Matched Products and Dynamical Models for Multiplex Networks

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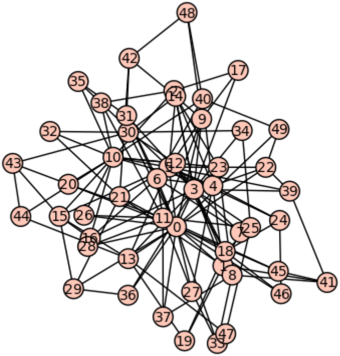
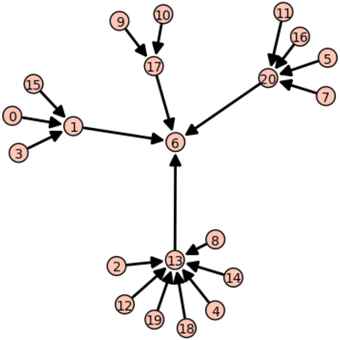
Dartmouth

Outline

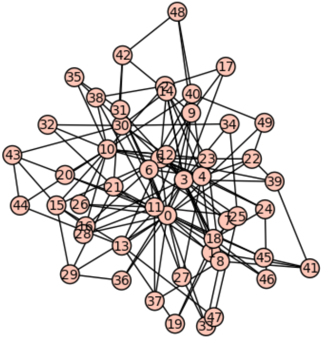
Complex Networks



Centrality



Centrality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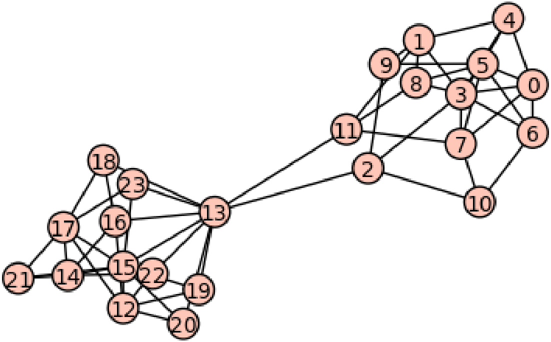
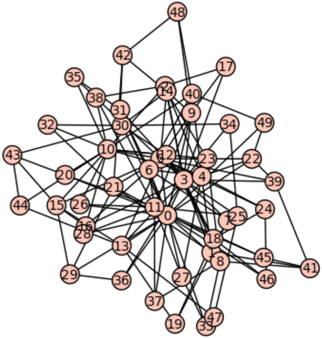
Clustering



Clustering



Diffusion Dynamics

Given a vector of initial heat values v the change in temperature at node i with respect to time is given by

$$\frac{dv_i}{dt} = -K \sum_{i \sim j} v_i - v_j$$

or

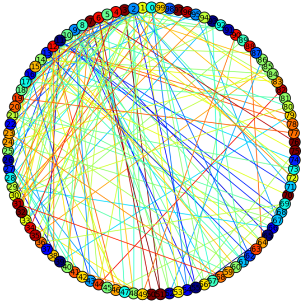
$$\frac{dv}{dt} = -KLv$$

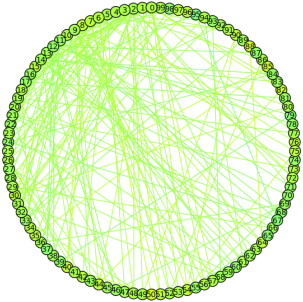
where L is the graph Laplacian. This is a symmetric, positive semi–definite matrix so the value at time t is

$$v(t) = \sum_{i=1}^{n} c_i v^i e^{-\lambda_i t}$$

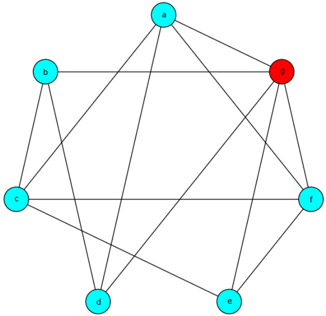
where the (v^i, λ_i) are eigenpairs for L.

Diffusion Animation

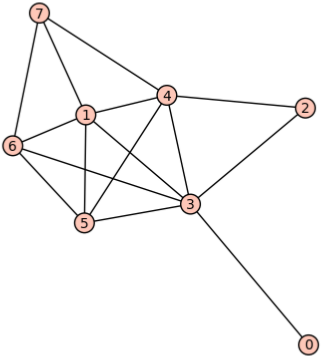




Random Walk Dynamics

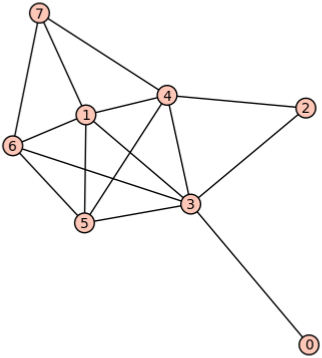


Degree Matrix



$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

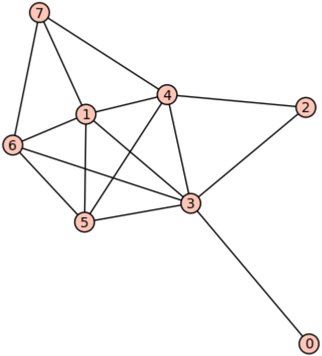
Degree Matrix



$$D = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

Degree Centrality: You are popular if you have many friends.

Adjacency Matrix



$$A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \end{bmatrix}$$

Eigenvector Centrality

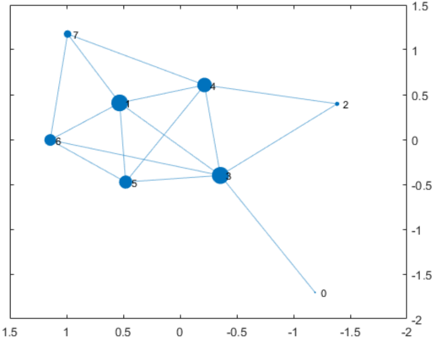
- Intuitively: You are popular if your friends are popular
- Formally: Your popularity should be proportional to the sum of your friends' popularities.
- Mathematically: Given a vector v whose entries represent the initial popularity of each node in the network we seek a solution to:

$$v_i = \lambda \sum_{i \sim j} v_j = \sum_{j=1}^{n} A_{i,j} v_j$$

or equivalently:

$$v = \lambda A v$$

Toy Eigenvector Centrality



$$v = \begin{bmatrix} 0.1052 \\ 0.4470 \\ 0.2022 \\ 0.4508 \\ 0.4155 \\ 0.3926 \\ 0.3683 \\ 0.2873 \end{bmatrix}$$

Dynamics on Networks

- Adjacency
  - Matrix: A
  - Symmetric, binary
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  - Discretized version of Laplacian heat diffusion
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- Laplacian
  - Matrix: $L = D - A$
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  - Discretized version of Laplacian heat diffusion
  - **Fiedler value**: smallest non–zero eigenvalue
- Random Walk
  - Matrix: AD^{-1}
  - Stochastic, regular if G is connected
  - Transition matrix of associated Markov process
  - Convergence governed by second largest eigenvalue

Spectral Graph Theory

Fan Chung: *Spectral Graph Theory*, AMS, (1997).

