Orthogonal tensor decomposition

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Largely based on 2012 arXiv report “Tensor decompositions for learning latent variable models”, with Anandkumar, Ge, Kakade, and Telgarsky.

The basic decomposition problem

Notation: For a vector $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$,

$$\vec{x} \otimes \vec{x} \otimes \vec{x}$$

denotes the 3-way array (call it a "tensor") in $\mathbb{R}^{n \times n \times n}$ whose $(i, j, k)^{\text{th}}$ entry is $x_i x_j x_k$.

Problem: Given $T \in \mathbb{R}^{n \times n \times n}$ with the promise that

$$T = \sum_{t=1}^{n} \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$$

for some orthonormal basis $\{\vec{v}_t\}$ of $\mathbb{R}^n$ (w.r.t. standard inner product) and positive scalars $\{\lambda_t > 0\}$, approximately find $\{(\vec{v}_t, \lambda_t)\}$ (up to some desired precision).

Basic questions

1. Is $\{(\vec{v}_t, \lambda_t)\}$ uniquely determined?
2. If so, is there an efficient algorithm for finding the decomposition?
3. What if T is perturbed by some small amount?

   Perturbed problem: Same as the original problem, except instead of T, we are given $T + E$ for some "error tensor" E.

   How "large" can E be if we want ε precision?

Analogous matrix problem

Matrix problem: Given $M \in \mathbb{R}^{n \times n}$ with the promise that

$$M = \sum_{t=1}^{n} \lambda_t \; \vec{v}_t \; \vec{v}_t^{\top}$$

for some orthonormal basis $\{\vec{v}_t\}$ of $\mathbb{R}^n$ (w.r.t. standard inner product) and positive scalars $\{\lambda_t > 0\}$, approximately find $\{(\vec{v}_t, \lambda_t)\}$ (up to some desired precision).

Analogous matrix problem

- We're promised that M is symmetric and positive definite, so requested decomposition is an **eigendecomposition.**

  In this case, an eigendecomposition **always exists**, and **can be found efficiently.**

  It is **unique** if and only if the $\{\lambda_i\}$ are distinct.

- What if M is perturbed by some small amount?

Perturbed matrix problem: Same as the original problem, except instead of M, we are given $M + E$ for some "error matrix" E (assume to be symmetric).

  Answer provided by **matrix perturbation theory** (*e.g.*, Davis-Kahan), which requires $\|E\|_2 < \min_{i \neq j} |\lambda_i - \lambda_j|$.

Back to the original problem

Problem: Given $T \in \mathbb{R}^{n \times n \times n}$ with the promise that

$$T = \sum_{t=1}^{n} \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$$

for some orthonormal basis $\{\vec{v}_t\}$ of $\mathbb{R}^n$ (w.r.t. standard inner product) and positive scalars $\{\lambda_t > 0\}$, approximately find $\{(\vec{v}_t, \lambda_t)\}$ (up to some desired precision).

Such decompositions **do not necessarily exist**, even for symmetric tensors.

Where the decompositions do exist, the Perturbed problem asks if they are "robust".

Main ideas

Easy claim: Repeated application of a certain quadratic operator based on T (a "power iteration") recovers a single $(\vec{v}_t, \lambda_t)$ up to any desired precision.

Self-reduction: Replace T with $T - \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$.

- Why?: $T - \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t \; = \; \sum_{\tau \neq t} \lambda_\tau \, \vec{v}_\tau \otimes \vec{v}_\tau \otimes \vec{v}_\tau$.
- Catch: We don't recover $(\vec{v}_t, \lambda_t)$ exactly, so we actually can only replace T with

$$T - \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t + E_t$$

  for some "error tensor" E_t.
- Therefore, must anyway deal with perturbations.

Rest of this talk

1. Identifiability of decomposition $\{(\vec{v}_t, \lambda_t)\}$ from T.
2. A decomposition algorithm based on tensor power iteration.
3. Error analysis of decomposition algorithm.

Identifiability of the decomposition

Orthonormal basis $\{\vec{v}_t\}$ of $\mathbb{R}^n$, positive scalars $\{\lambda_t > 0\}$:

$$T = \sum_{t=1}^{n} \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$$

In what sense is $\{(\vec{v}_t, \lambda_t)\}$ uniquely determined?

