Inference and Representation

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Approximate marginal inference

- Given the joint $p(x_1, \ldots, x_n)$ represented as a graphical model, how do we perform **marginal inference**, e.g. to compute $p(x_1 \mid e)$?
- We showed in Lecture 4 that doing this exactly is NP-hard
- Nearly all *approximate inference* algorithms are either:
  1. Monte-carlo methods (e.g., Gibbs sampling, likelihood reweighting, MCMC)
  2. **Variational algorithms (e.g., mean-field, loopy belief propagation)**

Variational methods

- **Goal**: Approximate difficult distribution $p(\mathbf{x} \mid \mathbf{e})$ with a new distribution $q(\mathbf{x})$ such that:
  1. $p(\mathbf{x} \mid \mathbf{e})$ and $q(\mathbf{x})$ are "close"
  2. Computation on $q(\mathbf{x})$ is easy
- How should we measure distance between distributions?
- The **Kullback-Leibler divergence** (KL-divergence) between two distributions p and q is defined as

$$D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$

  (measures the expected number of extra bits required to describe *samples from* $p(\mathbf{x})$ using a code based on q instead of p)
- $D(p \,\|\, q) \geq 0$ for all p, q, with equality if and only if $p = q$
- Notice that KL-divergence is **asymmetric**

KL-divergence *(see Section 2.8.2 of Murphy)*

$$D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$

- Suppose p is the true distribution we wish to do inference with
- What is the difference between the solution to

$$\arg\min_q D(p\|q)$$

(called the *M-projection* of q onto p) and

$$\arg\min_q D(q\|p)$$

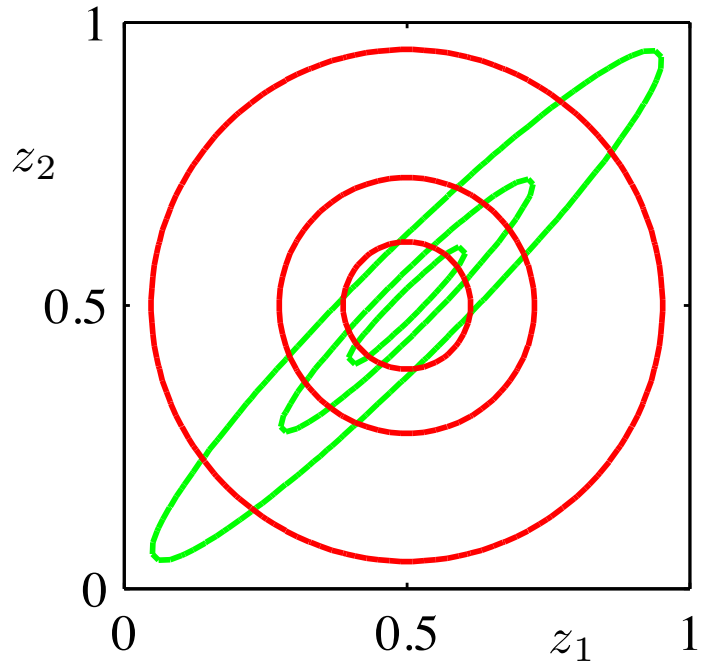
(called the *I-projection*)?

- These two will differ only when q is minimized over a restricted set of probability distributions $Q = \{q_1, \ldots\}$, and in particular when $p \notin Q$

KL-divergence – M-projection

$$q^* = \arg\min_{q \in Q} D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$

For example, suppose that $p(\mathbf{z})$ is a 2D Gaussian and Q is the set of all Gaussian distributions with diagonal covariance matrices:



