Probabilistic Graphical Models

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Today's lecture

1. MAP inference as an integer linear program
2. Linear programming relaxations for MAP inference
3. Efficiently solving the dual

MAP as an integer linear program (ILP)

- MAP as a discrete optimization problem is

$$\arg\max_{\mathbf{x}} \sum_{i \in V} \theta_i(x_i) + \sum_{ij \in E} \theta_{ij}(x_i, x_j).$$

- To turn this into an integer linear program, we introduce variables

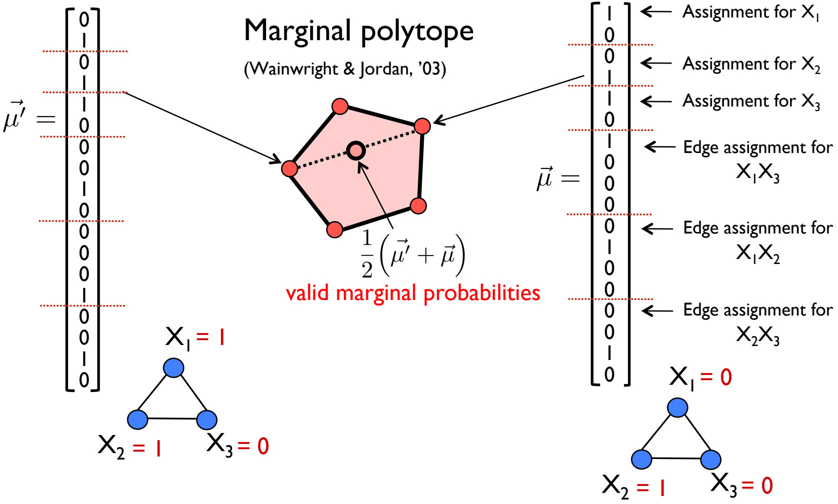
  1. $\mu_i(x_i)$, one for each $i \in V$ and state x_i
  2. $\mu_{ij}(x_i, x_j)$, one for each edge $ij \in E$ and pair of states x_i, x_j

- The objective function is then

$$\max_{\mu} \sum_{i \in V} \sum_{x_i} \theta_i(x_i)\mu_i(x_i) + \sum_{ij \in E} \sum_{x_i, x_j} \theta_{ij}(x_i, x_j)\mu_{ij}(x_i, x_j)$$

- What is the dimension of μ, if binary variables?

Visualization of feasible μ vectors



What are the constraints?

- Force every "cluster" of variables to choose a local assignment:

$$
\begin{aligned}
\mu_i(x_i) &\in \{0,1\} \quad \forall i \in V, x_i \\
\sum_{x_i} \mu_i(x_i) &= 1 \quad \forall i \in V \\
\mu_{ij}(x_i, x_j) &\in \{0,1\} \quad \forall ij \in E, x_i, x_j \\
\sum_{x_i, x_j} \mu_{ij}(x_i, x_j) &= 1 \quad \forall ij \in E
\end{aligned}
$$

- Enforce that these local assignments are globally consistent:

$$
\begin{aligned}
\mu_i(x_i) &= \sum_{x_j} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_i \\
\mu_j(x_j) &= \sum_{x_i} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_j
\end{aligned}
$$

MAP as an integer linear program (ILP)

$$\mathrm{MAP}(\theta) = \max_{\mu} \sum_{i \in V} \sum_{x_i} \theta_i(x_i)\mu_i(x_i) + \sum_{ij \in E} \sum_{x_i, x_j} \theta_{ij}(x_i, x_j)\mu_{ij}(x_i, x_j)$$

subject to:

$$\begin{aligned}
\mu_i(x_i) &\in \{0,1\} \quad \forall i \in V, x_i \\
\sum_{x_i} \mu_i(x_i) &= 1 \quad \forall i \in V \\
\mu_i(x_i) &= \sum_{x_j} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_i \\
\mu_j(x_j) &= \sum_{x_i} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_j
\end{aligned}$$

- Many extremely good off-the-shelf solvers, such as CPLEX and Gurobi

Linear programming relaxation for MAP

Integer linear program was:

$$\mathrm{MAP}(\theta) = \max_{\mu} \sum_{i \in V} \sum_{x_i} \theta_i(x_i)\mu_i(x_i) + \sum_{ij \in E} \sum_{x_i, x_j} \theta_{ij}(x_i, x_j)\mu_{ij}(x_i, x_j)$$

subject to

$$\begin{aligned}
\mu_i(x_i) &\in \{0, 1\} \quad \forall i \in V, x_i \\
\sum_{x_i} \mu_i(x_i) &= 1 \quad \forall i \in V \\
\mu_i(x_i) &= \sum_{x_j} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_i \\
\mu_j(x_j) &= \sum_{x_i} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_j
\end{aligned}$$