Claim: $\{\vec{v}_t\}$ are the n isolated local maximizers of certain cubic form $f_T : \mathbb{B}^n \to \mathbb{R}$, and $f_T(\vec{v}_t) = \lambda_t$.

Aside: multilinear form

There is a natural trilinear form associated with T:

$$(\vec{x}, \vec{y}, \vec{z}) \mapsto \sum_{i,j,k} T_{i,j,k}\, x_i y_j z_k.$$

For matrices M, it looks like

$$(\vec{x}, \vec{y}) \mapsto \sum_{i,j} M_{i,j}\, x_i y_j = \vec{x}^\top M \vec{y}.$$

Review: Rayleigh quotient

Recall Rayleigh quotient for matrix $M := \sum_{t=1}^{n} \lambda_t \ \vec{v}_t \vec{v}_t^{\top}$ (assuming $\vec{x} \in \mathbb{S}^{n-1}$):

$$R_M(\vec{x}) \ := \ \vec{x}^{\top} M \vec{x} \ = \ \sum_{t=1}^{n} \lambda_t \ (\vec{v}_t^{\top} \vec{x})^2 .$$

Every $\vec{v}_t$ such that $|\lambda_t| = \max!$ is a maximizer of R_M.

(These are also the only local maximizers.)

The natural cubic form

Consider the function $f_T : \mathbb{B}^n \to \mathbb{R}$ given by

$$\vec{x} \;\mapsto\; f_T(\vec{x}) = \sum_{i,j,k} T_{i,j,k}\, x_i x_j x_k .$$

For our promised $T = \sum_{t=1}^{n} \lambda_t\; \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$, f_T becomes

$$\begin{aligned} f_T(\vec{x}) &= \sum_{t=1}^{n} \lambda_t \sum_{i,j,k} \left(\vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t\right)_{i,j,k} x_i x_j x_k \\ &= \sum_{t=1}^{n} \lambda_t \sum_{i,j,k} (\vec{v}_t)_i (\vec{v}_t)_j (\vec{v}_t)_k x_i x_j x_k \\ &= \sum_{t=1}^{n} \lambda_t\, (\vec{v}_t^{\top} \vec{x})^3 . \end{aligned}$$

Observation: $f_T(\vec{v}_t) = \lambda_t$.

Variational characterization

Claim: Isolated local maximizers of f_T on $\mathbb{B}^n$ are $\{\vec{v}_t\}$.

Objective function (with constraint):

$$\vec{x} \mapsto \inf_{\lambda \geq 0} \sum_{t=1}^{n} \lambda_t \, (\vec{v}_t^{\top} \vec{x})^3 - 1.5\lambda(\|\vec{x}\|_2^2 - 1).$$

First-order condition for local maxima:

$$\sum_{t=1}^{n} \lambda_t \, (\vec{v}_t^{\top} \vec{x})^2 \; \vec{v}_t = \lambda \vec{x}.$$

Second-order condition for isolated local maxima:

$$\vec{w}^{\top} \left(2 \sum_{t=1}^{n} \lambda_t \, (\vec{v}_t^{\top} \vec{x}) \vec{v}_t \vec{v}_t^{\top} - \lambda I \right) \vec{w} < 0, \qquad \vec{w} \perp \vec{x}.$$

Intuition behind variational characterization

May as well assume $\vec{v}_t$ is t^{th} coordinate basis vector, so

$$\max_{\vec{x}\in\mathbb{R}^n} f_T(\vec{x}) = \sum_{t=1}^{n} \lambda_t \, x_t^3 \quad \text{s.t.} \quad \sum_{t=1}^{n} x_t^2 \leq 1.$$

Intuition: Suppose $\mathrm{supp}(\vec{x}) = \{1, 2\}$, and $x_1, x_2 > 0$.

$$f_T(\vec{x}) = \lambda_1 x_1^3 + \lambda_2 x_2^3 < \lambda_1 x_1^2 + \lambda_2 x_2^2 \leq \max\{\lambda_1, \ \lambda_2\}.$$

Better to have $|\mathrm{supp}(\vec{x})| = 1$, *i.e.*, picking $\vec{x}$ to be a coordinate basis vector. ∎

Aside: canonical polyadic decomposition

Rank-K **canonical polyadic decomposition** (CPD) of T
(also called PARAFAC, CANDECOMP, or CP):

$$T = \sum_{i=1}^{K} \sigma_i \, \vec{u}_i \otimes \vec{v}_i \otimes \vec{w}_i .$$

[Harshman/Jennrich, 1970; Kruskal, 1977; Leurgans et al., 1993].