"Roughly speaking, half of the main problems of spectral theory lie in deriving bounds on the distributions of eigenvalues. The other half concern the impact and consequences of the eigenvalue bounds as well as their applications."

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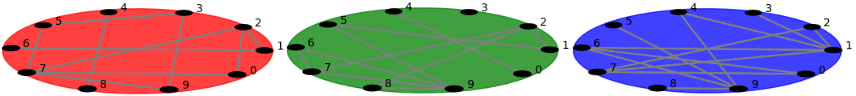
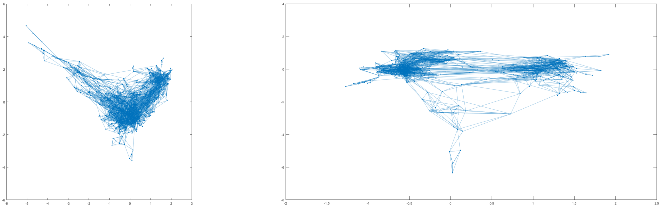


(a) Family (b) Colleagues (c) Facebook

Karnataka Village Data[1]



(a) Village 5 (b) Village 61

Figure: Karnataka Villages

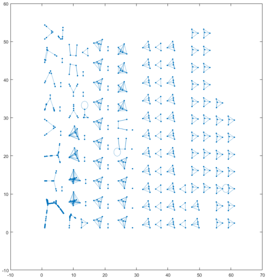
[1] A. Banerjee, A.G. Chandrasekhar, E. Duflo, and M.O. Jackson, The Diffusion of Microfinance. Science, (2013).

Village Layers

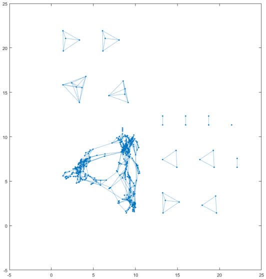
Layer Description	Village 5			Village 61		
	Density	Comp.	Giant %	Density	Comp.	Giant %
Borrow Money	.0082	26	.8354	.0108	15	.9188
Give Advice	.0077	49	.5892	.0098	34	.7377
Help Make Decisions	.0076	61	.1277	.0100	24	.8562
Borrow Kerosene or Rice	.0085	21	.8338	.0113	14	.9171
Lend Kerosene or Rice	.0086	22	.8308	.0113	14	.9255
Lend Money	.0081	14	.7908	.0107	17	.9036
Medical Advice	.0075	84	.2938	.0106	14	.9306
Friends	.0089	15	.9277	.0105	22	.8714
Relatives	.0085	29	.7231	.0105	26	.5448
Attend Temple With	.0073	117	.0462	.0089	108	.0372
Visit Their Home	.0087	15	.9185	.0116	11	.9475
Visit Your Home	.0088	16	.9108	.0117	11	.9492
Aggregate	.0121	3	.9862	.0155	8	.9679

Table: Layer information for two of the Karnataka Villages.

Medical Advice

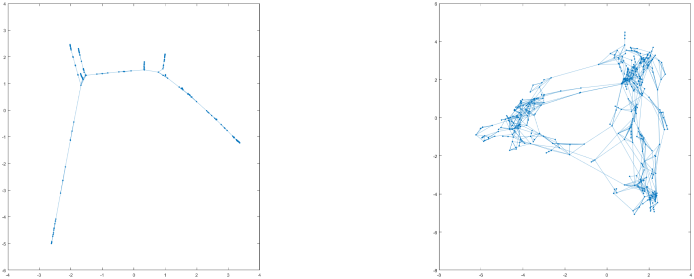


(a) Village 5



(b) Village 61

Medical Advice



(a) Village 5 (b) Village 61

World Trade Web[2]

Figure: World trade networks

[2] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

WTW Layers

Layer	Description	Volume	% Total	Transitivity
0	Food and live animals	291554437	5.1	.82
1	Beverages and tobacco	48046852	0.9	.67
2	Crude materials	188946835	3.3	.79
3	Mineral fuels	565811660	10.0	.62
4	Animal and vegetable oils	14578671	0.3	.64
5	Chemicals	535703156	9.5	.83
6	Manufactured Goods	790582194	13.9	.87
7	Machinery	2387828874	42.1	.85
8	Miscellaneous manufacturing	736642890	13.0	.83
9	Other commodities	107685024	1.9	.56
All	Aggregate Trade	5667380593	100	.93

Table: Layer information for the 2000 World Trade Web.

Multiplex Representations

Given an n node multiplex $M = (V, (E_1, E_2, \ldots, E_k))$ there are several ways to represent the data with a single network.

- **Disjoint Layers:** Form an nk node network with no connections between the edge sets: $\coprod_{i=1}^{k} (V, E_i)$.

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- **Aggregate:**
  - **Weighted:** Form an n node weighted graph whose edge set is a multiset $(V, \dot{\cup}_{i=1}^{k} E_i)$.
  - **Thresholded:** Fix a parameter $\ell \geq 1$ and form an n node network $(V, \{(i,j) : (i,j) \in E_m \text{ for at least } \ell \text{ values of } 1 \leq m \leq k\}$

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- **Matched Sum***: Start with the disjoint layers model and then connect all copies of the same node. That is, $n_i^\alpha \sim n_j^\beta$ if and only if:
  - $\alpha = \beta$ and $(i,j) \in E_\alpha$
  - or $\alpha \neq \beta$ and $i = j$.

*Manlio De Domenico, Albert Solé-Ribalta, Emanuele Cozzo, Mikko Kivelä, Yamir Moreno, Mason A. Porter, Sergio Gómez, and Alex Arenas, *Mathematical formulation of multilayer networks*, Physical Review X 3 (2013), 4, 041022.

Multiplex Null Models

Definition

The multiplex null model with n nodes and k edge sets, where each layer is an Erdos–Renyi (ER) graph with connection probability p_i will be denoted:

$$MER(n, k, (p_1, p_2, \ldots, p_k)).$$

Definition

The multiplex null model with n nodes and k edge sets, where each layer is an Stochastic Block Model (SBM) graph with connection matrix P_i and partition z_i will be denoted:

$$MSBM(n, k, (z_1, z_2, \ldots, z_k), (P_1, P_2, \ldots, P_k)).$$

Disjoint Layers

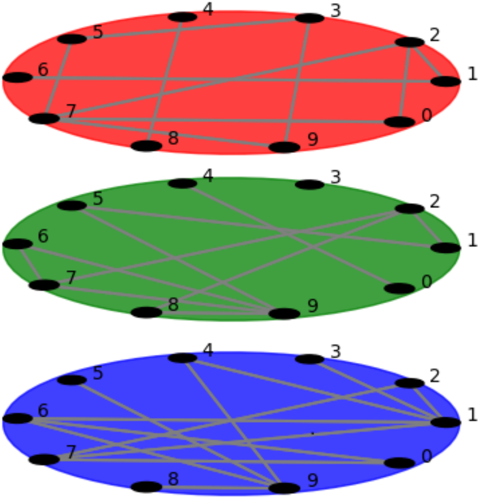
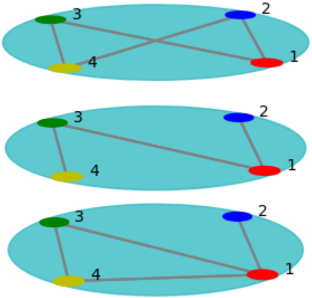
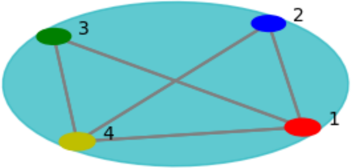