Relax integrality constraints, allowing the variables to be **between** 0 and 1:

$$\mu_i(x_i) \in [0, 1] \quad \forall i \in V, x_i$$

Linear programming relaxation for MAP

Linear programming relaxation is:

$$\mathrm{LP}(\theta) = \max_{\mu} \sum_{i \in V} \sum_{x_i} \theta_i(x_i)\mu_i(x_i) + \sum_{ij \in E} \sum_{x_i, x_j} \theta_{ij}(x_i, x_j)\mu_{ij}(x_i, x_j)$$

$$\begin{aligned}
\mu_i(x_i) &\in [0,1] \quad \forall i \in V, x_i \\
\sum_{x_i} \mu_i(x_i) &= 1 \quad \forall i \in V \\
\mu_i(x_i) &= \sum_{x_j} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_i \\
\mu_j(x_j) &= \sum_{x_i} \mu_{ij}(x_i, x_j) \quad \forall ij \in E, x_j
\end{aligned}$$

- Linear programs can be solved **efficiently**!
- Since the LP relaxation maximizes over a **larger** set of solutions, its value can only be larger!

$$\mathrm{MAP}(\theta) \leq \mathrm{LP}(\theta)$$

Dual decomposition

- Consider the original discrete optimization problem:

$$\mathrm{MAP}(\theta) = \max_{\mathbf{x}} \sum_{i \in V} \theta_i(x_i) + \sum_{ij \in E} \theta_{ij}(x_i, x_j).$$

- If we push the maximizations *inside* the sums, the value can only *increase*:

$$\mathrm{MAP}(\theta) \leq \sum_{i \in V} \max_{x_i} \theta_i(x_i) + \sum_{ij \in E} \max_{x_i, x_j} \theta_{ij}(x_i, x_j)$$

- Recall from your homework that you can always *reparameterize* a distribution by operations like

$$\begin{aligned} \theta_i^{\mathrm{new}}(x_i) &= \theta_i^{\mathrm{old}}(x_i) + f(x_i) \\ \theta_{ij}^{\mathrm{new}}(x_i, x_j) &= \theta_{ij}^{\mathrm{old}}(x_i, x_j) - f(x_i) \end{aligned}$$

for **any** function $f(x_i)$, without changing the distribution

Dual decomposition

- Define:

$$\begin{aligned}\tilde{\theta}_i(x_i) &= \theta_i(x_i) + \sum_{ij \in E} \delta_{j \to i}(x_i) \\ \tilde{\theta}_{ij}(x_i, x_j) &= \theta_{ij}(x_i, x_j) - \delta_{j \to i}(x_i) - \delta_{i \to j}(x_j)\end{aligned}$$

- It is easy to verify that

$$\sum_i \theta_i(x_i) + \sum_{ij \in E} \theta_{ij}(x_i, x_j) = \sum_i \tilde{\theta}_i(x_i) + \sum_{ij \in E} \tilde{\theta}_{ij}(x_i, x_j) \quad \forall \mathbf{x}$$

- Thus, we have that:

$$\mathrm{MAP}(\theta) = \mathrm{MAP}(\tilde{\theta}) \le \sum_{i \in V} \max_{x_i} \tilde{\theta}_i(x_i) + \sum_{ij \in E} \max_{x_i, x_j} \tilde{\theta}_{ij}(x_i, x_j)$$

- Every value of δ gives a different upper bound on the value of the MAP!
- The **tightest** upper bound can be obtained by minimizing the r.h.s. with respect to δ!