p=Green, q^*=Red

KL-divergence – I-projection

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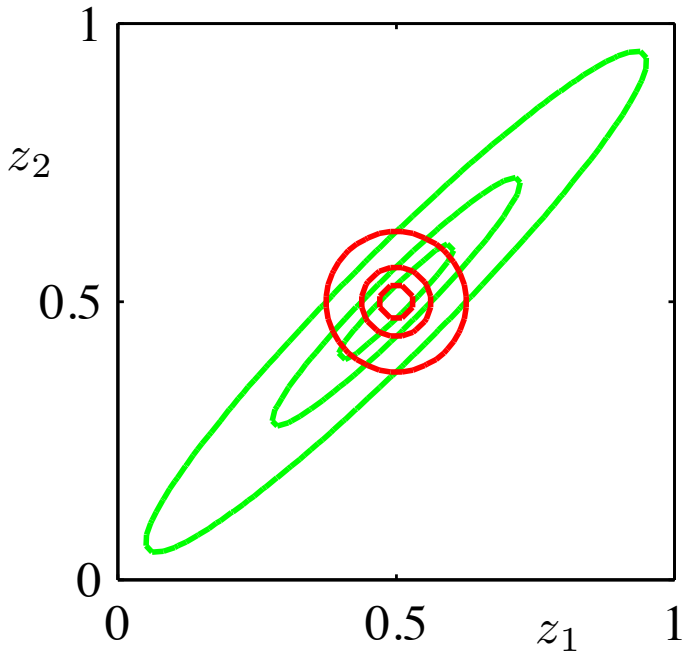
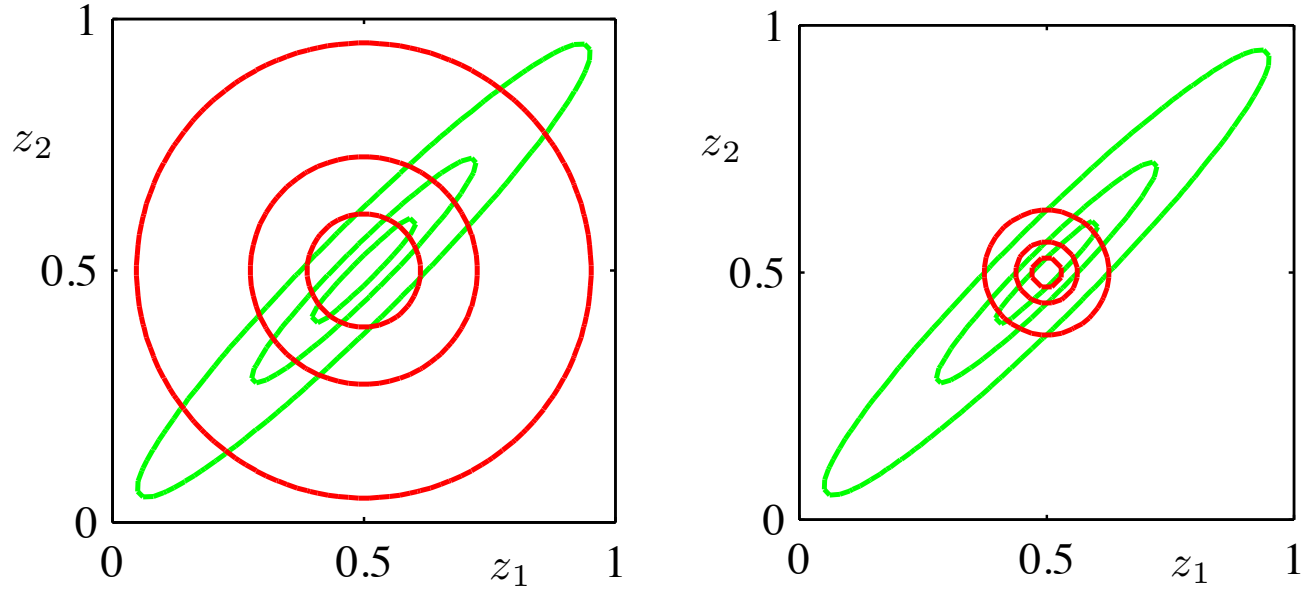


p=Green, q^*=Red

KL-divergence (single Gaussian)

In this simple example, both the M-projection and I-projection find an approximate $q(\mathbf{x})$ that has the correct mean (i.e. $E_p[\mathbf{z}] = E_q[\mathbf{z}]$):

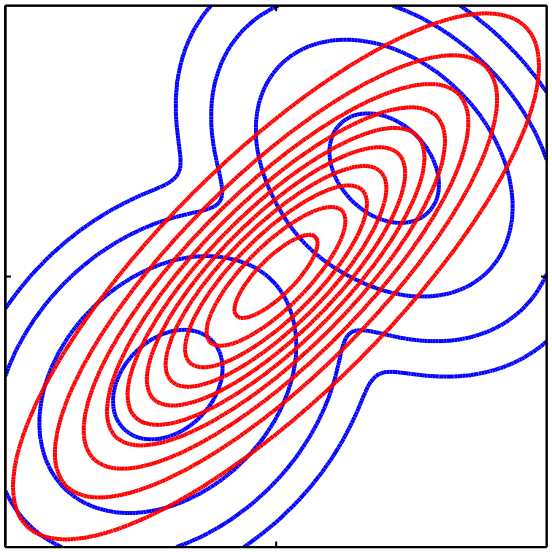


What if $p(\mathbf{x})$ is multi-modal?

KL-divergence – M-projection (mixture of Gaussians)

$$q^* = \arg\min_{q \in Q} D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$

Now suppose that $p(\mathbf{x})$ is mixture of two 2D Gaussians and Q is the set of all 2D Gaussian distributions (with arbitrary covariance matrices):