Figure: Disjoint Layers

Aggregate Representations

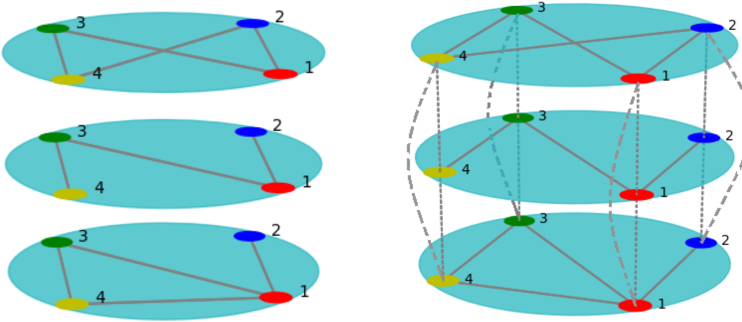


(a) Disjoint Layers



(b) Aggregate

Matched Sum



(a) Disjoint Layers (b) Matched Sum

Adjacency Matrices

We can represent the matched sum with a supra–adjacency matrix:

$$\begin{bmatrix} A^1 & wI_n & \cdots & wI_n & wI_n \\ wI_n & A^2 & \cdots & wI_n & wI_n \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ wI_n & wI_n & \cdots & A^{k-1} & wI_n \\ wI_n & wI_n & \cdots & wI_n & A^k \end{bmatrix}$$

where the A^α are the adjacency matrices of the individual layers and w is a connection strength parameter.

Network Properties of the Matched Sum

Although the original motivation for the matched sum was dynamical (the supra–Laplacian*) many applications of the supra adjacency methods are equivalent to studying the matched sum as a single network.

Our first task is to observe how the matched sum behaves under standard network measures. We use the $MER(n, k, (p, p, \ldots, p)$ as the main object of study and are particularly interested in the behavior as $k \to \infty$ as the number of inter–layer edges has an increasingly large effect on the global structure.

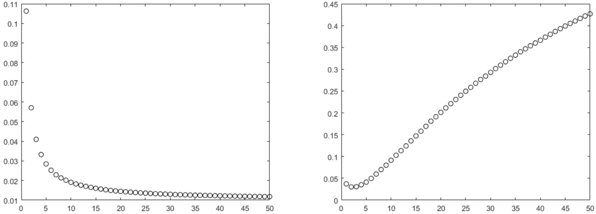
*S. Gómez, A. Díaz-Guilera, J. Gómez-Gardeñes, C. J. Pérez-Vicente, Y. Moreno, and A. Arenas, *Diffusion dynamics on multiplex networks*, Physical Review Letters 110 (2013), 2, 028701.

Matched Sum Properties

Proposition

For $MER(n, k, (p, p, \ldots, p))$ the expected density and expected local clustering coefficient are:

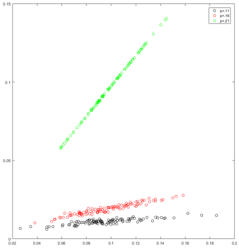
$$\frac{pk\binom{n}{2} + n\binom{k}{2}}{\binom{nk}{2}} \approx \frac{p}{k} + \frac{1}{n} \qquad \text{and} \qquad \frac{\binom{k-1}{2} + p^3 n^2}{\binom{pn+(k-1)}{2}}.$$



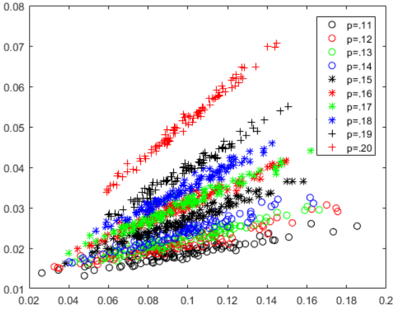
(a) Density

(b) Clustering Coefficient

MER Eigenvector Centrality



(a) MER(100,3,(.11,.16,.21))



(b) MER(100,10,(.11,.12,.13,.14,.15,.16,.17,.18,.19,.20))

MER Clustering

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If (V, E_i) is connected for all i, the k partition that separates all layers from each other is a local minimum for the (ratio) cut.

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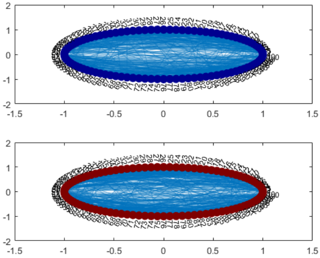
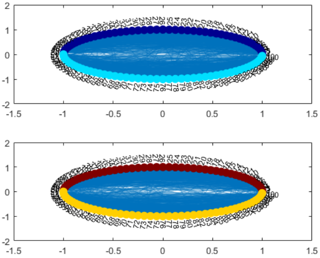
If (V, E_i) is connected for all i, the k partition that separates all layers from each other is a local maximum for modularity.

MSBM Clustering

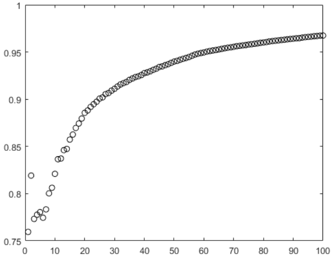


(a) Two Clusters

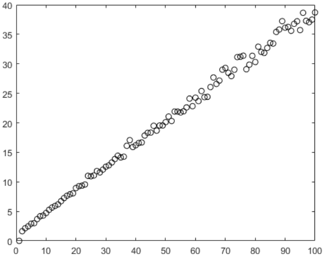


(b) Four Clusters

Random Walk Convergence

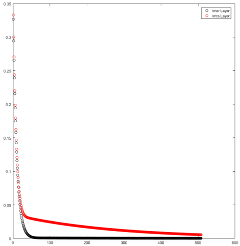


(a) Random Walk: Convergence

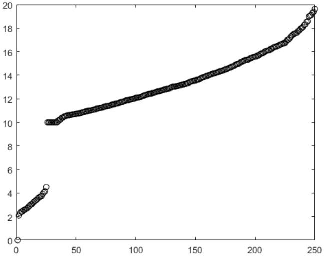


(b) Projected Walk: Steps to Escape

Matched Sum Diffusion



(a) Inter vs. Intra



(b) supra–Laplacian Eigenvalues

Dynamics Setup

Given a multiplex $M = (V, (E_1, E_2, \dots, E_K))$ and a collection of operators D_i associated to (V, E_i) we wish to construct a method for extending the dynamics to the global structure. We begin by letting D be the operator that acts diagonally on each respective component by D_i. That is, given a $1 \times nk$ vector with values associated to each element n_i^α we define the action:

$$Dv = \begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & \cdots & 0 \\ \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & D_k \end{bmatrix} v = \begin{bmatrix} D_1 v^1 \\ D_2 v^2 \\ \vdots \\ D_k v^k \end{bmatrix}$$