Dual decomposition

- We obtain the following **dual** linear program: $L(\delta) =$

$$\sum_{i \in V} \max_{x_i} \Big(\theta_i(x_i) + \sum_{ij \in E} \delta_{j \to i}(x_i) \Big) + \sum_{ij \in E} \max_{x_i, x_j} \Big(\theta_{ij}(x_i, x_j) - \delta_{j \to i}(x_i) - \delta_{i \to j}(x_j) \Big),$$

$$\mathrm{DUAL\text{-}LP}(\theta) = \min_{\delta} L(\delta)$$

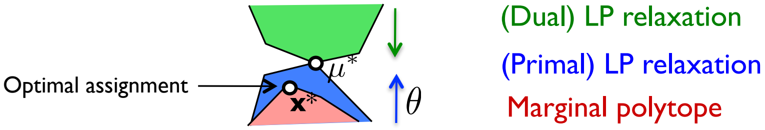
- We showed two ways of upper bounding the value of the MAP assignment:

$$\begin{aligned} \mathrm{MAP}(\theta) &\leq \mathrm{LP}(\theta) & (1) \\ \mathrm{MAP}(\theta) &\leq \mathrm{DUAL\text{-}LP}(\theta) \leq L(\delta) & (2) \end{aligned}$$

- The dual LP allows us to upper bound the value of the MAP assignment without solving a LP to optimality
- Although we derived these linear programs in seemingly very different ways, in turns out that:

$$\mathrm{LP}(\theta) = \mathrm{DUAL\text{-}LP}(\theta)$$

Linear programming duality



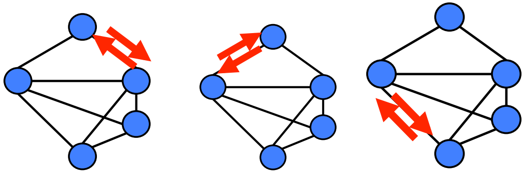
$$\mathrm{MAP}(\theta) \leq \mathrm{LP}(\theta) = \mathrm{DUAL\text{-}LP}(\theta) \leq L(\delta)$$

Solving the dual efficiently

- Many ways to solve the dual linear program, i.e. minimize with respect to δ:

$$\sum_{i \in V} \max_{x_i} \Big(\theta_i(x_i) + \sum_{ij \in E} \delta_{j \to i}(x_i) \Big) + \sum_{ij \in E} \max_{x_i, x_j} \Big(\theta_{ij}(x_i, x_j) - \delta_{j \to i}(x_i) - \delta_{i \to j}(x_j) \Big),$$

- One option is to use the subgradient method, as you saw in Lecture 3
- Can also solve using **block coordinate-descent**, which gives algorithms that look very much like max-sum belief propagation:



Max-product linear programming (MPLP) algorithm

Input: A set of factors $\theta_i(x_i), \theta_{ij}(x_i, x_j)$

Output: An assignment $x_1, \ldots, x_n$ that approximates the MAP

Algorithm:

- Initialize $\delta_{i\to j}(x_j) = 0, \quad \delta_{j\to i}(x_i) = 0, \quad \forall ij \in E, x_i, x_j$
- Iterate until small enough change in $L(\delta)$:

  For each edge $ij \in E$ (sequentially), perform the updates:

$$\begin{aligned} \delta_{j\to i}(x_i) &= -\frac{1}{2}\delta_i^{-j}(x_i) + \frac{1}{2}\max_{x_j}\left[\theta_{ij}(x_i,x_j) + \delta_j^{-i}(x_j)\right] \quad \forall x_i \\ \delta_{i\to j}(x_j) &= -\frac{1}{2}\delta_j^{-i}(x_j) + \frac{1}{2}\max_{x_i}\left[\theta_{ij}(x_i,x_j) + \delta_i^{-j}(x_i)\right] \quad \forall x_j \end{aligned}$$

  where $\delta_i^{-j}(x_i) = \theta_i(x_i) + \sum_{ik\in E, k\neq j} \delta_{k\to i}(x_i)$

- Return $x_i \in \arg\max_{\hat{x}_i} \tilde{\theta}_i^{\delta}(\hat{x}_i)$

Other approaches to solve MAP

- Local search
  - Greedily search over the space of assignments
  - Start from an arbitrary assignment (e.g., random). Iterate:
  - Choose a variable. Change a new state for this variable to maximize the value of the resulting assignment
- Branch-and-bound
  - Exhaustive search over space of assignments, pruning branches that can be provably shown not to contain a MAP assignment
  - Can use the LP relaxation or its dual to obtain upper bounds
  - Lower bound obtained from value of any assignment found along the way
- Branch-and-cut (most powerful method; used by CPLEX)
  - Same as branch-and-bound, except spend more time getting tighter bounds
  - Adds *cutting-planes* to cut off fractional solutions of the LP relaxation, making the upper bound tighter