Machine Learning and Computational Statistics, Spring 2014

Problem Set 6: Bayesian methods, naive Bayes, and hidden Markov models

Due: Monday, April 28, 2014 at 5pm (sent to akshaykumar@nyu.edu)

Important: *See problem set policy on the course web site.* You must show **all** of your work and be rigorous in your writeups to obtain full credit. If your solution is handwritten, please scan and e-mail as a PDF to Akshay.

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1. (10 points) **Medical diagnosis**.

   You go for your annual checkup and have several lab tests performed. A week later your doctor calls you and says she has good and bad news. The bad news is that you tested positive for a marker of a serious disease, and that the test is 98% accurate (i.e. the probability of testing positive given that you have the disease is 0.98, as is the probability of testing negative given that you don't have the disease). The good news is that this is a rare disease, striking only 1 in 30,000 people. Why is it good news that the disease is rare? What are the chances that you actually have the disease?

2. (10 points) **naive Bayes**.

   In this problem you will show that naive Bayes corresponds to a linear classifier. Consider using a naive Bayes algorithm for binary prediction (two classes), where the features $x_1, \dots, x_k$ are also binary valued. Let $\theta_c = \Pr(Y = c)$ and $\theta_{ci} = \Pr(X_i = 1 \mid Y = c)$ for $c \in \{0, 1\}$. It will be helpful to use the following form for the joint distribution:

$$\Pr(Y = 1, x_1, \dots, x_k \,;\, \vec{\theta}) = \theta_1 \prod_{i=1}^{k} \theta_{1i}^{x_i} (1 - \theta_{1i})^{1 - x_i} \tag{1}$$

$$\Pr(Y = 0, x_1, \dots, x_k \,;\, \vec{\theta}) = \theta_0 \prod_{i=1}^{k} \theta_{0i}^{x_i} (1 - \theta_{0i})^{1 - x_i} \tag{2}$$

   For a naive Bayes model given by parameters $\vec{\theta}$, demonstrate a weight vector $\mathbf{w}$ and offset b such that for any new example $\mathbf{x}$,

$$\arg\max_y \Pr(y \mid \mathbf{x} \,;\, \vec{\theta}) \quad = \quad \arg\max_y y\,(\mathbf{w} \cdot \mathbf{x} + b)\,,$$

   where $\vec{\theta}$ refers to all parameters, including both θ_c and θ_{ci}.

   Hint: Use Bayes' rule to obtain the posterior, and then take its logarithm (noticing that this is a monotonic transformation which does not change the argmax).

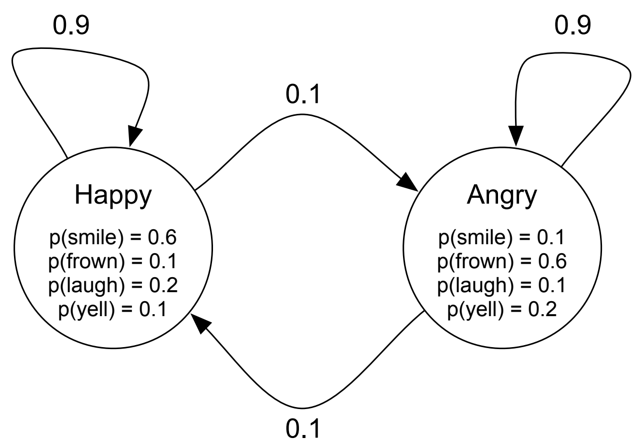
   Thus, if one had a sufficient amount of data, one would prefer to directly learn a linear model using logistic regression or a SVM rather than using naive Bayes, since the former consider a strictly larger hypothesis class than the latter. With limited numbers of training points (or settings where some features may be missing) naive Bayes may be preferable.

3. (20 points) **Hidden Markov models**.

Andy lives a simple life. Some days he is Angry and some days he is Happy. But he hides his emotional state, and so all we can observe is whether he smiles, frowns, laughs, or yells. Andy's best friend is utterly confused about whether Andy is actually happy or angry and decides to model his emotional state using a hidden Markov model.

Let $X_d \in \{\text{Happy, Angry}\}$ denote Andy's emotional state on day d, and let $Y_d \in \{\text{smile, frown, laugh, yell}\}$ denote the observation made about Andy on day d. **Assume that on day 1 Andy is in the Happy state**, i.e. $X_1 = \text{Happy}$. Furthermore, assume that Andy transitions between states exactly once per day (staying in the same state is an option) according to the following distribution: $p(X_{d+1} = \text{Happy} \mid X_d = \text{Angry}) = 0.1$, $p(X_{d+1} = \text{Angry} \mid X_d = \text{Happy}) = 0.1$, $p(X_{d+1} = \text{Angry} \mid X_d = \text{Angry}) = 0.9$, and $p(X_{d+1} = \text{Happy} \mid X_d = \text{Happy}) = 0.9$.

The observation distribution for Andy's Happy state is given by $p(Y_d = \text{smile} \mid X_d = \text{Happy}) = 0.6, p(Y_d = \text{frown} \mid X_d = \text{Happy}) = 0.1, p(Y_d = \text{laugh} \mid X_d = \text{Happy}) = 0.2$, and $p(Y_d = \text{yell} \mid X_d = \text{Happy}) = 0.1$. The observation distribution for Andy's Angry state is $p(Y_d = \text{smile} \mid X_d = \text{Angry}) = 0.1, p(Y_d = \text{frown} \mid X_d = \text{Angry}) = 0.6, p(Y_d = \text{laugh} \mid X_d = \text{Angry}) = 0.1$, and $p(Y_d = \text{yell} \mid X_d = \text{Angry}) = 0.2$. **All of this is summarized in the following figure**:



Each question below is worth 5 points. Be sure to show all of your work!

(a) What is $p(X_2 = \text{Happy})$?

(b) What is $p(Y_2 = \text{frown})$?

(c) What is $p(X_2 = \text{Happy} \mid Y_2 = \text{frown})$?

(d) What is $p(Y_{80} = \text{yell})$?

(e) Assume that $Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = \text{frown}$. What is the most likely sequence of the states? That is, compute the MAP assignment $\arg\max_{x_1,\ldots,x_5} p(X_1 = x_1, \ldots, X_5 = x_5 \mid Y_1 = Y_2 = Y_3 = Y_4 = Y_5 = \text{frown})$.