| **6.841: Advanced Complexity Theory** | Fall 2012 |
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| Problem Set 5 – PCPs and Counting | |
| *Prof. Dana Moshkovitz/TA: Henry Yuen* | *Due Date: December 6, 2012* |

**Turn in your solution to each problem on a separate piece of paper.** Mark the top of each sheet with the following: (1) your name, (2) the question number, (3) the names of any people you worked with on the problem. We encourage you to spend time on each problem individually before collaborating!

## Problem 1 – PCP basics

Recall that $PCP_{c,s}[r,q]_\Sigma$ is the class of all decision problems $L$ that have a probabilistically checkable proof that can be verified by a polynomial time Turing Machine that uses $r$ random bits to make $q$ queries to the given proof (over the alphabet $\Sigma$), and has completeness $c$ and soundness $s$ (i.e. if $x \in L$, then the verifier accepts with probability $c$, and if $x \notin L$, then the verifier accepts with probability $s$).

Show the following:

**(a)** $PCP_{c,s}[r,q]_\Sigma \subseteq PCP_{c,s}[r, q \cdot \log|\Sigma|]_{\{0,1\}}$.

**(b)** $PCP_{1,s}[r,q]_{\{0,1\}} \subseteq PCP_{1,1-\frac{1}{q}+\frac{s}{q}}[r+\log q, 2]_{\{0,1\}^q}$.

**(c)** If $|\Sigma| \leq \text{poly}(n)$, then $PCP_{c,s}[O(\log n), 1]_\Sigma \subseteq \mathsf{P}$.

**(d)** If $|\Sigma|^q \leq \text{poly}(n)$, then $PCP_{1,(\frac{1}{|\Sigma|})^q}[O(\log n), q]_\Sigma \subseteq \mathsf{P}$.

**(e)** Recall from class that a *Projection Game* can be described as a bipartite graph $G = (A, B, E)$ with a constraint on each edge; furthermore, the constraints are *projections*, meaning that for each edge $e = (a, b)$, there is a projection $\pi_e : \Sigma_A \to \Sigma_B$ such that if $a$ is assigned some value $\alpha \in \Sigma_A$, then $b$ must be assigned value $\pi_e(\alpha) \in \Sigma_B$ to satisfy the constraint on $e$. Projection Games where all the projection functions $\pi_e$ are bijections are called *Unique Games*.

Show that the following computational problem is efficiently solvable: given an instance of Unique Games (i.e. a bipartite graph $G$ along with the constraints and projections), decide whether there exists an assignment to the vertices such that all the constraints are satisfied.

[The famed *Unique Games Conjecture* says that, however, the problem of distinguishing between the case that $1-\delta$ fraction of the constraints are satisfiable and the case that at most $\delta$ fraction of the constraints are satisfiable is $\mathsf{NP}$-hard, for all $\delta > 0$.]

## Problem 2 – Dictator Testing with the Not-All-Equals Predicate

In this problem you will analyze a dictator test that's based on the *Not-all-equals* (NAE) predicate: $\text{NAE}(a,b,c) = 1$ iff $\neg(a = b = c)$. Recall from class that a function $f : \{\pm 1\}^n \to \{\pm 1\}$ is a *dictator* if there exists an index $i$ such that $f(x) = x_i$. Consider the following test for $f$:

1. Choose $x, y, z \in \{\pm 1\}^n$ by choosing the triples $(x_i, y_i, z_i)$ uniformly and independently from the set of 6 assignments such that $\text{NAE}(x_i, y_i, z_i) = 1$.
2. Accept if $\text{NAE}(f(x), f(y), f(z)) = 1$, otherwise reject.

**(a)**. Show that this test has perfect completeness, i.e., all dictator functions $f$ pass with probability 1.

**(b).** Show that

$$\Pr[\text{NAE}(f) \text{ passes}] = \frac{3}{4} - \frac{3}{4}\sum_{S\subseteq[n]} \left(-\frac{1}{3}\right)^{|S|} \widehat{f}(S)^2.$$

Hint: Use the Fourier expansion of the NAE predicate.

**(c).** Prove that if $f$ passes the test with probability at least $1-\epsilon$, then $\sum_{|S|=1} \widehat{f}(S)^2 \geq 1 - O(\epsilon)$.

**(d)** Recall from class that in Hastad's test, we can assume that $f$ is given to us in a *folded* manner, meaning $f(-x) = -f(x)$. Using the theorem by Friedgut, Kalai, and Naor that was mentioned in class, show that if a folded $f$ passes the test with probability $1-\epsilon$, there exists a dictator function or anti-dictator function $d$ such that $\Pr_x[f(x) = d(x)] \geq 1 - O(\epsilon)$ (an anti-dictator function is such that $d(x) = -x_i$ for some $i$).

## Problem 3 – Hardness of approximate counting

Fix some small $\epsilon < 1$. Consider the following computational problem: given a polynomial time computable function $f : \{0,1\}^n \to \mathbb{Z}$, estimate $\sum_x f(x)$ to within a factor $(1 \pm \epsilon)$. Show that given an oracle for this problem, one can solve any $\#\mathsf{P}$ problem in polynomial time.