Dynamics on Multiplex Networks

- Two types of interactions
  - Within the individual layers
  - Between the layers
- Effects should "pass through" nodes
- Two step iterative model

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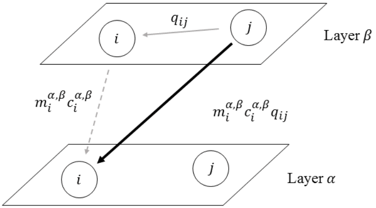

Matrix Realization

The matrix associated to the total operator also takes a convenient block form:

$$\begin{bmatrix} C^{1,1}D_1 & C^{1,2}D_2 & \cdots & C^{1,k}D_k \\ C^{2,1}D_1 & C^{2,2}D_2 & \cdots & C^{2,k}D_k \\ \vdots & \vdots & \vdots & \vdots \\ C^{k,1}D_1 & C^{k,2}D_2 & \cdots & C^{k,k}D_k \end{bmatrix}$$

Where the $\{C^{\alpha,\beta}\}$ are the diagonal proportionality matrices with diagonal given by $(m_1^{\alpha,\beta}c_1^{\alpha,\beta},\ldots,m_n^{\alpha,\beta}c_n^{\alpha,\beta})$. When $m_i^{\alpha,\beta}=1$ for all α, β, and i we say the operator is **closed**.

Choice of Coefficients

- Equidistribution $(\mathfrak{D}_e)$
  - $c_i^{\alpha,\beta} = \frac{1}{k}$
  - $C^{\alpha,\beta} = \frac{1}{k} I$
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- Ranked Layers $(\mathfrak{D}_h)$
    - $c_i^{\alpha,\beta} = c^{\alpha}$
    - $C^{\alpha,\beta} = c^{\alpha} I$
    - Global layer rankings
    - Villages

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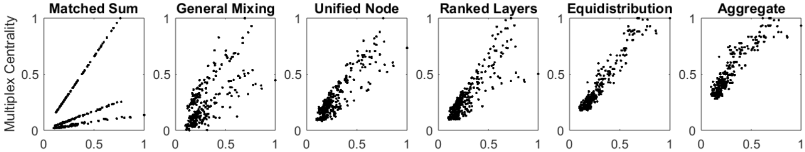
  - Starting Point/Aggregate
- Ranked Layers $(\mathfrak{D}_h)$
  - $c_i^{\alpha,\beta} = c^\alpha$
  - $C^{\alpha,\beta} = c^\alpha I$
  - Global layer rankings
  - Villages
- Unified Node $(\mathfrak{D}_u)$
  - $c_i^{\alpha,\beta} = c_i^\alpha$
  - $C^{\alpha,\beta} = C^\alpha$
  - Local node rankings
  - WTW

Choice of Coefficients

- Equidistribution $(\mathfrak{D}_e)$
  - $c_i^{\alpha,\beta} = \frac{1}{k}$
  - $C^{\alpha,\beta} = \frac{1}{k} I$
  - Starting Point/Aggregate
- Ranked Layers $(\mathfrak{D}_h)$
  - $c_i^{\alpha,\beta} = c^{\alpha}$
  - $C^{\alpha,\beta} = c^{\alpha} I$
  - Global layer rankings
  - Villages
- Unified Node $(\mathfrak{D}_u)$
  - $c_i^{\alpha,\beta} = c_i^{\alpha}$
  - $C^{\alpha,\beta} = C^{\alpha}$
  - Local node rankings
  - WTW
- General Model $(\mathfrak{D})$
  - $c_i^{\alpha,\beta} = c_i^{\alpha,\beta}$
  - Pairwise comparisons between node copies
  - Anything goes/Matched Sum

Eigenvector Centrality Comparison

Eigenvector Centrality Comparison



Layer Eigenvectors

Proposition

We consider the models $\mathfrak{D}_u, \mathfrak{D}_h$, and $\mathfrak{D}_e$ and assume that the C^α are invertible for $\mathfrak{D}_u$ and that the $c^\alpha \neq 0$ for $\mathfrak{D}_h$. Then,

1. *(Unified Node Model) Let $D_a = D^1C^1 + \ldots D^kC^k$ and $\{(\lambda_i, w_i)\}$ be its eigendata. If $\lambda_i \neq 0$, (λ_i, v_i) is an eigenvalue/eigenvector pair for $\mathfrak{D}_u$ where*
$$v_i = (C^1 w_i, \ldots, C^k w_i)^T.$$
2. *(Ranked Layers Model) Let $D_a = m^1c^1D^1 + \cdots + m^kc^kD^k$ and $\{(\lambda_i, w_i)\}$ be its eigendata. If $\lambda_i \neq 0$, (λ_i, v_i) is an eigenvalue/eigenvector pair for $\mathfrak{D}_h$ where*
$$v_i = \left(m^1c^1w_i, \ldots, m^kc^kw_i\right)^T.$$
3. *(Equi-distribution Model) Let $D_a = \frac{1}{k}(D^1 + \cdots + D^k)$ and $\{(\lambda_i, w_i)\}$ be its eigendata. If $\lambda_i \neq 0$ (λ_i, v_i) is an eigenvalue/eigenvector pair for $\mathfrak{D}_e$ where $v_i = \frac{1}{k}(w_i, w_i, \ldots, w_i)^T$.*

uth

Preserved Properties

Proposition

If the mixing matrices are closed, the the following properties are preserved in our operator:

- *Stochasticity*
- *Irreducibility*
- *Primitivity*
- *If we are additionally in the unified node case, the operator also preserves positive (negative) (semi)–definiteness.*

Multiplex Random Walks

The random walk defined by these dynamics has transition probabilities:

$$v_i^\alpha \to v_j^\beta = \begin{cases} \frac{c_j^{\beta,\alpha}}{\deg(v_i^\alpha)} & \text{if } v_i^\alpha \sim v_j^\alpha \\ 0 & \text{otherwise.} \end{cases}$$

Multiplex Random Walks

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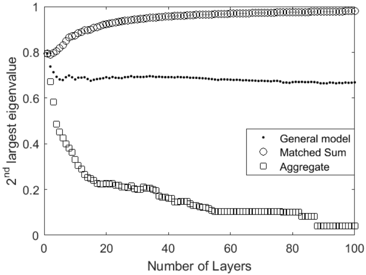

Multiplex Random Walks

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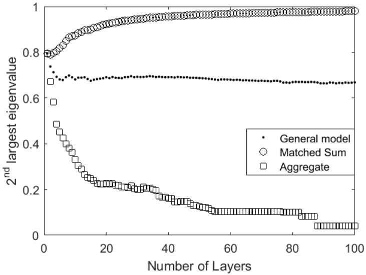


*Manlio De Domenico, Albert Solé-Ribalta, Sergio Gómez, and Alex Arenas, *Navigability of interconnected networks under random failures*, PNAS 111 (2014), 23, 8351.

**I. Trpevski, A. Stanoev, A. Koseska, and L. Kocarev, *D*iscrete-time distributed consensus on multiplex networks, New Journal of Physics 16 (2014), 11, 113063.