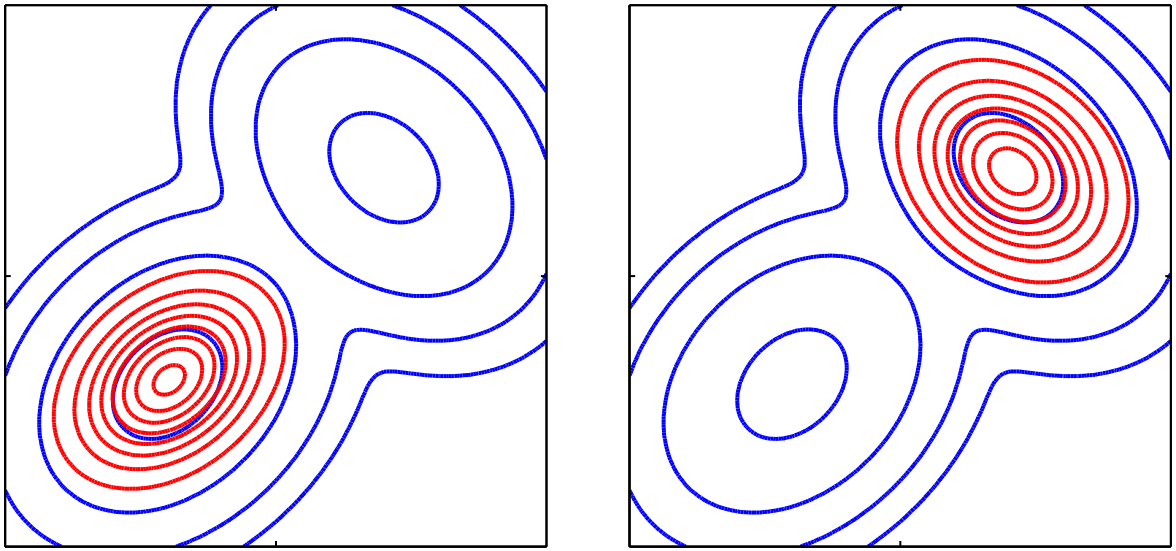


p=Blue, q^*=Red

M-projection yields distribution $q(\mathbf{x})$ with the correct mean and covariance.

KL-divergence – I-projection (mixture of Gaussians)

$$q^* = \arg\min_{q \in Q} D(q\|p) = \sum_{\mathbf{x}} q(\mathbf{x}) \log \frac{q(\mathbf{x})}{p(\mathbf{x})}.$$



p=Blue, q^*=Red (two local minima!)

Unlike M-projection, the I-projection does not always yield the correct moments.

Q: $D(p\|q)$ is convex – so why are there local minima?

A: using a *parametric* form for q (i.e., a Gaussian). Not convex in μ, Σ.

M-projection does moment matching

- Recall that the M-projection is:

$$q^* = \arg\min_{q \in Q} D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$

- Suppose that Q is an exponential family ($p(\mathbf{x})$ can be arbitrary) and that we perform the M-projection, finding q^*

- **Theorem:** The expected sufficient statistics, with respect to $q^*(\mathbf{x})$, are *exactly* the marginals of $p(\mathbf{x})$:

$$E_{q^*}[\mathbf{f}(\mathbf{x})] = E_p[\mathbf{f}(\mathbf{x})]$$

- Thus, solving for the M-projection (exactly) is just as hard as the original inference problem

M-projection does moment matching

- Recall that the M-projection is:
$$q^* = \arg \min_{q(\mathbf{x};\eta) \in Q} D(p\|q) = \sum_{\mathbf{x}} p(\mathbf{x}) \log \frac{p(\mathbf{x})}{q(\mathbf{x})}.$$
- **Theorem:** $E_{q^*}[\mathbf{f}(\mathbf{x})] = E_p[\mathbf{f}(\mathbf{x})]$.
- **Proof:** Look at the first-order optimality conditions.

$$\begin{aligned}
\partial_{\eta_i} D(p\|q) &= -\partial_{\eta_i} \sum_{\mathbf{x}} p(\mathbf{x}) \log q(\mathbf{x}) \\
&= -\partial_{\eta_i} \sum_{\mathbf{x}} p(\mathbf{x}) \log \Big\{ h(\mathbf{x}) \exp\{\eta \cdot \mathbf{f}(\mathbf{x}) - \ln Z(\eta)\} \Big\} \\
&= -\partial_{\eta_i} \sum_{\mathbf{x}} p(\mathbf{x}) \Big\{ \eta \cdot \mathbf{f}(\mathbf{x}) - \ln Z(\eta) \Big\} \\
&= -\sum_{\mathbf{x}} p(\mathbf{x}) f_i(\mathbf{x}) + E_{q(\mathbf{x};\eta)}[f_i(\mathbf{x})] \quad (\text{since } \partial_{\eta_i} \ln Z(\eta) = E_q[f_i(\mathbf{x})]) \\
&= -E_p[f_i(\mathbf{x})] + E_{q(\mathbf{x};\eta)}[f_i(\mathbf{x})] = 0.
\end{aligned}$$

- **Corollary**: Even computing the gradients is hard (can't do gradient descent)

Most variational inference algorithms make use of the I-projection

Variational methods

- Suppose that we have an arbitrary graphical model:

$$p(\mathbf{x};\theta) = \frac{1}{Z(\theta)} \prod_{\mathbf{c}\in C} \phi_c(\mathbf{x_c}) = \exp\Big(\sum_{\mathbf{c}\in C} \theta_c(\mathbf{x_c}) - \ln Z(\theta)\Big)$$