Number of parameters: $K \cdot (3n + 1)$ (compared to n^3 in general).

Fact: Our promised T has a rank-n CPD.

N.B.: *Overcomplete* ($K > n$) CPD is also interesting *and a possibility* as long as $K(3n + 1) \ll n^3$.

The quadratic operator

Easy claim: Repeated application of a certain quadratic operator (based on T) recovers a single $(\lambda_t, \vec{v}_t)$ up to any desired precision.

For any $T \in \mathbb{R}^{n \times n \times n}$ and $\vec{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, define the quadratic operator

$$\phi_T(\vec{x}) := \sum_{i,j,k} T_{i,j,k}\, x_j x_k\, \vec{e}_i \ \in \mathbb{R}^n$$

where $\vec{e}_i \in \mathbb{R}^n$ is the i^{th} coordinate basis vector.

$$\text{If } T = \sum_{t=1}^{n} \lambda_t\, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t\,, \ \text{ then } \phi_T(\vec{x}) = \sum_{t=1}^{n} \lambda_t\, \left(\vec{v}_t^{\top} \vec{x}\right)^2 \vec{v}_t.$$

An algorithm?

Recall: First-order condition for local maxima of $f_T(\vec{x}) = \sum_{t=1}^{n} \lambda_t \, (\vec{v}_t{}^\top \vec{x})^3$ for $\vec{x} \in \mathbb{B}^n$:

$$\phi_T(\vec{x}) = \sum_{t=1}^{n} \lambda_t \, (\vec{v}_t{}^\top \vec{x})^2 \; \vec{v}_t = \lambda \vec{x}$$

i.e., "eigenvector"-like condition.

Algorithm: Find $\vec{x} \in \mathbb{B}^n$ fixed under $\vec{x} \mapsto \phi_T(\vec{x}) / \|\phi_T(\vec{x})\|$.

(Ignoring numerical issues, can just repeatedly apply ϕ_T and defer normalization until later.)

N.B.: Gradient ascent also works [Kolda & Mayo, '11].

Tensor power iteration

[De Lathauwer *et al*, 2000]
Start with some $\vec{x}^{(0)}$, and for $j = 1, 2, \ldots$:

$$\vec{x}^{(j)} \; := \; \phi_T\left(\vec{x}^{(j-1)}\right) \; = \; \sum_{t=1}^{n} \lambda_t \left(\vec{v}_t^{\top} \vec{x}^{(j-1)}\right)^2 \vec{v}_t .$$

Claim: For almost all initial $\vec{x}^{(0)}$, the sequence $\left(\vec{x}^{(j)} / \|\vec{x}^{(j)}\|\right)_{j=1}^{\infty}$ converges *quadratically fast* to some $\vec{v}_t$.

Review: matrix power iteration

Recall matrix power iteration for matrix $M := \sum_{t=1}^{n} \lambda_t \; \vec{v}_t \vec{v}_t^{\top}$:

Start with some $\vec{x}^{(0)}$, and for $j = 1, 2, \ldots$:

$$\vec{x}^{(i)} \;:=\; M\vec{x}^{(j-1)} \;=\; \sum_{t=1}^{n} \lambda_t \left(\vec{v}_t^{\top} \vec{x}^{(j-1)}\right) \vec{v}_t .$$

i.e., component in $\vec{v}_t$ direction is scaled by λ_t.

If $\lambda_1 > \lambda_2 \geq \cdots$, then

$$\frac{\left(\vec{v}_1^{\top} \vec{x}^{(j)}\right)^2}{\sum_{t=1}^{n} \left(\vec{v}_t^{\top} \vec{x}^{(j)}\right)^2} \geq 1 - k\left(\frac{\lambda_2}{\lambda_1}\right)^{2j} .$$

i.e., converges *linearly* to $\vec{v}_1$ (assuming gap $\lambda_2/\lambda_1 < 1$).