Laplacian Dynamics

Under our dynamical model, where effects pass through node copies to other layers, the heat diffusion interpretation of the Laplacian can be derived from first principles:

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} \sum_{n_i^\beta \sim n_j^\beta} (v_i^\beta - v_j^\beta)$$

$$\frac{dv_i^\alpha}{dt} = -K \sum_{\beta=1}^{k} c_i^{\alpha,\beta} (Lv)_i^\beta$$

Laplacian Eigenvalue Bounds

Proposition

Let $\{\lambda_i\}$ *be the eigenvalues of* $\mathscr{D}$ *and* $\{\lambda_i^\alpha\}$ *be the eigenvalues of the* α*–layer Laplacian* D^α. *We have the following bounds for ranked layers model:*

- *Fiedler Value:*

$$\max_\alpha(\lambda_F^\alpha) \leq k\lambda_F \leq \lambda_F^m + \sum_{\beta \neq m} \lambda_1^\beta$$

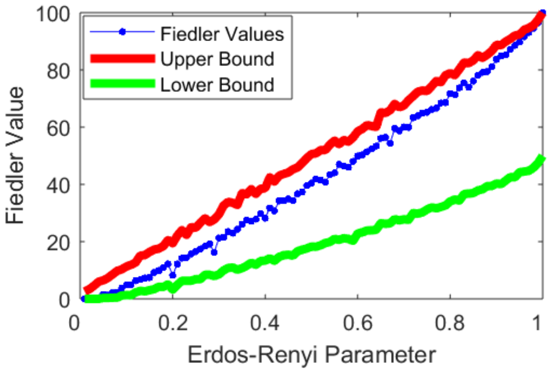
- *Leading Value:*

$$\max_i(\lambda_1^i) \leq k\lambda_1 \leq \sum_i \lambda_1^i$$

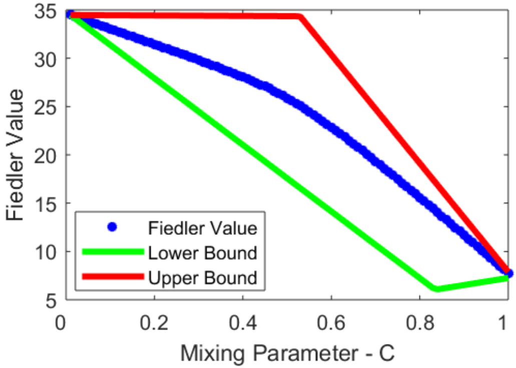
- *General Form:*

$$\max_i(\lambda_{n-j}^i) \leq k\lambda_{n-j} \leq \min_{J \vdash n+k-(j+1)} \left(\min_{\sigma \in S_n} \left(\sum_{\alpha=1}^{k} \lambda_{j_\alpha}^{\sigma(\alpha)} \right) \right)$$

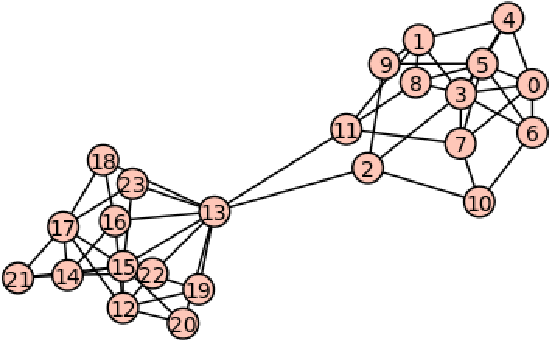
Bounds Example-p



Bounds Example - c



Clustering



Clustering Definitions

- Spectral clustering
  - Minimize Inter–Community Edges
  - Minimize $s^T L s$ with $s \in \{\pm 1\}^n$
- Modularity
  - Maximize Intra–Community Edges (compared to expectation)
  - Maximize $s^T B s$ with $s \in \{\pm 1\}^n$where $B_{i,j} = A_{i,j} - \frac{\deg(i)\deg(j)}{2m}$.
- Markov Stability
  - Given a partition $(V_1, V_2, \ldots, V_\ell)$ maximize (for fixed t) $r(t, V) = \sum_{i=1}^{\ell} \sum_{v_y, v_z \in V_i} C(t)_{y,z}$
  - Discrete: $C(t) = \Pi S^t - \pi^T \pi$ where π is the steady state vector and Π is the diagonal matrix of π
  - Continuous: $C(t) = \Pi e^{-t(I-S)} - \pi^T \pi$ where π is the steady state vector and Π is the diagonal matrix of π

Multiplex Cut Computations

Extending spectral clustering to the multiplex setting requires a variety of additional considerations. Let $s = (s_1, s_2, \ldots, s_k)$ be a $1 \times nk$ vector in $\{\pm 1\}^{nk}$ representing the community assignments.

- **Matched Sum** (w is the inter–layer weight parameter):

$$\sum_{\alpha=1}^{k} s_\alpha^T \mathcal{L}_{\frac{1}{2}(A^\alpha + (A^\alpha)^T)} s_\alpha + w \sum_{\alpha,\beta=1}^{k} \left((\vec{1})^T \vec{1} - s_\alpha^T s_\beta \right)$$

- **Dynamical** ($B^{\beta,\alpha} = C^{\beta,\alpha} A^\alpha + (C^{\alpha,\beta} A^\beta)^T$):

$$\sum_{\alpha} s_\alpha^T \mathcal{L}_{B^{\alpha,\alpha}} s_\alpha + \sum_{\alpha \neq \beta} \left((\vec{1})^T B^{\alpha,\beta} \vec{1} - s_\alpha^T B^{\alpha,\beta} s_\beta \right)$$

Experiments Outline

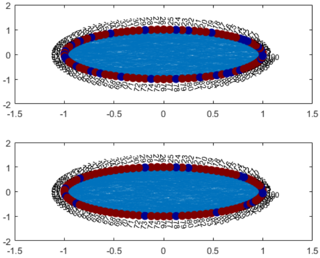
Layer Models

1. ER Layers
2. Aligned SBM Communities
3. Offset SBM Communities

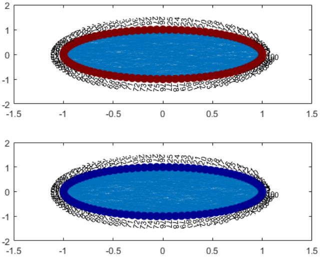
Parameters

1. Number of layers
2. Density of layers
3. Number of communities sought
4. Amount of offset
5. Inter–layer weight ω (Matched Sum)
6. Mixing Matrix C (Dynamical)