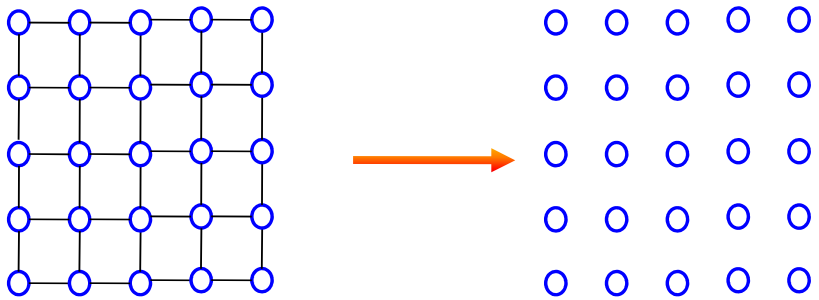
- All of the approaches begin as follows:

$$\begin{aligned}
D(q\|p) &= \sum_{\mathbf{x}} q(\mathbf{x}) \ln \frac{q(\mathbf{x})}{p(\mathbf{x})} \\
&= -\sum_{\mathbf{x}} q(\mathbf{x}) \ln p(\mathbf{x}) - \sum_{\mathbf{x}} q(\mathbf{x}) \ln \frac{1}{q(\mathbf{x})} \\
&= -\sum_{\mathbf{x}} q(\mathbf{x}) \big(\sum_{\mathbf{c}\in C} \theta_c(\mathbf{x_c}) - \ln Z(\theta)\big) - H(q(\mathbf{x})) \\
&= -\sum_{\mathbf{c}\in C} \sum_{\mathbf{x}} q(\mathbf{x}) \theta_c(\mathbf{x_c}) + \sum_{\mathbf{x}} q(\mathbf{x}) \ln Z(\theta) - H(q(\mathbf{x})) \\
&= -\sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + \ln Z(\theta) - H(q(\mathbf{x})).
\end{aligned}$$

Mean field algorithms for variational inference

$$\max_{q \in Q} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} q(\mathbf{x_c}) \theta_c(\mathbf{x_c}) + H(q(\mathbf{x})).$$

- Although this function is concave and thus in theory should be easy to optimize, we need some compact way of representing $q(\mathbf{x})$
- *Mean field* algorithms assume a factored representation of the joint distribution, e.g.



$$q(\mathbf{x}) = \prod_{i \in V} q_i(x_i) \qquad \text{(called } \textit{naive} \text{ mean field)}$$

Naive mean-field

- Suppose that Q consists of all fully factored distributions, of the form $q(\mathbf{x}) = \prod_{i \in V} q_i(x_i)$
- We can use this to simplify

$$\max_{q \in Q} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} q(\mathbf{x_c}) \theta_c(\mathbf{x_c}) + H(q)$$

- First, note that $q(\mathbf{x}_c) = \prod_{i \in c} q_i(x_i)$
- Next, notice that the joint entropy decomposes as a sum of local entropies:

$$\begin{aligned}
H(q) &= -\sum_{\mathbf{x}} q(\mathbf{x}) \ln q(\mathbf{x}) \\
&= -\sum_{\mathbf{x}} q(\mathbf{x}) \ln \prod_{i \in V} q_i(x_i) = -\sum_{\mathbf{x}} q(\mathbf{x}) \sum_{i \in V} \ln q_i(x_i) \\
&= -\sum_{i \in V} \sum_{\mathbf{x}} q(\mathbf{x}) \ln q_i(x_i) \\
&= -\sum_{i \in V} \sum_{x_i} q_i(x_i) \ln q_i(x_i) \sum_{\mathbf{x}_{V \setminus i}} q(\mathbf{x}_{V \setminus i} \mid x_i) = \sum_{i \in V} H(q_i).
\end{aligned}$$

Naive mean-field

- Suppose that Q consists of all fully factored distributions, of the form $q(\mathbf{x}) = \prod_{i \in V} q_i(x_i)$
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$$\max_{q \in Q} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} q(\mathbf{x_c}) \theta_c(\mathbf{x_c}) + H(q)$$

- First, note that $q(\mathbf{x}_c) = \prod_{i \in c} q_i(x_i)$
- Next, notice that the joint entropy decomposes as $H(q) = \sum_{i \in V} H(q_i)$.
- Putting these together, we obtain the following variational objective:

$$(*) \max_{q} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \prod_{i \in c} q_i(x_i) + \sum_{i \in V} H(q_i)$$

subject to the constraints

$$q_i(x_i) \geq 0 \quad \forall i \in V, x_i \in \mathrm{Val}(X_i)$$

$$\sum_{x_i \in \mathrm{Val}(X_i)} q_i(x_i) = 1 \quad \forall i \in V$$

Naive mean-field for pairwise MRFs

- How do we maximize the variational objective?

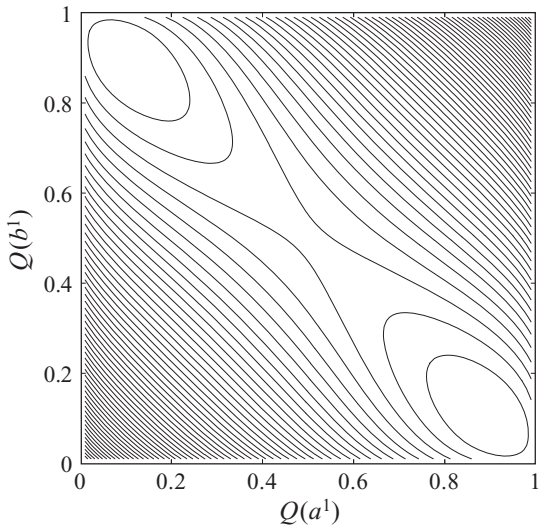
$$(*) \max_q \sum_{ij \in E} \sum_{x_i, x_j} \theta_{ij}(x_i, x_j) q_i(x_i) q_j(x_j) - \sum_{i \in V} \sum_{x_i} q_i(x_i) \ln q_i(x_i)$$