Tensor power iteration convergence analysis

Let $c_t := \vec{v}_t^{\top} \vec{x}^{(0)}$ (initial component in $\vec{v}_t$ direction); assume WLOG

$$\lambda_1 |c_1| > \lambda_2 |c_2| \geq \lambda_3 |c_3| \geq \cdots .$$

Then

$$\vec{x}^{(1)} = \sum_{t=1}^{n} \lambda_t \left(\vec{v}_t^{\top} \vec{x}^{(0)} \right)^2 \vec{v}_t = \sum_{t=1}^{n} \lambda_t c_t^2 \ \vec{v}_t$$

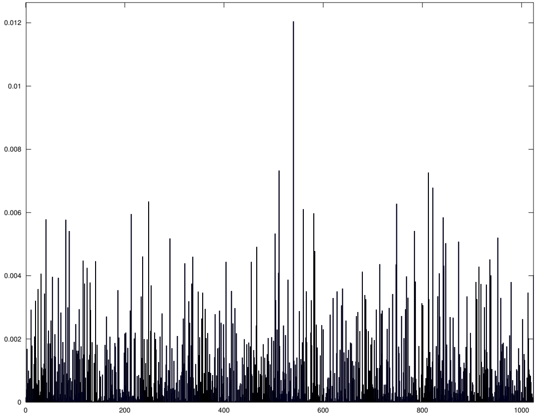
i.e., component in $\vec{v}_t$ direction is squared then scaled by λ_t.

Easy to show

$$\frac{\left(\vec{v}_1^{\top} \vec{x}^{(j)} \right)^2}{\sum_{t=1}^{n} \left(\vec{v}_t^{\top} \vec{x}^{(j)} \right)^2} \geq 1 - k \left(\frac{\lambda_1}{\max_{t \neq 1} \lambda_t} \right)^2 \left| \frac{\lambda_2 c_2}{\lambda_1 c_1} \right|^{2^{j+1}} .$$

Example

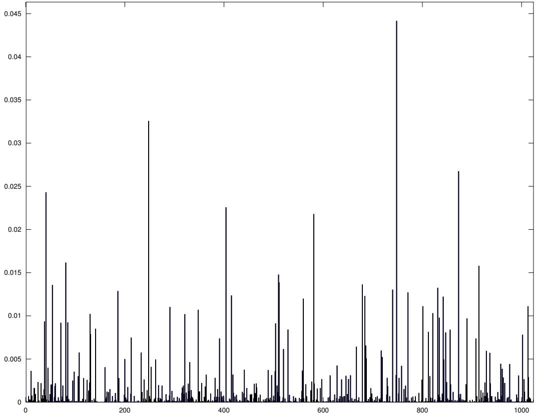
$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t^{\top} \vec{x}^{(0)})^2$ for $t = 1, 2, \ldots, 1024$

Example

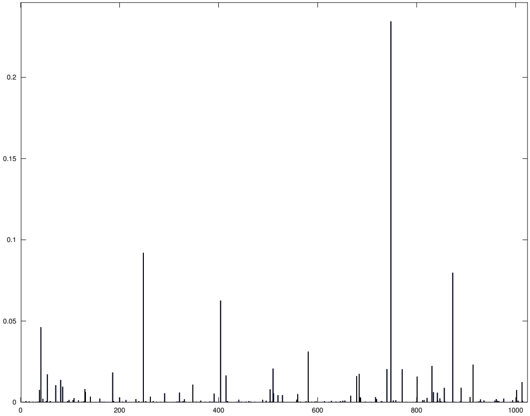
$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t^{\top}\vec{x}^{(1)})^2$ for $t = 1, 2, \ldots, 1024$

Example

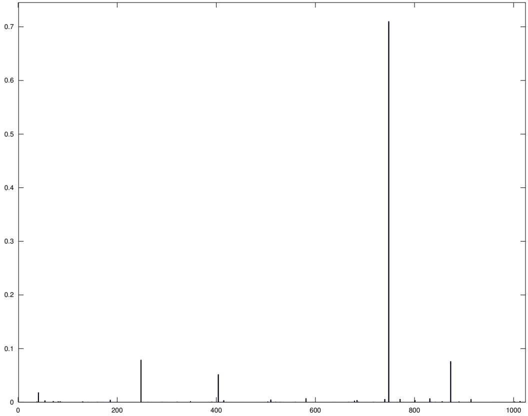
$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t^{\top} \vec{x}^{(2)})^2$ for $t = 1, 2, \ldots, 1024$

