APPENDIX: SUPPLEMENTARY MATERIAL
Tightness of LP Relaxations for Almost Balanced Models

In this Appendix, we provide the following.

- Background material:
  - ◇ §7 Derivation of the triangle inequalities
  - ◇ §8 Discussion of symmetry: flipping, polytope constraints and problem triangles
- The following Sections, which provide key results on the structure of weak and strong down edges, and together provide complete proofs of Theorems 8, 9 and 11 in the main paper:
  - ◇ §9 Locking components, and 0 or 1 singleton marginals
  - ◇ §10 Results on the Structure of Weak and Strong Down Edges in an Almost Attractive Model
  - ◇ §11 Specification of the Perturbation for all Singleton and Edge Marginals
  - ◇ §12 Demonstration of Consistency
  - ◇ §13 Gathering Results and Finalizing Proofs of Theorems 8, 9 and 11

Add a high level overview of the Sections and how they fit together.

7 Derivation of the Triangle Inequalities

Here we show how to derive the inequalities defining TRI, i.e. (9) and (10) together with the standard constraints for LOC (3), following the lift-and-project method as described in (Wainwright and Jordan, 2008, Example 8.7). We first 'lift' to the space of marginals over three variables, where we require that a well-defined probability distribution exists over every triplet of variables in the model. Next we 'project' the resulting constraints back down to our familiar space of singleton and pairwise marginals, defined (in the minimal representation) by a vector of dimension $d = n + m$, where n is the number of variables, each with a q_i term, and m is the number of edges, each with a q_{ij} term.

Recall that each set of terms $\{q_i, q_j, q_{ij}\}$, provided they are feasible in LOC, defines a valid probability distribution on the pair of variables q_i, q_j as shown in (4), which we reproduce here:

$$\begin{pmatrix} q(X_i=0,X_j=0) & q(X_i=0,X_j=1) \\ q(X_i=1,X_j=0) & q(X_i=1,X_j=1) \end{pmatrix} = \begin{pmatrix} 1+q_{ij}-q_i-q_j & q_j-q_{ij} \\ q_i-q_{ij} & q_{ij} \end{pmatrix}$$

Observe that 4 terms are required for a distribution over variables X_i and X_j, but given $\{q_i, q_j\}$, we have several constraints: all must sum to 1, which leaves 3 degrees of freedom; then in order to match the singleton marginals given by q_i and q_j, this removes 2 more degrees of freedom leaving just one, which here is specified by q_{ij}. Note that enforcing that all terms are nonnegative yields the LOC inequalities (3).

Similarly, when considering a distribution over 3 variables, say i, j and k, there are 8 terms but given $\{q_i, q_j, q_k, q_{ij}, q_{ik}, q_{jk}\}$, we must satisfy the following constraints: all must sum to 1, marginalizing out any one variable must give the appropriate pairwise term (3 constraints), and marginalizing out any two variables must give the appropriate singleton term (3 constraints). Thus just one free term remains (in fact, it is not hard to see that for a cluster of any size, there is always just one degree of freedom, given all lower order terms), which here we shall specify using $\alpha = q_{ijk} = q(X_i = 1, X_j = 1, X_k = 1)$.

Given $\{q_i, q_j, q_k, q_{ij}, q_{ik}, q_{jk}, \alpha = q_{ijk}\}$, it is straightforward to see that we may write down the probabilities of all 8 states as follows:

With $k = 0$,

$$\begin{pmatrix} q(X_i=0,X_j=0) & q(X_i=0,X_j=1) \\ q(X_i=1,X_j=0) & q(X_i=1,X_j=1) \end{pmatrix} = \begin{pmatrix} 1-q_i-q_j-q_k+q_{ij}+q_{ik}+q_{jk}-\alpha & q_j+\alpha-q_{ij}-q_{jk} \\ q_i+\alpha-q_{ij}-q_{ik} & q_{ij}-\alpha \end{pmatrix}$$

