Logic Minimization

Can realize an arbitrary logic function in sum-of-products or two-level form

$$F1 = \bar{A}\,\bar{B} + \bar{A}\,B\,D + \bar{A}\,B\,\bar{C}\,\bar{D} + A\,B\,C\,\bar{D} + A\,\bar{B} + A\,B\,D$$

$$F1 = \bar{B} + D + \bar{A}\,\bar{C} + A\,C$$

Of great interest to find a minimum sum-of-products representation

Definitions - 1

Basic definitions:

don't care

Let $B = \{0, 1\}$ and $Y = \{0, 1, 2\}$

Input variables: $X_1, X_2 \ldots X_n$

Output variables: $Y_1, Y_2 \ldots Y_m$

A logic function ff (or Boolean function, switching function) in n inputs and m outputs is the map

$$ff: B^n \longrightarrow Y^m$$

Definitions - 2

If $b \in B^n$ is mapped to a 2 then function is incompletely specified, else completely specified

For each output we define:

$\text{ON-SET}_i \subseteq B^n$, the set of all input values for which $ff_i(x) = 1$

$\text{OFF-SET}_i \subseteq B^n$, the set of all input values for which $ff_i(x) = 0$

$\text{DC-SET}_i \subseteq B^n$, the set of all input values for which $ff_i(x) = 2$

Example Function

<u>EXAMPLE:</u> Truth table form of an incompletely specified function

$ff: B^3 \longrightarrow Y^2$

X_1	X_2	X_3	Y_1	Y_2
0	0	0	1	1
0	0	1	1	0
0	1	0	0	1
0	1	1	0	1
1	0	0	1	0
1	0	1	1	2
1	1	0	1	1
1	1	1	2	1

Y_1: ON-SET_1 = {000, 001, 100, 101, 110}
OFF-SET_1 = {010, 011}
DC-SET_1 = {111}

Cube Representation

$$F1 = \overline{A}\,\overline{B} + \overline{A}\,B\,D + \overline{A}\,B\,\overline{C}\,\overline{D} + A\,B\,C\,\overline{D} + A\,\overline{B} + A\,B\,D$$

0 0 - -	1
0 1 - 1	1
0 1 0 0	1
1 1 1 0	1
1 0 - -	1
1 1 - 1	1

$$F1 = \overline{B} + D + \overline{A}\,\overline{C} + A\,C$$

- 0 - -	1
- - - 1	1
0 - 0 -	1
1 - 1 -	1

minimum representation

Operations on Logic Functions

(1) **Complement:** $f \longrightarrow \bar{f}$
interchange ON and OFF-SETS

(2) **Product (or intersection or logical AND)**
$h = f \bullet g$ **or** $h = f \cap g$

(3) **Sum (or union or logical OR):**
$h = f + g$ **or** $h = f \cup g$

(4) **Difference** $h = f - g = f \cap \bar{g}$

Prime Implicants

A cube p is an implicant of f if it does not intersect the OFF-SET of f

$p \subseteq f_{ON} \cup f_{DC}$ (or $p \cap f_{OFF} = 0$)

A prime implicant of f is an implicant p such that

(1) No other implicant q is such that $q \supset p$

(2) $f_{DC} \not\supset p$

A minterm is a fully specified implicant e.g., 011, 111 (not 01-)

Examples of Implicants/Primes

X_1	X_2	X_3	Y_1
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	2

000, 00- are implicants, but not primes (-0-)

1-1

0-0

Prime and Irredundant Covers

A cover is a set of cubes C such that

$C \supseteq f_{ON}$ and

$C \subseteq f_{ON} \cup f_{DC}$

A prime cover is a cover whose cubes are all prime implicants

An irredundant cover is a cover C such that removing any cube from C results in a set of cubes that is not a cover

Minimum covers

A minimum cover is a cover of minimum cardinality

Theorem: A minimum cover can always be found by restricting the search to prime and irredundant covers.

Given any minimum cover C

(a) if redundant, not minimum

(b) if any cube q is not prime, replace q with prime $p \supset q$ and it is a minimum prime cover

Example Covers

X_1	X_2	X_3	Y_1
0	0	0	1
0	0	1	1
0	1	0	0
0	1	1	0
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	2

0 0 -
1 0 - is a cover. Is it prime?
1 1 - Is it irredundant?

- 0 -
1 1 - is a cover. Is it prime?
Is it irredundant?
Is it minimum?

What is a minimum prime and irredundant cover for the function?

The Quine - McCluskey Method

Step 1: List all minterms in ON-SET and DC-SET

Step 2: Use a prescribed sequence of steps to find all the prime implicants of the function

Step 3: Construct the prime implicant table

Step 4: Find a minimum set of prime implicants that cover all the minterms

Example

0	0000	0,8	-000 E	8,9,10,11	10-- B
5	0101	5,7	01-1 D	10,11,14,15	1-1- A
7	0111	7,15	-111 C		
8	1000	8,9	100-		
9	1001	8,10	10-0		
10	1010	9,11	10-1		
11	1011	10,11	101-		
14	1110	10,14	1-10		
15	1111	11,15	1-11		
		14,15	111-		

A B C D E are prime implicants

Prime Implicant Table

Minterms (ON-SET only)

	A	B	C	D	E
0					X
5				X	
7			X	X	
8		X			X
9		X			
10	X	X			
11	X	X			
14	X	X			
15	X		X		

X's indicate minterms covered by PIs

Essential Prime Implicants

	A	B	C	D	E
0					X
5				X	
7			X	X	
8		X			X
9		X			
10	X	X			
11	X	X			
14	X	X			
15	X		X		

Row with a single X identifies an essential prime implicant (EPI)