Example

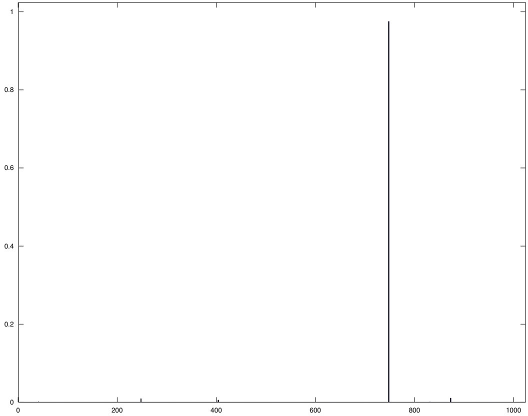
$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t^{\top}\vec{x}^{(3)})^2$ for $t = 1, 2, \ldots, 1024$

Example

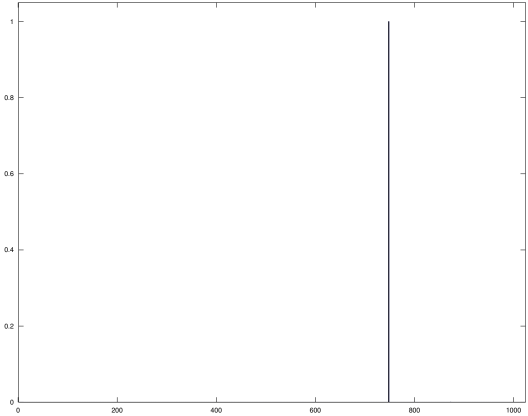
$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t{}^{\top} \vec{x}^{(4)})^2$ for $t = 1, 2, \ldots, 1024$

Example

$$n = 1024,\ \lambda_t \sim_{\text{u.a.r.}} [0, 1].$$



Value of $(\vec{v}_t^{\top} \vec{x}^{(5)})^2$ for $t = 1, 2, \ldots, 1024$

Matrix vs. tensor power iteration

Matrix power iteration:

1. Requires gap between largest and second-largest λ_t.
   (Property of the matrix only.)

2. Converges to top $\vec{v}_t$.

3. Linear convergence. (Need $O(\log(1/\epsilon))$ iterations.)

Tensor power iteration:

1. Requires gap between largest and second-largest $\lambda_t |c_t|$.
   (Property of the tensor and initialization $\vec{x}^{(0)}$.)

2. Converges to $\vec{v}_t$ for which $\lambda_t |c_t| = \max$! (could be any of them).

3. Quadratic convergence. (Need $O(\log\log(1/\epsilon))$ iterations.)

Initialization of tensor power iteration

Convergence of tensor power iteration requires **gap** between **largest** and **second-largest** $\lambda_t |\vec{v}_t^\top \vec{x}^{(0)}|$.

Example of bad initialization: Suppose $T = \sum_t \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t$, and $\vec{x}^{(0)} = \frac{1}{\sqrt{2}}(\vec{v}_1 + \vec{v}_2)$.

$$\begin{aligned} \phi_T(\vec{x}^{(0)}) &= (\vec{v}_1^\top \vec{x}^{(0)})^2 \vec{v}_1 + (\vec{v}_2^\top \vec{x}^{(0)})^2 \vec{v}_2 \\ &= \frac{1}{2}(\vec{v}_1 + \vec{v}_2) = \frac{1}{\sqrt{2}} \vec{x}^{(0)}. \end{aligned}$$

Fortunately, bad initialization points are atypical.

Full decomposition algorithm

Input: $T \in \mathbb{R}^{n \times n \times n}$.

Initialize: $\widetilde{T} := T$.

For $i = 1, 2, \ldots, n$:

1. Pick $\vec{x}^{(0)} \in \mathbb{S}^{n-1}$ unif. at random.
2. Run tensor power iteration with $\widetilde{T}$ starting from $\vec{x}^{(0)}$ for N iterations.
3. Set $\hat{v}_i := \vec{x}^{(N)} / \|\vec{x}^{(N)}\|$ and $\hat{\lambda}_i := f_{\widetilde{T}}(\hat{v}_i)$.
4. Replace $\widetilde{T} := \widetilde{T} - \hat{\lambda}_i \, \hat{v}_i \otimes \hat{v}_i \otimes \hat{v}_i$.

Output: $\{(\hat{v}_i, \hat{\lambda}_i) : i \in [n]\}$.

Actually: repeat Steps 1–3 several times, and take results of trial yielding largest $\hat{\lambda}_i$.

