Inference and Representation, Fall 2014

Problem Set 1: Bayesian networks

Due: Monday, September 15, 2014 at 10pm (as a PDF document sent to pg1338@nyu.edu)

Important: *See problem set policy on the course web site.*

---

1. Show that the statement
$$p(A, B|C) = p(A|C)p(B|C)$$
is equivalent to the statement
$$p(A|B, C) = p(A|C)$$
and also to
$$p(B|A, C) = p(B|C)$$
(you need to show both directions, i.e. that each statement implies the other).

2. This problem investigates the way in which conditional independence relationships affect the amount of information needed for probabilistic calculations. H, E_1, and E_2 are random variables and the notation $p(H)$ refers to the probability distribution for H, i.e. one number for every $h \in \text{Val}(H)$.

   (a) Suppose we wish to calculate $p(H|E_1, E_2)$, and we have no conditional independence information. Which of the following sets of numbers are sufficient for the calculation?

      i. $p(E_1, E_2), p(H), p(E_1|H), p(E_2|H)$.

      ii. $p(E_1, E_2), p(H), p(E_1, E_2|H)$.

      iii. $p(E_1|H), p(E_2|H), p(H)$.

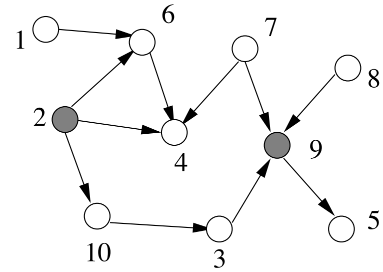
      Provide justification for your answer.

   (b) Suppose we know that E_1 and E_2 are conditionally independent given H. Now which of the above three sets are sufficient? Explain why.

3. **Bayesian networks must be acyclic.** Suppose we have a graph $\mathcal{G} = (V, E)$ and discrete random variables $X_1, \ldots, X_n$, and define
$$f(x_1, \ldots, x_n) = \prod_{v \in V} f_v(x_v|x_{pa(v)}),$$
where $pa(v)$ refers to the parents of variable X_v in $\mathcal{G}$ and $f_v(x_v \mid x_{pa(v)})$ specifies a distribution over X_v for every assignment to X_v's parents, i.e. $0 \leq f_v(x_v \mid x_{pa(v)}) \leq 1$ for all $x_v \in \text{Vals}(X_v)$ and $\sum_{x_v \in \text{Vals}(X_v)} f_v(x_v \mid x_{pa(v)}) = 1$. Recall that this is precisely the definition of the joint probability distribution associated with the Bayesian network $\mathcal{G}$, where the f_v are the conditional probability distributions.

   Show that if $\mathcal{G}$ has a directed cycle, f may no longer define a valid probability distribution. In particular, give an example of a cyclic graph $\mathcal{G}$ and distributions f_v such that $\sum_{x_1,\ldots,x_n} f(x_1, \ldots, x_n) \neq 1$. (A valid probability distribution must be non-negative and sum to one.) This is why Bayesian networks must be defined on *acyclic* graphs.

4. **D-separation.** Consider the Bayesian network shown in the below figure:

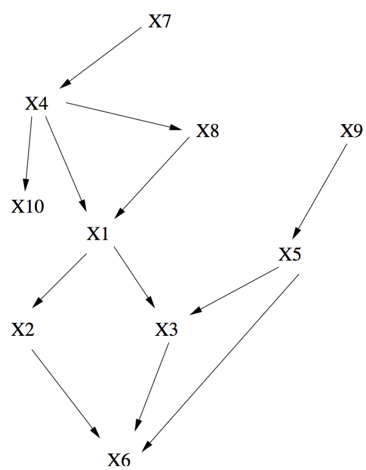


   (a) For what pairs (i, j) does the statement $X_i \perp X_j$ hold? (Do not assume any conditioning in this part.)

   (b) Suppose that we condition on $\{X_2, X_9\}$, shown shaded in the graph. What is the largest set A for which the statement $X_1 \perp X_A \mid \{X_2, X_9\}$ holds? The Bayes ball algorithm for d-separation given in Section 10.5.1 of Murphy's book may be helpful.

   (c) What is the largest set B for which $X_8 \perp X_B \mid \{X_2, X_9\}$ holds?

5. **Markov blanket.** Let $\mathcal{X} = \{X_1, ..., X_n\}$ be a set of random variables with distribution p given by the following graph.



   (a) Consider the variable X_1. What is the minimal subset of the variables, $A \subseteq \mathcal{X} - \{X_1\}$, such that $(X_1 \perp \mathcal{X} - A - \{X_1\} | A)$? Justify your answer.

   (b) Now, generalize this to any BN defined by (G, p). Specifically, consider variable X_i. What is the *Markov blanket* of X_i? Namely, the minimal subset of variables $A \subseteq \mathcal{X} - \{X_i\}$ such that $(X_i \perp \mathcal{X} - A - \{X_i\} \mid A)$? Prove that this subset is necessary and sufficient.

   (Hint: Think about the variables that X_i cannot possibly be conditionally independent of, and then think some more).

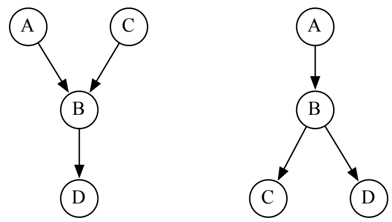
6. Consider the following distribution over 3 binary variables X, Y, Z:

$$p(x, y, z) \quad = \quad \begin{cases} 1/12 & x \oplus y \oplus z = 0 \\ 1/6 & x \oplus y \oplus z = 1 \end{cases}$$

where $\oplus$ denotes the XOR function.

Show that there is no directed acyclic graph G such that $I_{d-sep}(G) = I(p)$.