Essential PI's E, D, B, A $\Rightarrow$ Form minimum cover

Dominating Rows

In general EPIs do not form a cover

At Step 4, we need to select PIs to add to the EPIs so as to form a minimum cover

	A	B	C	D	F	G
1		X	X			
8	X		X			
9	X	X	X			
24	X					X
25	X	X			X	X
27				X	X	

Row 9 dominates 8
Row 25 dominates 24

Can remove 9 since covering 8 implies covering of 9

Dominating Columns

	A	B	C	D	F	G
1		X	X			
8	X		X			
9	X	X	X			
24	X					X
25	X	X			X	X
27				X	X	

F **dominates** D

Can remove D **since** F **covers all minterms** D **covers**

Can this happen in the original table?

May happen after removal of EPIs

Step 4 Issues

Removal of dominating columns or dominated rows may introduce columns with single X's.

– Need to iterate

A cover may still not be formed after all essential elements and dominance relations have been removed

– Need to branch over possible solutions

Recursive Branching (Step 4)

(a) Select EPIs, remove dominated columns and dominating rows iteratively till table does not change

(b) If the size of the selected set (+ lower bound) exceeds or equals best solution so far, return from this level of recursion. If no elements left to be covered, declare selected set as the best solution recorded.

(c) Select (heuristically) a branching column.

Recursive Branching (Step 4) - 2

(d) Add this PI to selected set and recur

- On the sub-table resulting from deleting the column and all rows covered by this column. Add the column to the selected set.
- On the sub-table resulting from deleting the column without adding it to the selected set.

Example

	A	B	C	D	E	F	G	H
0	X							X
1	X	X						
5		X	X					
7			X	X				
8							X	X
10						X	X	
14					X	X		
15				X	X			

No essential primes, dominated rows or columns.

Select prime A

Example - 2

	B	C	D	E	F	G	H
5	X	X					
7		X	X				
8						X	X
10					X	X	
14				X	X		
15			X	X			

Selected set = { A }

B **is dominated by** C

H **is dominated by** G

Removed B, H

Example - 3

	C	D	E	F	G
5	X				
7	X	X			
8					X
10				X	X
14			X	X	
15		X	X		

C, G essential to this table

Selected set = {A, C, G}

	D	E	F
14		X	X
15	X	X	

Selected set = {A, C, G, E}

Example - 4

	B	C	D	E	F	G	H
0							X
1	X						
5	X	X					
7		X	X				
8						X	X
10					X	X	
14				X	X		
15			X	X			

Selected set = { }

Essential primes
<u>in this table</u> are B, H

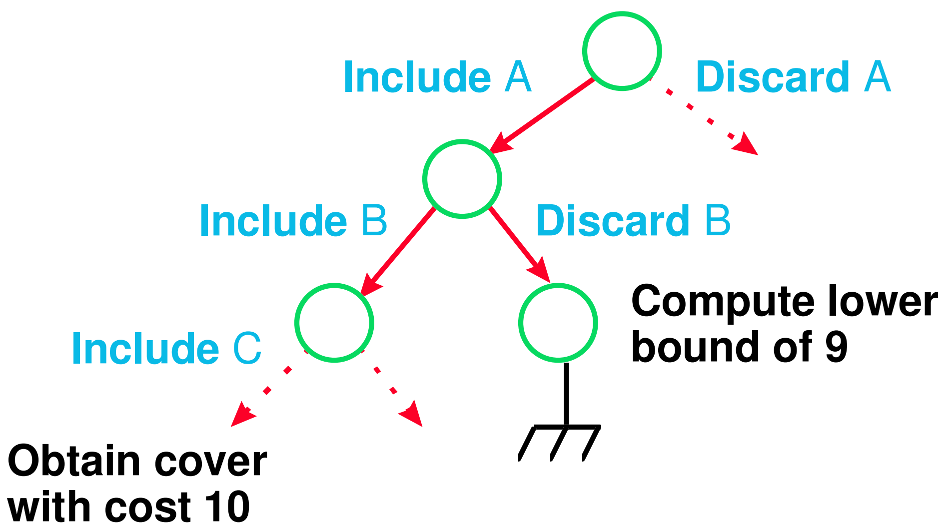
Selected set = {B, H}

	C	D	E	F	G
7	X	X			
10				X	X
14			X	X	
15		X	X		

Selected set
= {B, H, D, F}

Espresso-Exact (1987)

Efficient lower bounding at Step 4(b) to terminate unprofitable searches high in the recursion



Size of selected set + Lower bound $\leq$ best solution already known, quit level of recursion

Lower Bounding

	A	B	C	D	E	F
0	X				X	
1	X		X			X
4					X	
6		X	X			
8		X				
10				X		X
12				X		X

Lower bound: Maximal independent set of rows all of which are pairwise disjoint

Maximal independent set = {1, 4, 8} or {0, 6, 10}

Need to select at least one PI/column to cover each row.

NOTE: Finding <u>maximum</u> independent set is itself worst-case exponential

Complexity of Q-M based Methods

(1) There exist functions for which the number of prime implicants is $O(3^n)$ (n is number of inputs)

(2) Given a PI table, recursive branching could require $O(2^m)$ time (m is the number of PIs)

Current logic minimizers able to find exact solutions for functions with 20-25 input variables

$\Rightarrow$ Need heuristic methods for larger functions