- This is a non-concave optimization problem, with many local maxima!
- Nonetheless, we can greedily maximize it using **block coordinate ascent**:
  1. Iterate over each of the variables $i \in V$. For variable i,
  2. Fully maximize (*) with respect to $\{q_i(x_i), \forall x_i \in \mathrm{Val}(X_i)\}$.
  3. Repeat until convergence.
- Constructing the Lagrangian, taking the derivative, setting to zero, and solving yields the update: (*shown on blackboard*)

$$q_i(x_i) \leftarrow \frac{1}{Z_i} \exp\left\{\theta_i(x_i) + \sum_{j \in N(i)} \sum_{x_j} q_j(x_j) \theta_{ij}(x_i, x_j)\right\}$$

How accurate will the approximation be?

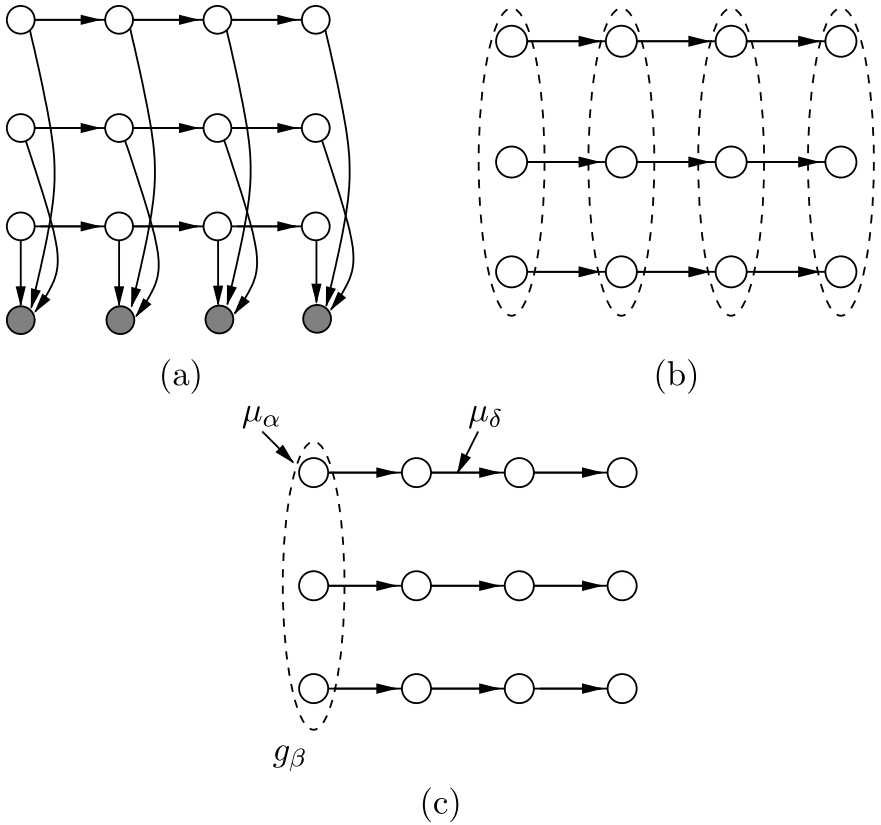
- Consider a distribution which is an XOR of two binary variables A and B: $p(a, b) = 0.5 - \epsilon$ if $a \neq b$ and $p(a, b) = \epsilon$ if $a = b$
- The contour plot of the variational objective is:



- Even for a single edge, mean field can give very wrong answers!
- Interestingly, once $\epsilon > 0.1$, mean field has a single maximum point at the uniform distribution (thus, exact)

Structured mean-field approximations

- Rather than assuming a fully-factored distribution for q, we can use a *structured* approximation, such as a spanning tree
- For example, for a factorial HMM, a good approximation may be a product of chain-structured models:



Recall our starting place for variational methods...

- Suppose that we have an arbitrary graphical model:

$$p(\mathbf{x};\theta) = \frac{1}{Z(\theta)} \prod_{\mathbf{c}\in C} \phi_c(\mathbf{x_c}) = \exp\Big(\sum_{\mathbf{c}\in C} \theta_c(\mathbf{x_c}) - \ln Z(\theta)\Big)$$

- All of the approaches begin as follows:

$$\begin{aligned}
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&= -\sum_{\mathbf{x}} q(\mathbf{x}) \ln p(\mathbf{x}) - \sum_{\mathbf{x}} q(\mathbf{x}) \ln \frac{1}{q(\mathbf{x})} \\
&= -\sum_{\mathbf{x}} q(\mathbf{x}) \big(\sum_{\mathbf{c}\in C} \theta_c(\mathbf{x_c}) - \ln Z(\theta)\big) - H(q(\mathbf{x})) \\
&= -\sum_{\mathbf{c}\in C} \sum_{\mathbf{x}} q(\mathbf{x}) \theta_c(\mathbf{x_c}) + \sum_{\mathbf{x}} q(\mathbf{x}) \ln Z(\theta) - H(q(\mathbf{x})) \\
&= -\sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + \ln Z(\theta) - H(q(\mathbf{x})).
\end{aligned}$$