Aside: direct minimization

Can also consider directly minimizing

$$\left\| T - \sum_{t=1}^{n} \hat{\lambda}_t \ \hat{v}_t \otimes \hat{v}_t \otimes \hat{v}_t \right\|_F^2$$

via local optimization (*e.g.*, coord. descent, alternating least squares).

Decomposition algorithm via tensor power iteration can be viewed as **orthgonal greedy algorithm** for minimizing above objective [Zhang & Golub, '01].

Aside: implementation for bag-of-words models

Let $\vec{f}^{(i)}$ be empirical word frequency vector for document i:

$$(\vec{f}^{(i)})_j = \frac{\text{\# times word } j \text{ appears in document } i}{\text{length of document } i}$$

Matrix of word-pair frequencies (from m documents)

$$\widehat{\text{Pairs}} \approx \frac{1}{m}\sum_{i=1}^{m} \vec{f}^{(i)} \otimes \vec{f}^{(i)} \longrightarrow \sum_{t=1}^{K} \vec{\mu}_t \otimes \vec{\mu}_t.$$

Tensor of word-triple frequencies (from m documents)

$$\widehat{\text{Triples}} \approx \frac{1}{m}\sum_{i=1}^{m} \vec{f}^{(i)} \otimes \vec{f}^{(i)} \otimes \vec{f}^{(i)} \longrightarrow \sum_{t=1}^{K} \vec{\mu}_t \otimes \vec{\mu}_t \otimes \vec{\mu}_t.$$

Aside: implementation for bag-of-words models

Use inner product system given by $\langle \vec{x}, \vec{y} \rangle := \vec{x}^\top \widehat{\text{Pairs}}^\dagger \vec{y}$.

Why?: If $\widehat{\text{Pairs}} = \sum_{t=1}^{K} \vec{\mu}_t \otimes \vec{\mu}_t$, then $\langle \vec{\mu}_i, \vec{\mu}_j \rangle = \mathbb{1}_{\{i=j\}}$.
$\Rightarrow \; \{\vec{\mu}_i\}$ are orthonormal under this inner product system.

Power iteration step:

$$\phi_{\widehat{\text{Triples}}}(\vec{x}) \; := \; \frac{1}{m} \sum_{i=1}^{m} \langle \vec{x}, \vec{f}^{(i)} \rangle^2 \, \vec{f}^{(i)} \; = \; \frac{1}{m} \sum_{i=1}^{m} \left(\vec{x}^\top \widehat{\text{Pairs}}^\dagger \vec{f}^{(i)} \right)^2 \vec{f}^{(i)}.$$

1. First compute $\vec{y} := \widehat{\text{Pairs}}^\dagger \vec{x}$ (use low-rank factors of $\widehat{\text{Pairs}}$).
2. Then compute $(\vec{y}^\top \vec{f}^{(i)})^2 \, \vec{f}^{(i)}$ for all documents i, and add them up (all sparse operations).

Final running time $\propto$ # topics $\times$ (model size $+$ input size).

Effect of errors in tensor power iterations

Suppose we are given $\widehat{T} := T + E$, with

$$T = \sum_{t=1}^{n} \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t, \qquad \varepsilon := \sup_{\vec{x} \in \mathbb{S}^{n-1}} \|\phi_E(\vec{x})\|.$$

What can we say about the resulting $\hat{v}_i$ and $\hat{\lambda}_i$?

Perturbation analysis

Theorem: If $\varepsilon \leq O(\frac{\min_t \lambda_t}{n})$, then with high probability, a modified variant of the full decomposition algorithm returns $\{(\hat{v}_i, \hat{\lambda}_i) : i \in [n]\}$ with

$$\|\hat{v}_i - \vec{v}_i\| \leq O(\varepsilon/\lambda_i), \qquad |\hat{\lambda}_i - \lambda_i| \leq O(\varepsilon), \qquad i \in [n].$$

Essentially third-order analogue of Wedin's theorem for SVD of matrices, but specific to fixed-point iteration algorithm.

Similar analysis holds for variational characterization.