With $k = 1$,

$$\begin{pmatrix} q(X_i=0,X_j=0) & q(X_i=0,X_j=1) \\ q(X_i=1,X_j=0) & q(X_i=1,X_j=1) \end{pmatrix} = \begin{pmatrix} q_k+\alpha-q_{ik}-q_{jk} & q_{jk}-\alpha \\ q_{ik}-\alpha & \alpha \end{pmatrix}$$

We have the inequalities that all 8 expressions must be nonnegative. Now to project back down to our original space, α must be eliminated, which can be achieved using Fourier-Motzkin elimination (Schrijver, 1998) as follows: (i) first express all inequalities in $\leq$ form with α (the variable to be eliminated) isolated; then (ii) combine $\leq \alpha$ constraints with $\alpha \leq$ constraints in pairs in order to yield a new inequality without α.

Working through this algebra yields exactly the constraints of LOC and TRI, i.e. (3), (9) and (10). As one example, to obtain the first inequality of (9), which is that $q_i + q_{jk} \geq q_{ij} + q_{ik}$, combine the inequality from the bottom left of the upper matrix, i.e. $q_i + \alpha - q_{ij} - q_{ik} \geq 0 \Leftrightarrow q_{ij} + q_{ik} - q_i \leq \alpha$, with the inequality from the top right of the lower matrix, i.e. $q_{jk} - \alpha \geq 0 \Leftrightarrow \alpha \leq q_{jk}$.

8 Symmetry: Flipping, Polytope Constraints and Problem Triangles

The minimal representation can sometimes obscure the underlying symmetry of the system. We demonstrate that the constraints for each of the local and triplet polytopes may be obtained by starting with just one constraint then flipping variables and applying the constraint to the flipped models. (This illustrates the symmetry but note that it is not true that having all constraints is the same as having just one constraint.)

Suppose we have a model including variables X_i and X_j with an edge (i,j) between them, together with a pseudo-marginal vector q. If X_i is flipped then we consider the model with $Y_i = 1 - X_i$ and $Y_j = X_j$. Let the new equivalent pseudo-marginal vector be q'. Clearly $q'_i = 1 - q_i$ and $q'_j = q_j$. For the edge marginal, observe that

$$\text{Original edge marginal} \quad \begin{pmatrix} q(X_i=0,X_j=0) & q(X_i=0,X_j=1) \\ q(X_i=1,X_j=0) & q(X_i=1,X_j=1) \end{pmatrix} = \begin{pmatrix} 1+q_{ij}-q_i-q_j & q_j - q_{ij} \\ q_i - q_{ij} & q_{ij} \end{pmatrix},$$

$$\text{New edge marginal} \quad \begin{pmatrix} q'(Y_i=0,Y_j=0) & q'(Y_i=0,Y_j=1) \\ q'(Y_i=1,Y_j=0) & q'(Y_i=1,Y_j=1) \end{pmatrix} = \begin{pmatrix} 1+q'_{ij}-q'_i-q'_j & q'_j - q'_{ij} \\ q'_i - q'_{ij} & q'_{ij} \end{pmatrix}.$$

To equate terms, note that $Y_i = 1$ or 0 corresponds to $X_i = 0$ or 1, so the row order has been reversed. Hence, $q'_{ij} = q_j - q_{ij}$.

The constraints that $0 \leq q_i \leq 1 \ \forall i \in \mathcal{V}$, and $0 \leq q_{ij} \leq 1 \ \forall (i,j) \in \mathcal{E}$ are base constraints that hold without considering multiple variables.

8.1 Local Polytope LOC

Let us start with the following one constraint (other choices would also work),

$$q_{ij} \leq q_i.$$

Flipping X_i and applying the above constraint to the new model yields

$$q'_{ij} \leq q'_i \quad \Leftrightarrow \quad q_j - q_{ij} \leq 1 - q_i \quad \Leftrightarrow \quad q_{ij} \geq q_i + q_j - 1.$$

Now take the last constraint above and flip X_j to obtain

$$q_i - q_{ij} \geq q_i + 1 - q_j - 1 \quad \Leftrightarrow \quad q_{ij} \leq q_j.$$

Observe that we have obtained all the local polytope constraints.