7. Consider the following two networks:



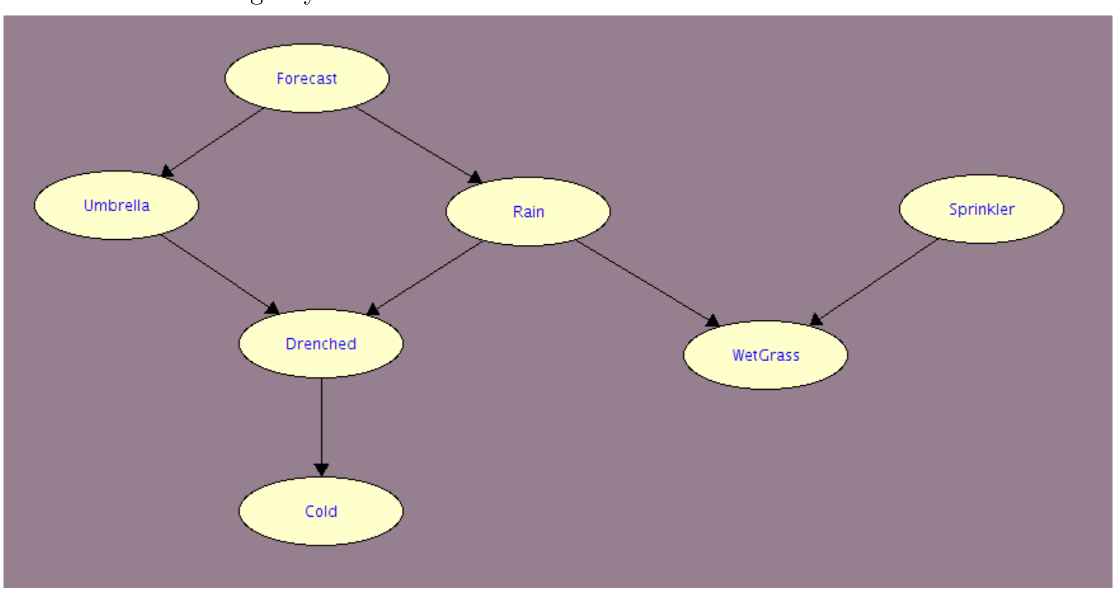
(a) (b)

Two networks G, G' are *I-equivalent* if their structures encode exactly the same independence statements, i.e. $I(G) = I(G')$. For each of the above two networks, determine whether there can be any other Bayesian network that is I-equivalent to it. Justify your answer, listing all of the I-equivalent Bayesian networks (if any).

8. For this question, you will use **SamIam: Sensitivity Analysis, Modeling, Inference and More**, which is a free tool for inference using graphical models. The installation files, installation instructions and documentation are available on `http://reasoning.cs.ucla.edu/samiam/`.

Note: On some systems, when you first launch SamIam, you may not be able to create new files or load existing network files. Follow these steps to resolve the issue:
- Under the Preferences menu, go to Preferences.
- Go to Global tab.
- Change 'User interface look and feel' to 'Nimbus'.

Consider the following Bayesian network structure:



The description of the nodes is given in the following table:

Node	Values	Description
Forecast	{WillRain, WillNotRain}	Prediction of whether it will rain or not
Umbrella	{Yes, No}	Whether you are carrying an umbrella along or not
Rain	{Yes, No}	Whether it actually rains or not
Sprinkler	{On, Off}	Whether the sprinkler was On or Off last night
Drenched	{Yes, No}	Whether you get drenched in the rain
WetGrass	{Yes, No}	Whether the grass is wet or not
Cold	{Yes, No}	Whether you catch cold or not

For the above network, answer the following questions using SamIam. You should use the "Shenoy-Shafer" inference algorithm, which performs exact inference. It's a good idea to set monitors on all of the variables (these show you the nodes' marginal probabilities), so that you can watch the effect on all of the variables of any changes that you make.

(a) After inputing the above network into SamIam, construct conditional probability distribution (CPD) tables for each of the nodes, assigning probabilities to events that agree with intuition. Include a print out of all of the CPD tables in your solutions, and use these CPD tables to answer the below questions.

(b) Perturb the CPD of `Forecast` and observe how the marginal probability of `Cold` changes. Give the initial and final CPDs of `Forecast` and the corresponding marginal probabilities for `Cold`. Explain your observations intuitively.

(c) Given that you observe `Cold`=Yes, how does the probability of `Rain` change (that is, compare P(Rain|Cold=Yes) with P(Rain))? Give the initial and final probability distributions of rain (P(Rain) and P(Rain|Cold=Yes)). Explain intuitively.

(d) Given that you observe that the grass is wet, how does the probability of cold change? What about if the grass is observed *not* to be wet? Give the initial and final marginal probabilities for `Cold` in both cases. Explain the change intuitively. Mention the active trail(s) which allow flow of inference in this scenario, i.e. the path(s) that the ball travels to get from `Cold` to `WetGrass` using the Bayes Ball algorithm.

(e) Given that we observe evidence that the sprinkler was off, how does the probability of cold change? What if, in addition, we observe that the grass is wet? Give the initial and final marginal distributions for `Cold` in each case. Explain your observations using terminology of Bayesian networks. Also give an intuitive explanation.

(f) Modify your CPTs to represent an almost perfect forecast (that is, P(Rain=Yes |Forecast=WillRain) $\approx$ 1.0, P(Rain=No | Forecast = WillNotRain) $\approx$ 1.0, P(Umbrella=Yes | Forecast=WillRain) $\approx$ 1.0). How does this modification change the relation between `Rain` and `Cold`? Explain intuitively. Make sure you test your theory by setting evidence on `Rain` and seeing its influence on `Cold`.