The log-partition function

- Since $D(q\|p) \geq 0$, we have

$$-\sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + \ln Z(\theta) - H(q(\mathbf{x})) \geq 0,$$

  which implies that

$$\ln Z(\theta) \geq \sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + H(q(\mathbf{x})).$$

- Thus, *any* approximating distribution $q(\mathbf{x})$ gives a lower bound on the log-partition function (for a BN, this is the log probability of the observed variables)
- Recall that $D(q\|p) = 0$ if and only if $p = q$.Thus, if we allow ourselves to optimize over *all* distributions, we have:

$$\ln Z(\theta) = \max_q \sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + H(q(\mathbf{x})).$$

Re-writing objective in terms of moments

$$\begin{aligned}
\ln Z(\theta) &= \max_q \sum_{\mathbf{c}\in C} E_q[\theta_c(\mathbf{x_c})] + H(q(\mathbf{x})) \\
&= \max_q \sum_{\mathbf{c}\in C} \sum_{\mathbf{x}} q(\mathbf{x})\theta_c(\mathbf{x_c}) + H(q(\mathbf{x})) \\
&= \max_q \sum_{\mathbf{c}\in C} \sum_{\mathbf{x}_c} q(\mathbf{x_c})\theta_c(\mathbf{x_c}) + H(q(\mathbf{x})).
\end{aligned}$$

- Now assume that $p(\mathbf{x})$ is in the exponential family, and let $\mathbf{f}(\mathbf{x})$ be its sufficient statistic vector
- Define $\mu_q = E_q[\mathbf{f}(\mathbf{x})]$ to be the *marginals* of $q(\mathbf{x})$
- We can re-write the objective as

$$\ln Z(\theta) = \max_{\mu\in M} \max_{q:E_q[\mathbf{f}(\mathbf{x})]=\mu} \sum_{\mathbf{c}\in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c})\mu_c(\mathbf{x_c}) + H(q(\mathbf{x})),$$

where M, the **marginal polytope**, consists of all valid marginal vectors

Re-writing objective in terms of moments

- Next, push the max over q instead to obtain:

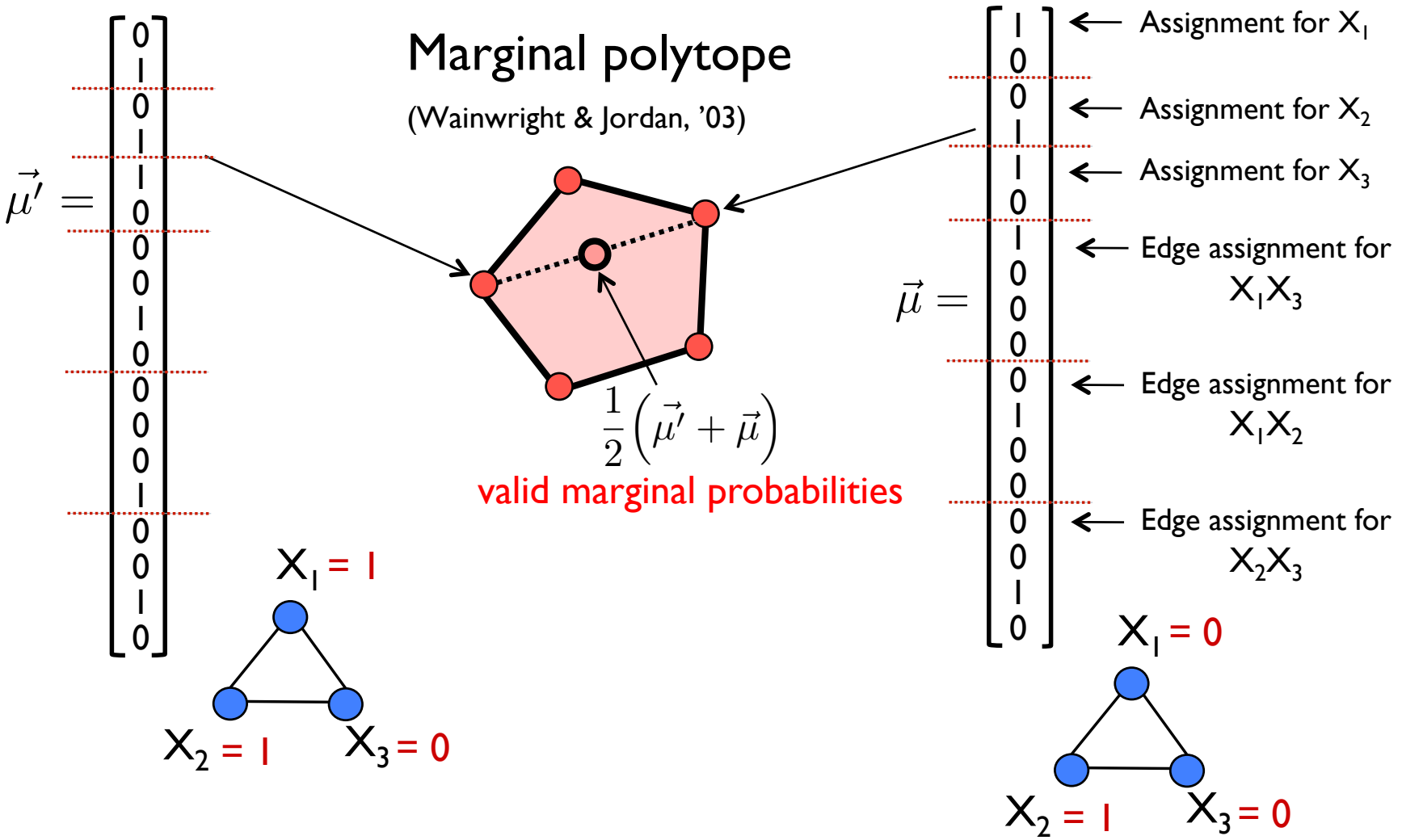
$$\begin{aligned} \ln Z(\theta) &= \max_{\mu \in M} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \mu_c(\mathbf{x_c}) + H(\mu), \text{ where} \\ H(\mu) &= \max_{q: E_q[\mathbf{f}(\mathbf{x})] = \mu} H(q) \qquad \leftarrow \text{Does this look familiar?} \end{aligned}$$