ER Layers Clusters

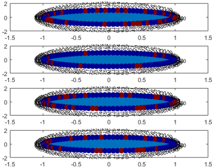


(a) Dynamical

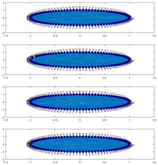


(b) Matched Sum

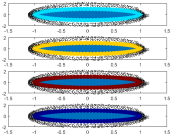
ER Layers Clusters



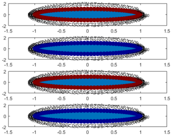
(a) Dynamical



(b) Dynamical

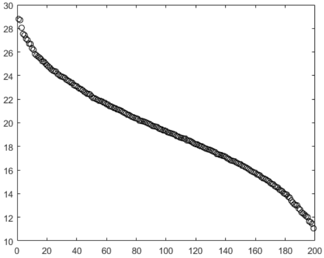


(c) Matched Sum

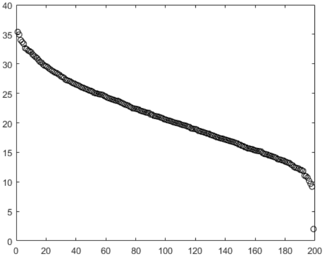


(d) Matched Sum

ER Layers Eigenvalues

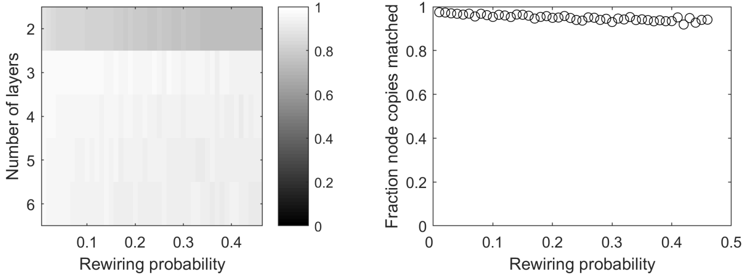


(a) Dynamical



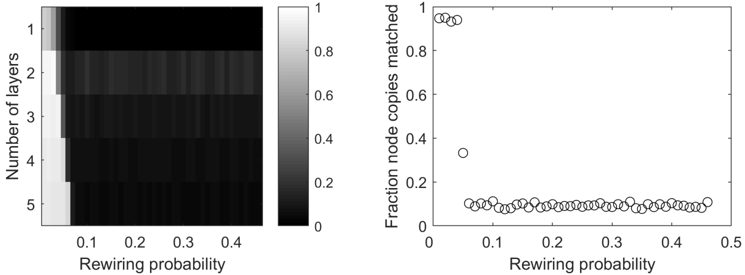
(b) Matched Sum

ER Layers Match Proportion



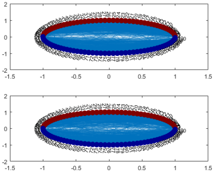
(a) Dynamical

ER Layers Match Proportion

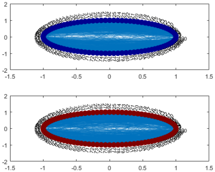


(a) Matched Sum

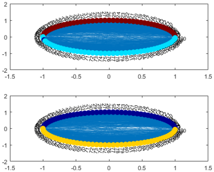
Aligned SBM Layer Clusters



(b) Dynamical

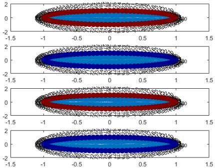


(c) Matched Sum 2

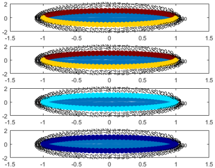


(d) Matched Sum 4

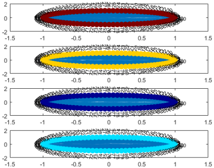
Aligned SBM Layer Clusters



(e) Matched Sum 2

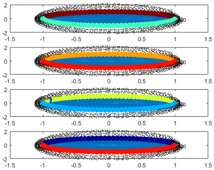


(f) Matched Sum 4

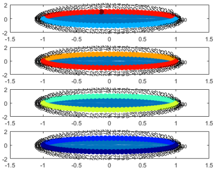


(g) Matched Sum 4

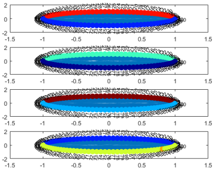
Aligned SBM Layer Clusters



(h) Matched Sum 8

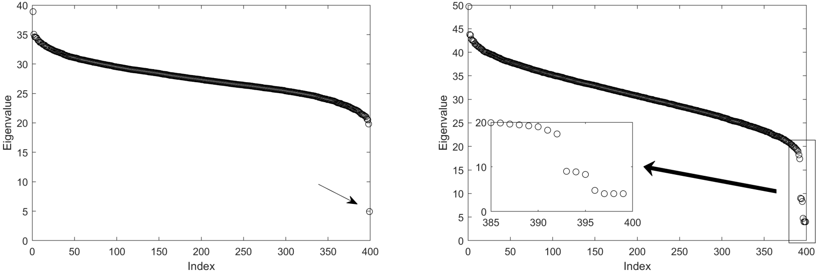


(i) Matched Sum 8



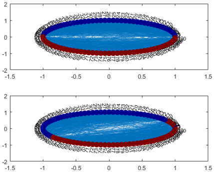
(j) Matched Sum 8

Aligned SBM Layer Eigenvalues

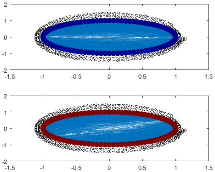


(a) Dynamical (b) Matched Sum

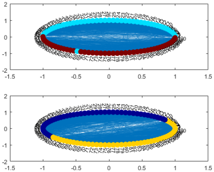
Offset SBM Layer Clusters



(a) Dynamical

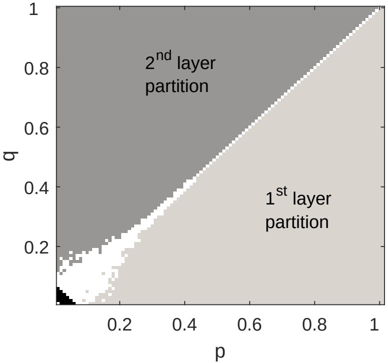


(b) Matched Sum 2



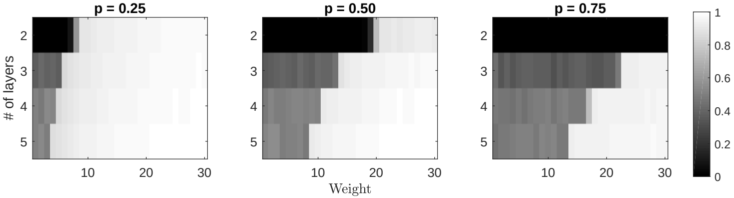
(c) Matched Sum 4

Dynamical Mixing Clusters



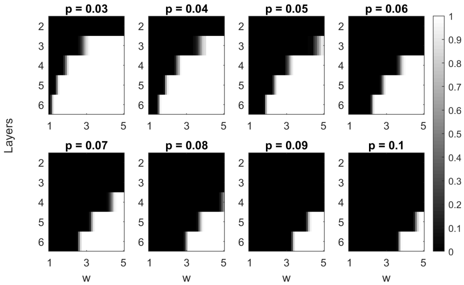
(a) Offset (1,25)

Matched Sum Weighting ER



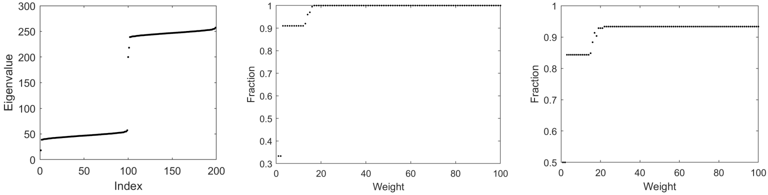
(a) ER Layers

Matched Sum Weighting SBM



(b) SBM Layers

Matched Sum Offset Weighting



(a) Eigenvalues (b) Offset (1,1,1,10) (c) Offset (1,5,10,15)

Multiplex Modularity

There are several natural ways to extend modularity to the multiplex setting.