Effect of errors in tensor power iterations

Quadratic operator $\phi_{\widehat{T}}$ with $\widehat{T}$:

$$\phi_{\widehat{T}}(\vec{x}) \;=\; \sum_{t=1}^{n} \lambda_t \left(\vec{v}_t^{\top}\vec{x}\right)^2 \vec{v}_t \;+\; \phi_E(\vec{x}).$$

Claim: If $\varepsilon \leq O(\frac{\min_t \lambda_t}{n})$ and $N \geq \Omega(\log(n) + \log\log \frac{\max_t \lambda_t}{\varepsilon})$, then N steps of tensor power iteration on $T + E$ (with good initialization) gives

$$\|\hat{v}_i - \vec{v}_i\| \leq O(\varepsilon/\lambda_i), \qquad |\hat{\lambda}_i - \lambda_i| \leq O(\varepsilon).$$

Deflation

(For simplicity, assume $\lambda_1 = \cdots = \lambda_n = 1$.)

Using tensor power iteration on $\widehat{T} := T + E$:
Approximate (say) $\vec{v}_1$ with $\hat{v}_1$ up to error $\|\vec{v}_1 - \hat{v}_1\| \leq \varepsilon$.

Deflation danger: To find next $\vec{v}_t$, use

$$\widehat{T} - \hat{v}_1 \otimes \hat{v}_1 \otimes \hat{v}_1 = \sum_{t=2}^{n} \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t + E + \left(\vec{v}_1 \otimes \vec{v}_1 \otimes \vec{v}_1 - \hat{v}_1 \otimes \hat{v}_1 \otimes \hat{v}_1\right).$$

Now error seems to be of size $2\varepsilon \ldots$ exponential explosion?

How do the errors look?

$$E_1 := \vec{v}_1 \otimes \vec{v}_1 \otimes \vec{v}_1 - \hat{v}_1 \otimes \hat{v}_1 \otimes \hat{v}_1$$

- Take any direction $\vec{x}$ orthogonal to $\vec{v}_1$:

$$\begin{aligned} \|\phi_{E_1}(\vec{x})\| &= \|(\vec{v}_1{}^\top \vec{x})^2 \vec{v}_1 - (\hat{v}_1{}^\top \vec{x})^2 \hat{v}_1\| \\ &= \|(\hat{v}_1{}^\top \vec{x})^2 \hat{v}_1\| \\ &= ((\hat{v}_1 - \vec{v}_1)^\top \vec{x})^2 \\ &\le \|\hat{v}_1 - \vec{v}_1\|^2 \le \varepsilon^2. \end{aligned}$$

- Effect of $E + E_1$ in directions orthogonal to $\vec{v}_1$ is just $(1 + o(1))\varepsilon$.

Deflation analysis

Upshot: all errors due to "deflation" have only lower-order effects on ability to find subsequent $\vec{v}_t$.

Analogous statement for matrix power iteration is **not true.**

Recap and remarks

- Orthogonally diagonalizable tensors have very nice *identifiability*, *computational*, and *robustness* properties.
  - Many analogues to matrix SVD, but also many important differences arising from non-linearity.
  - Greedy algorithm for finding the decomposition can be rigorously analyzed and shown to be effective and efficient.
- Many other approaches to moment-based estimation (*e.g.*, subspace ID / OOMs, local optimization).

Other stuff I didn't talk about

1. Overcomplete tensor decomposition: $K > n$ components in $\mathbb{R}^n$.

$$T = \sum_{t=1}^{K} \lambda_t \, \vec{v}_t \otimes \vec{v}_t \otimes \vec{v}_t.$$

    - ICA/blind source separation [Cardoso, 1991; Goyal *et al*, 2014]
    - Mixture models [Bhaskara *et al*, 2014; Anderson *et al*, 2014]
    - Dictionary learning [Barak *et al*, 2014]
    - …

2. General Tucker decompositions (CPD is a special case).
    - Exploit other structure (*e.g.*, sparsity)

Questions?

Tensor product of vectors spaces

What is the tensor product $V \bigotimes W$ of vector spaces V and W?

- Define objects $E_{\vec{v},\vec{w}}$ for $\vec{v} \in V$ and $\vec{w} \in W$.
- Declare equivalences
  - $E_{\vec{v}_1+\vec{v}_2,\vec{w}} \sim E_{\vec{v}_1,\vec{w}} + E_{\vec{v}_2,\vec{w}}$
  - $E_{\vec{v},\vec{w}_1+\vec{w}_2} \sim E_{\vec{v},\vec{w}_1} + E_{\vec{v},\vec{w}_2}$
  - $c\, E_{\vec{v},\vec{w}} \sim E_{c\vec{v},\vec{w}} \sim E_{\vec{v},c\vec{w}}$ for $c \in \mathbb{R}$.
- Pick any bases B_V for V, and B_W for W.