- For discrete random variables, the **marginal polytope** M is given by

$$\begin{aligned} M &= \left\{ \mu \in \mathbb{R}^d \mid \mu = \sum_{\mathbf{x} \in \mathcal{X}^m} p(\mathbf{x}) \mathbf{f}(\mathbf{x}) \text{ for some } p(\mathbf{x}) \geq 0, \sum_{\mathbf{x} \in \mathcal{X}^m} p(\mathbf{x}) = 1 \right\} \\ &= \texttt{conv}\left\{ \mathbf{f}(\mathbf{x}), \mathbf{x} \in \mathcal{X}^m \right\} \quad (\texttt{conv} \text{ denotes the convex hull operation}) \end{aligned}$$

- For a discrete-variable MRF, the sufficient statistic vector $\mathbf{f}(\mathbf{x})$ is simply the concatenation of indicator functions for each clique of variables that appear together in a potential function
- For example, if we have a pairwise MRF on binary variables with $m = |V|$ variables and $|E|$ edges, $d = 2m + 4|E|$

Marginal polytope for discrete MRFs



Relaxation

$$\ln Z(\theta) = \max_{\mu \in M} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \mu_c(\mathbf{x_c}) + H(\mu)$$

- We still haven't achieved anything, because:
  1. The marginal polytope M is complex to describe (in general, exponentially many vertices and facets)
  2. $H(\mu)$ is very difficult to compute or optimize over
- We now make two approximations:
  1. We replace M with a *relaxation* of the marginal polytope, e.g. the local consistency constraints M_L
  2. We replace $H(\mu)$ with a function $\tilde{H}(\mu)$ which approximates $H(\mu)$

Local consistency constraints

- Force every "cluster" of variables to choose a local assignment:

$$\begin{aligned} \mu_i(x_i) &\geq 0 \quad \forall i \in V, x_i \\ \sum_{x_i} \mu_i(x_i) &= 1 \quad \forall i \in V \\ \mu_{ij}(x_i, x_j) &\geq 0 \quad \forall ij \in E, x_i, x_j \\ \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) &= 1 \quad \forall ij \in E \end{aligned}$$

- Enforce that these local assignments are globally consistent:

$$\begin{aligned} \mu_i(x_i) &= \sum_{x_j} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_i \\ \mu_j(x_j) &= \sum_{x_i} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_j \end{aligned}$$

- The *local consistency polytope*, M_L is defined by these constraints
- **Theorem:** The local consistency constraints *exactly* define the marginal polytope for a tree-structured MRF

Entropy for tree-structured models

- Suppose that p is a tree-structured distribution, so that we are optimizing only over marginals $\mu_{ij}(x_i, x_j)$ for $ij \in T$
- The solution to $\arg\max_{q:E_q[\mathbf{f}(\mathbf{x})]=\mu} H(q)$ is a tree-structured MRF (c.f. lecture 10, maximum entropy estimation)
- The entropy of q as a function of its marginals can be shown to be

$$H(\vec{\mu}) = \sum_{i \in V} H(\mu_i) - \sum_{ij \in T} I(\mu_{ij})$$

where

$$\begin{aligned} H(\mu_i) &= -\sum_{x_i} \mu_i(x_i) \log \mu_i(x_i) \\ I(\mu_{ij}) &= \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) \log \frac{\mu_{ij}(x_i, x_j)}{\mu_i(x_i)\mu_j(x_j)} \end{aligned}$$

- Can we use this for non-tree structured models?

Bethe-free energy approximation

- The Bethe entropy approximation is (for any graph)

$$H_{bethe}(\vec{\mu}) = \sum_{i \in V} H(\mu_i) - \sum_{ij \in E} I(\mu_{ij})$$

- This gives the following variational approximation:

$$\max_{\mu \in M_L} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \mu_c(\mathbf{x_c}) + H_{bethe}(\vec{\mu})$$

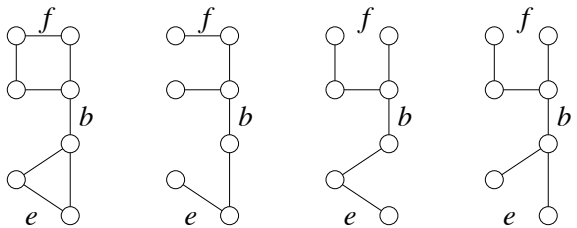
- For non tree-structured models this is not concave, and is hard to maximize
- Loopy belief propagation, if it converges, finds a saddle point!