8.2 Triplet Polytope TRI

Consider any triplet of variables X_i, X_j, X_k. Let us start with the following one constraint,

$$q_i + q_{jk} \geq q_{ij} + q_{ik}.$$

Flip X_i to obtain

$$1 - q_i + q_{jk} \geq q_j - q_{ij} + q_k - q_{ik} \quad \Leftrightarrow \quad q_{ij} + q_{jk} + q_{ik} \geq q_i + q_j + q_k - 1. \tag{11}$$

Take the last constraint above and flip X_j to obtain

$$q_i - q_{ij} + q_k - q_{jk} + q_{ik} \geq q_i + 1 - q_j + q_k - 1 \quad \Leftrightarrow \quad q_j + q_{ik} \geq q_{ij} + q_{jk}.$$

Instead, take (11) and flip X_k to obtain

$$q_{ij} + q_j - q_{jk} + q_i - q_{ik} \geq q_i + q_j + 1 - q_k - 1 \quad \Leftrightarrow \quad q_k + q_{ij} \geq q_{ik} + q_{jk}.$$

Observe that all the triplet polytope constraints may be obtained.

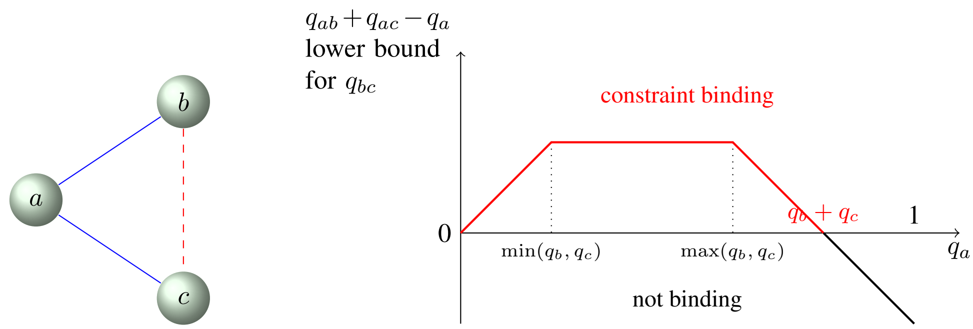


Figure 7: A triangle abc with two attractive edges $a-b$ and $a-c$, and one repulsive edge $b-c$, together with a graph of the relevant triangle constraint $q_a + q_{bc} \geq q_{ab} + q_{ac}$ as q_a is varied, holding fixed q_b and q_c while recomputing LOC-optimum edge marginals for q_{ab} and q_{ac}. The constraint is binding where the plot is red, and not where it is black. Here we consider $q_b + q_c < 1$, hence on LOC, $q_{bc} = 0$, and $q_{ab} = \min(q_a, q_b), q_{ac} = \min(q_a, q_c)$. Observe that $q_a = q_b + q_c$ is the one new case that causes trouble (e.g. if just q_a is perturbed, the constraint becomes binding just on one side leading to an asymmetric response). There can also be difficulties at the vertices at $q_a \in \{\min(q_b, q_c), \max(q_b, q_c)\}$ but these would be locking edges from a to b or c, hence are already covered by the LOC cases. When $q_a = q_b + q_c$, observe that any sufficiently small perturbation of singleton marginals up and down by a vector within the following two dimensional space will work symmetrically for edge marginals: $(\delta a, \delta b, \delta c) = \alpha(1, 1, 0) + \beta(1, 0, 1)$. In particular, this includes a small perturbation of $(\delta a, \delta b, \delta c) = \pm\epsilon(0, 1, -1)$.

