

Derivation of Dual of Pairwise LP Relaxation

In this note we show how to derive the dual of the pairwise LP relaxation using the technique of Lagrangian relaxation, or dual decomposition. Consider the primal LP,

$$\max_{\mu \geq 0} \sum_i \sum_{x_i} \mu_i(x_i) \theta_i(x_i) + \sum_{ij} \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) \theta_{ij}(x_i, x_j)$$

subject to:

$$\begin{aligned} \sum_{x_j} \mu_{ij}(x_i, x_j) &= \mu_i(x_i), \quad \forall ij \in E, x_i \\ \sum_{x_i} \mu_{ij}(x_i, x_j) &= \mu_j(x_j), \quad \forall ij \in E, x_j \\ \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) &= 1, \quad \forall ij \in E \end{aligned} \quad (1)$$

$$\sum_{x_i} \mu_i(x_i) = 1, \quad \forall i \in V. \quad (2)$$

We introduce the Lagrange multipliers $\delta_{ji}(x_i)$ and $\delta_{ij}(x_j)$ for the first two constraints. Leaving the last two equality constraints and the non-negativity constraints explicit, we obtain the following equivalent optimization problem:

$$\begin{aligned} \min_{\delta} \max_{\mu \geq 0} \quad & \sum_i \sum_{x_i} \mu_i(x_i) \theta_i(x_i) + \sum_{ij \in E} \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) \theta_{ij}(x_i, x_j) \\ & + \sum_{ij \in E} \sum_{x_i} \delta_{ji}(x_i) \left(\mu_i(x_i) - \sum_{x_j} \mu_{ij}(x_i, x_j) \right) \\ & + \sum_{ij \in E} \sum_{x_j} \delta_{ij}(x_j) \left(\mu_j(x_j) - \sum_{x_i} \mu_{ij}(x_i, x_j) \right) \end{aligned}$$

subject to 1 and 2. Re-arranging the objective, we get

$$\begin{aligned} \min_{\delta} \max_{\mu \geq 0} \quad & \sum_i \sum_{x_i} \mu_i(x_i) \left(\theta_i(x_i) + \sum_{j \in N(i)} \delta_{ji}(x_i) \right) \\ & + \sum_{ij \in E} \sum_{x_i, x_j} \mu_{ij}(x_i, x_j) \left(\theta_{ij}(x_i, x_j) - \delta_{ji}(x_i) - \delta_{ij}(x_j) \right) \end{aligned} \quad (3)$$

Finally, we analytically solve the maximization with respect to $\mu \geq 0$ and the normalization constraints from 1 and 2 to obtain the dual objective:

$$J(\delta) = \sum_i \max_{x_i} \left(\theta_i(x_i) + \sum_{j \in N(i)} \delta_{ji}(x_i) \right) + \sum_{ij} \max_{x_i, x_j} \left(\theta_{ij}(x_i, x_j) - \delta_{ji}(x_i) - \delta_{ij}(x_j) \right)$$

The dual linear program is then: $\min_{\delta} J(\delta)$.