1. Full Rewiring – Form a single network and apply classic modularity.
2. Intra-Layer rewiring – Compute the individual modularity matrices and then combine with the matched sum or dynamical mixing matrices.
3. Markov stability using new random walk.

* J.-C. Delvenne, S. N. Yaliraki, and M. Barahona *Stability of graph communities across time scales*, PNAS, (2010), 107 (29) 12755-12760.
** Peter J. Mucha, Thomas Richardson, Kevin Macon, Mason A. Porter, and Jukka-Pekka Onnela, *Community structure in time-dependent, multiscale, and multiplex networks*, Science 328 (2010),5980.
*** L. Jeub, M. Mahoney, P. Mucha, and M. Porter, *A* local perspective on community structure in multilayer networks, Network Science 5 (2017), 2, 144163.

**** Zijing Liu and Mauricio Barahona, Geometric multiscale community detection: Markov stability and vector partitioning, Journal of Complex Networks (2018), 6(2), 157–172.

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Markov Stability

Our multiplex random walk operator is intrinsically directed, with the transition probability from $n_i^\alpha \to n_j^\beta$ given by

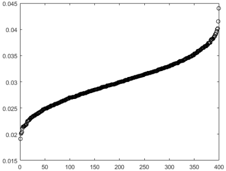
$$\begin{cases} \frac{c^\beta}{\deg(n_i^\alpha)} & n_i^\alpha \sim n_j^\alpha \\ 0 & n_i^\alpha \not\sim n_j^\alpha \end{cases}.$$

In the ranked layers case, the random walk can be reduced to studying an n state Markov process with transition probabilities given by the projected walk where we only observe the progress of the walker between objects, not a the level of node copies. The weights on this aggregate are given by

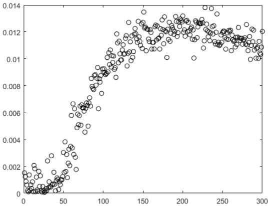
$$2W_{i,j} = \begin{cases} 0 & n_i^\alpha \not\sim n_j^\alpha \text{ and } n_i^\beta \not\sim n_j^\beta \\ \deg(n_j^\beta) & n_i^\alpha \sim n_j^\alpha \text{ and } n_i^\beta \not\sim n_j^\beta \\ \deg(n_j^\alpha) & n_i^\alpha \not\sim n_j^\alpha \text{ and } n_i^\beta \sim n_j^\beta \\ \deg(n_j^\alpha) + \deg(n_j^\beta) & n_i^\alpha \sim n_j^\alpha \text{ and } n_i^\beta \sim n_j^\beta \end{cases}.$$

Dartmouth

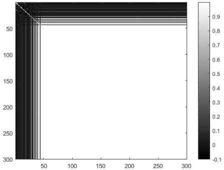
SBM Stability Spectral Gap



(a) Continuous Eigenvalues

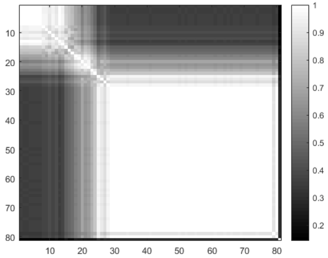


(b) Continuous Spectral Gap

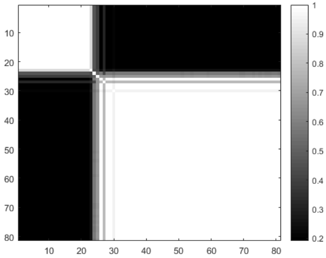


(c) Continuous Rand Index

Offset SBM Markov Stability

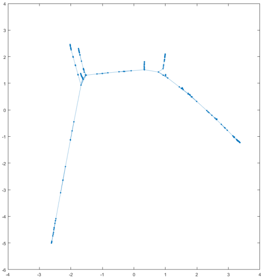


(a) Offsets (1,1,1,15)

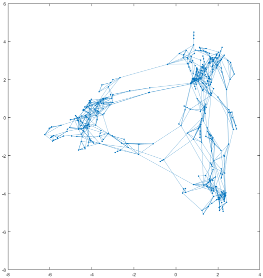


(b) Offsets (1,5,10,15)

Medical Advice

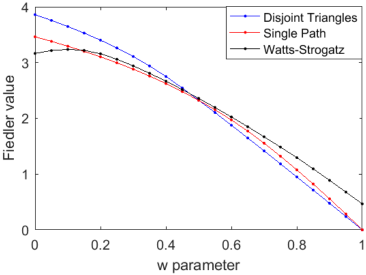


(a) Village 4

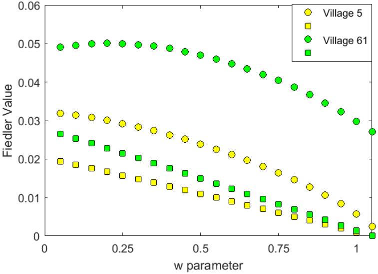


(b) Village 61

Social Diffusion

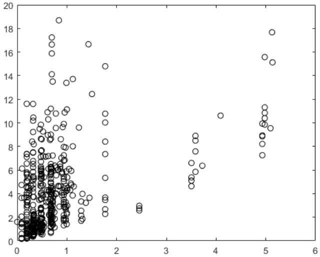


(a) Synthetic

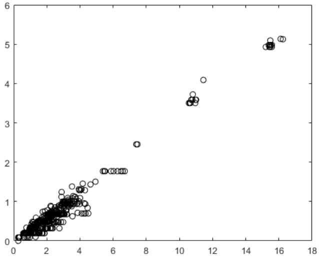


(b) Villages

Village 5 Dynamical Diffusion Centrality



(a) Medical vs. Aggregate



(b) Multiplex vs. Medical $c = \frac{7}{24}$

World Trade Web[2]

Figure: World trade networks

[2] R. Feenstra, R. Lipsey, H. Deng, A.C. Ma, and H. Mo. World Trade Flows: 1962-2000. NBER Working Paper 11040, (2005).

Random Walk Model

- Unified Node Model: Random walk dynamics. At each time step, each country collects its in–flowing dollars and then redistributes them proportional to its outflow values on each layer.

Random Walk Model

- Unified Node Model: Random walk dynamics. At each time step, each country collects its in–flowing dollars and then redistributes them proportional to its outflow values on each layer.
- Natural centrality measure: Random Walk Betweeness Centrality (RWBC)*
- The RWBC of node i is defined by summing over all pairs (j, k) the probability that a random walk beginning at node j passes through node i before reaching node k.

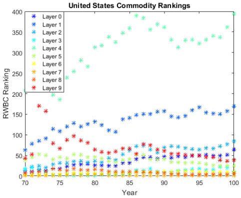
* Newman, M.E.J.: A measure of betweenness centrality based on random walks. Social Networks 27(1), 39-54 (2005).