  $V \bigotimes W :=$ span of $\{E_{\vec{v},\vec{w}} : \vec{v} \in B_V, \vec{w} \in B_W\}$, modulo equivalences (eliminating dependence on choice of bases).
- Can check that $V \bigotimes W$ is a vector space.
- $\vec{v} \otimes \vec{w}$ (tensor product of $\vec{v} \in V$ and $\vec{w} \in W$) is the equivalence class of $E_{\vec{v},\vec{w}}$ in $V \bigotimes W$.

Tensor algebra perspective

From tensor algebra: Since $\{\vec{v}_t \;:\; t \in [n]\}$ is a basis for $\mathbb{R}^n$,
$\left\{\vec{v}_i \otimes \vec{v}_j \otimes \vec{v}_k \;:\; i, j, k \in [n]\right\}$ is a *basis* for $\mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n$
("$\bigotimes$" denotes the tensor product of vector spaces)

Every tensor $T \in \mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n$ has a unique representation in this basis:

$$T = \sum_{i,j,k} c_{i,j,k} \, \vec{v}_i \otimes \vec{v}_j \otimes \vec{v}_k$$

N.B.: $\dim(\mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n) = n^3$.

Aside: general bases for $\mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n$

Pick any bases $\left(\{\vec{\alpha}_i\}, \ \{\vec{\beta}_i\}, \ \{\vec{\gamma}_i\}\right)$ for $\mathbb{R}^n$
(not necessary orthonormal). $\Rightarrow$ Basis for $\mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n$:

$$\left\{\vec{\alpha}_i \otimes \vec{\beta}_j \otimes \vec{\gamma}_k \ : \ 1 \leq i, j, k \leq n\right\}.$$

Every tensor $T \in \mathbb{R}^n \bigotimes \mathbb{R}^n \bigotimes \mathbb{R}^n$ has a unique representation in this basis:

$$T = \sum_{i,j,k} c_{i,j,k} \ \vec{\alpha}_i \otimes \vec{\beta}_j \otimes \vec{\gamma}_k.$$

A tensor T such that $c_{i,j,k} \neq 0 \Rightarrow i = j = k$ is called *diagonal*:

$$T = \sum_{i=1}^{n} c_{i,i,i} \ \vec{\alpha}_i \otimes \vec{\beta}_i \otimes \vec{\gamma}_i.$$

Claim: A tensor T can be diagonal w.r.t. at most one basis.

Aside: canonical polyadic decomposition

Rank-K **canonical polyadic decomposition** (CPD) of T
(also called PARAFAC, CANDECOMP, or CP):

$$T = \sum_{i=1}^{K} \sigma_i \, \vec{u}_i \otimes \vec{v}_i \otimes \vec{w}_i .$$

Number of parameters: $K \cdot (3n + 1)$ (compared to n^3 in general).

Fact: If T is diagonal w.r.t. bases then it has a rank-K CPD with $K \leq n$.

Diagonal w.r.t. bases $\equiv$ "non-overcomplete" CPD.

N.B.: *Overcomplete* ($K > n$) CPD is also interesting *and a possibility* as long as $K(3n + 1) \ll n^3$.

Initialization of tensor power iteration

Let $t_{\max} := \arg\max_t \lambda_t$, and draw $\vec{x}^{(0)} \in \mathbb{S}^{n-1}$ unif. at random.

- Most coefficients of $\vec{x}^{(0)}$ are around $1/\sqrt{n}$; largest is around $\sqrt{\log(n)/n}$.
- Almost surely, a gap exists:

$$\max_{t \neq t_{\max}} \frac{\lambda_t |\vec{v}_t^{\top} \vec{x}^{(0)}|}{\lambda_{t_{\max}} |\vec{v}_{t_{\max}}^{\top} \vec{x}^{(0)}|} < 1.$$

- With probability $\geq 1/n^{1.2}$, the gap is non-negligible:

$$\max_{t \neq t_{\max}} \frac{\lambda_t |\vec{v}_t^{\top} \vec{x}^{(0)}|}{\lambda_{t_{\max}} |\vec{v}_{t_{\max}}^{\top} \vec{x}^{(0)}|} < 0.9.$$

Try $O(n^{1.3})$ initializers; chances are at least one is good.
(Very conservative estimate only; can be *much* better than this.)