Concave relaxation

- Let $\tilde{H}(\mu)$ be an *upper bound* on $H(\mu)$, i.e. $H(\mu) \leq \tilde{H}(\mu)$
- As a result, we obtain the following **upper bound** on the log-partition function:

$$\ln Z(\theta) \leq \max_{\mu \in M_L} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \mu_c(\mathbf{x_c}) + \tilde{H}(\mu)$$

- An example of a **concave** entropy upper bound is the **tree-reweighted** approximation (Jaakkola, Wainwright, & Wilsky, '05), given by specifying a distribution over spanning trees of the graph



Letting $\{\rho_{ij}\}$ denote edge appearance probabilities, we have:

$$H_{TRW}(\vec{\mu}) = \sum_{i \in V} H(\mu_i) - \sum_{ij \in E} \rho_{ij} I(\mu_{ij})$$

Comparison of LBP and TRW

We showed two approximation methods, both making use of the *local consistency constraints* M_L on the marginal polytope:

1. Bethe-free energy approximation (for pairwise MRFs):

$$\max_{\mu \in M_L} \quad \sum_{ij \in E} \sum_{x_i, x_j} \mu_{ij}(x_i, x_j)\theta_{ij}(x_i, x_j) + \sum_{i \in V} H(\mu_i) - \sum_{ij \in E} I(\mu_{ij})$$

   - Not concave. Can use concave-convex procedure to find local optima
   - Loopy BP, if it converges, finds a saddle point (often a local maxima)

2. Tree re-weighted approximation (for pairwise MRFs):

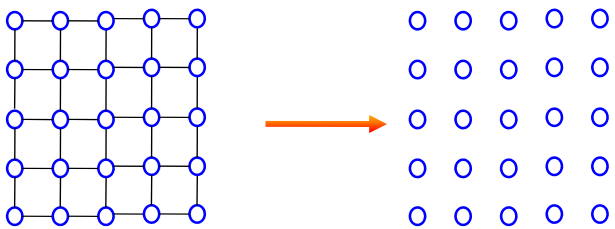
$$(*) \max_{\mu \in M_L} \quad \sum_{ij \in E} \sum_{x_i, x_j} \mu_{ij}(x_i, x_j)\theta_{ij}(x_i, x_j) + \sum_{i \in V} H(\mu_i) - \sum_{ij \in E} \rho_{ij} I(\mu_{ij})$$

   - $\{\rho_{ij}\}$ are edge appearance probabilities (must be consistent with some set of spanning trees)
   - This is concave! Find global maximiza using projected gradient ascent
   - Provides an upper bound on log-partition function, i.e. $\ln Z(\theta) \leq (*)$

Two types of variational algorithms: Mean-field and relaxation

$$\max_{q \in Q} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} q(\mathbf{x_c}) \theta_c(\mathbf{x_c}) + H(q(\mathbf{x})).$$

- Although this function is concave and thus in theory should be easy to optimize, we need some compact way of representing $q(\mathbf{x})$
- *Relaxation* algorithms work directly with *pseudomarginals* which may not be consistent with any joint distribution
- *Mean-field* algorithms assume a factored representation of the joint distribution, e.g.



$$q(\mathbf{x}) = \prod_{i \in V} q_i(x_i) \qquad \text{(called } \textit{naive} \text{ mean field)}$$

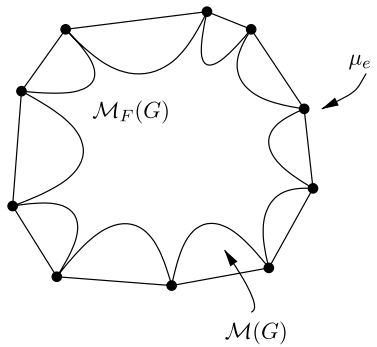
Naive mean-field

- Using the same notation as in the rest of the lecture, naive mean-field is:

$$(*) \max_{\mu} \sum_{\mathbf{c} \in C} \sum_{\mathbf{x_c}} \theta_c(\mathbf{x_c}) \mu_c(\mathbf{x_c}) + \sum_{i \in V} H(\mu_i) \quad \text{subject to}$$

$$\begin{aligned} \mu_i(x_i) &\geq 0 \quad \forall i \in V, x_i \in \text{Val}(X_i) \\ \sum_{x_i \in \text{Val}(X_i)} \mu_i(x_i) &= 1 \quad \forall i \in V \\ \mu_c(\mathbf{x}_c) &= \prod_{i \in c} \mu_i(x_i) \end{aligned}$$

- Corresponds to optimizing over an *inner bound* on the marginal polytope:



- We obtain a *lower bound* on the partition function, i.e. $(*) \leq \ln Z(\theta)$