Global Aggregate Rankings

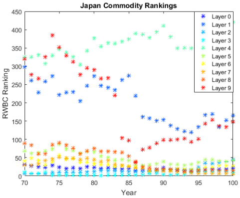
Year	1970	1980	1990	2000
1	US	US	US	US
2	Germany	Germany	Germany	Germany
3	Canada	Japan	Japan	Japan
4	UK	UK	France	China
5	Japan	France	UK	UK
6	France	Saudi Arabia	Italy	France

Table: RWBC values for the aggregate WTW.

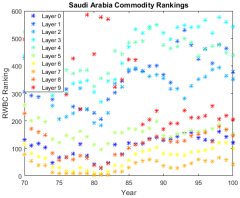
Commodity Rankings



(a) US Commodities



(b) Japan Commodities



(c) Saudi Arabia Commodities.

Dartmouth

Full Multiplex RWBC

Ranking	Country	Layer
1	US	7
2	Germany	7
3	China	7
4	UK	7
5	Japan	7
6	US	8
7	Canada	7
8	France	7
9	Japan	3
10	US	6
12	US	3
13	Netherlands	7
14	Germany	6
15	Italy	7

Table: Multiplex RWBC values for the 2000 WTW.

Commodity Appearance

Layer	Ranking	Country
0	22	Japan
1	199	Germany
2	47	China
3	9	Japan
4	184	Australia
5	23	Germany
6	10	US
7	1	US
8	6	US
9	39	US

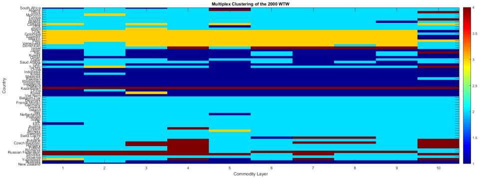
Table: First appearance of each layer in the rankings.

Ranking Movement

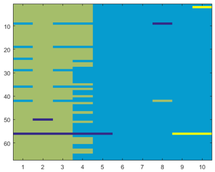
Layer 7 Ranking	Country	Multiplex Ranking
1	USA	1
2	Japan	5
3	Germany	2
4	China	3
5	France	8
6	UK	4
7	South Korea	18
8	Canada	7
9	Malaysia	16
10	Mexico	20

Table: Comparison of the relative rankings of the RWBC on Layer 7 versus the multiplex RWBC.

WTW Clustering



(a) Spectral Clustering



(b) Continuous Markov

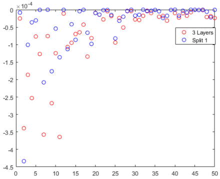
Multiplex References

- M. Kivela, A. Arenas, M. Barthelemy, J. Gleeson, Y. Moreno, and M. Porter: *Multilayer networks*, Journal of Complex Networks, 1–69, (2014).
- D. DeFord and S. Pauls: *A new framework for dynamical models on multiplex networks*, Journal of Complex Networks, to appear, (2018), 29 pages.
- D. DeFord and S. Pauls: *Spectral clustering methods for multiplex networks*, submitted, arXiv:1703.05355, (2017), 39 pages.
- D. DeFord: *Multiplex dynamics on the world trade web*,In Proc. 6th International Conference on Complex Networks and Applications, Studies in Computational Intelligence, Springer, 1111–1123, (2018).

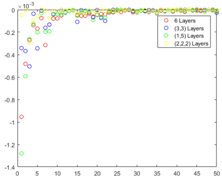
That's all...

Thank You!

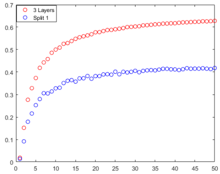
Layer Splitting Partitions



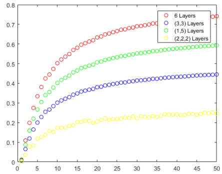
(a) Dynamical $k = 3$



(b) Dynamical $k = 6$

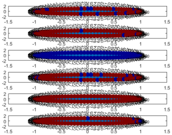


(c) Matched Sum $k = 3$

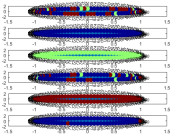


(d) Matched Sum $k = 6$

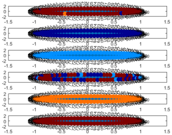
MER Matched Sum



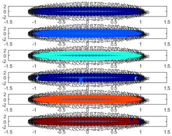
(a) 1 Eigenector



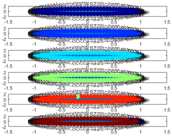
(b) 2 Eigenectors



(c) 3 Eigenectors

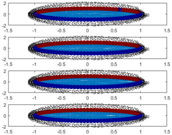


(d) 4 Eigenectors

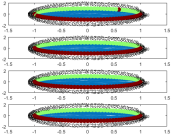


(e) 5 Eigenectors

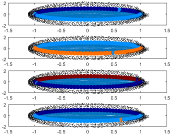
MSBM Matched Sum



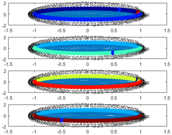
(a) 1 Eigenector



(b) 2 Eigenectors



(c) 3 Eigenectors



(d) 4 Eigenectors

Matched Product

Definition

Given a graph C with k labeled nodes called the **structure graph** and an ordered set of k layer graphs $(G_1, G_2, \ldots, G_k)$ each with n labeled nodes we defined the **matched product** $\boxed{C}(G_1, G_2, \ldots, G_k)$ of the $\{G_i\}$ with respect to C as the graph with vertex set $\cup V_i$ and edges between two nodes v_i^α and v_j^β if either:

- $\alpha = \beta$ and $i \neq j$ and $v_i \sim v_j$ in G_α or
- $\alpha \neq \beta$ and $i = j$ and $v_\alpha \sim v_\beta$ in C.

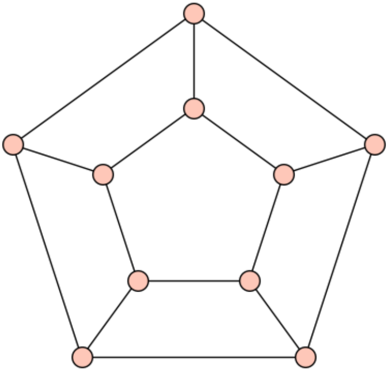
Other Products

Proposition

There are labelings of the graphs below such that the following hold:

1. *The cartesian product of G and H can be represented by* $\boxed{H}(G, G, \ldots, G)$
2. *The rooted product of G and H can be represented by* $\boxed{H}(G, E_n, E_n, \ldots, E_n)$
3. *The hierarchical product G and H with subset $\{a_i\} \subset H$ can be represented by* $\boxed{H}(G_1, G_2, \ldots, G_k)$ *where* $G_i = \begin{cases} G & \textit{if } i \in \{a_i\} \\ E_n & \textit{otherwise} \end{cases}$.

Vertex Labeling



(a) Cylinder Graph (b) Petersen Graph

Figure: Both of these graphs can be constructed as $\boxed{P_2}(C_5, C_5)$ with different labelings of the cycles.

Multiplex Special Cases

Given a collection of layers $(G_1, G_2, \ldots, G_k)$ we can use this notation to describe the common multiplex representations:

- Disjoint layers: $\boxed{E_k}(G_1, G_2, \ldots, G_k)$
- Matched sum: $\boxed{K_k}(G_1, G_2, \ldots, G_k)$
- Temporal multiplex: $\boxed{P_k}(G_1, G_2, \ldots, G_k)$