = 2, 3, \ldots \), is \( 1, 2, 6 \), and \( 24 \).

Conclusions and Extensions

Our combinational rearrangement model can be applied to many subjects, both inside mathematics and the physical sciences. Using combinatorial arguments to generate and study these results is a valuable technique for describing the structure underlying these problems. We also hope to be able to extend our results by considering the following problems:

1. Adapting our model to analyze problems in crystal physics combinatorially. Combinatorial rearrangements are useful in understanding the structure of systems of the type that are modeled by the Fibonacci numbers. These chess problems can also be stated in terms of digraphs. Examples of these chess piece rearrangements provided us with more interesting and important problems.

2. As an extension of the original problem, we asked the following question. Given a 3 × 3 square with sides \( a, b, c, d, e, \) and \( f \), how many rearrangements are possible if piece i must exactly make legal moves? At most one move? Can these problems be solved with recurrence equations? We begin by considering rearrangements of kings, queens, bishops, and rooks. We then extend this idea to Consider the number of rearrangements of certain tiles on a board, and how they can be applied to all varieties of problems.

References