8.3 Symmetry of Problem Triangles in TRI

Consider Figure 7. If $q_b + q_c < 1$ and $q_a = q_b + q_c$, with $a - b$ and $a - c$ strong up edges and $b - c$ a strong down edge, then this is a problem triangle of type (i) as described in §4: it has 3 strong edges with $a - b$ and $a - c$ strong up, and $b - c$ strong down; in addition, $b + c < 1$ and $a = b + c$. We shall show that the other 3 types of problem triangle described in §4 may be obtained from this one by flipping variables.

The following observations are easily checked:
Flipping a variable flips each of its incident edges between strong up $\leftrightarrow$ strong down.
Since flipping variables always changes an even number of edges, any flipping of our original problem triangle yields a triangle with three strong edges including an odd number of strong down edges, i.e. a strong frustrated triangle.

First, flip a to yield a triangle with 3 strong down edges and singleton marginals $a' = 1 - a, b' = b, c' = c$. Now $a = b + c \Leftrightarrow a' + b' + c' = 1$, i.e. problem triangle type (iii). Note that we have $b' + c' = a < 1$; also $a = b + c$ hence $a > b$ and $a > c$, which implies that $a' + b' < 1$ and $a' + c' < 1$.

Now flip all variables to give $a'' = 1-a', b'' = 1-b', c'' = 1-c'$. This again yields a triangle with 3 strong down edges but now $a''+b'' = 1-a'+1-b' > 1$, and similarly $a''+c'' > 1, b''+c'' > 1$. We have $a''+b''+c'' = 1-a'+1-b'+1-c' = 2$, i.e. problem triangle type (iv).

Finally, flip a'' to yield $a''' = 1-a'', b''' = 1-b'', c''' = 1-c''$ forming a strong triangle with edges incident to a''' strong up and $b''' - c'''$ strong down. Now $a'' + b'' + c'' = 2 \Leftrightarrow 1 - a''' + b''' + c''' = 2 \Leftrightarrow a''' = b''' + c''' - 1$, with $b''' + c''' > 1$, i.e. problem triangle type (ii).

9 Locking Components, and 0 or 1 Singleton Marginals

We first analyze locking components, see §9.2 for variables with 0 or 1 singleton marginals.

9.1 Locking Components

On TRI, given marginals q_i, q_j, q_{ij}, we say that variables i and j are *locked up* if $q_i = q_j$ and $q_{ij} = \min(q_i, q_j)$, i.e. they have the same singleton marginal and there is a strong up edge between them. Similarly, we say that variables i and j are *locked down* if $q_i = 1 - q_j$ and $q_{ij} = \max(0, q_i + q_j - 1)$, i.e. they have 'opposite' singleton marginals and there is a strong down edge between them. In either case, we say that the edge (i, j) is *locking* (either up or down).

We say that a cycle is *strong frustrated* if it is composed of strong edges with an odd number of strong down edges.

Define a *locking component* to be a component of the model that is connected when considering only locking edges. This means that between any 2 variables in the locking component, there exists some path between them composed only of locking edges. In general, this path might be long but the next result shows that in TRI, in fact it is always of length 1. In addition, we see that a locking component contains no strong frustrated cycle.

Lemma 14. *In TRI, within any locking component, all pairs of variables are adjacent via locking edges; further, there are no strong frustrated triangles, and hence no strong frustrated cycles.*

Proof. For the first part, the following result is sufficient, since given a path between any 2 variables in the component, this will allow us iteratively to find a path shorter by one edge, until we get the edge directly between them:

Suppose variable A is adjacent to B which is adjacent to C, each via a locking edge. We shall show that A is adjacent to C via a locking edge so as always to avoid a strong frustrated triangle. Let B have singleton marginal x. We shall consider all marginals, where A means singleton marginal for A etc., AB means edge marginal for edge $A - B$ etc. There are 3 cases:

1. $A - B$ is locking up, $B - C$ is locking up. $A : x, B : x, C : x, AB : x, BC : x$. Now triangle inequality $B + AC \geq AB + BC$ gives $AC = x$, i.e. $A - C$ is locking up.
2. $A - B$ is locking up, $B - C$ is locking down. $A : x, B : x, C : 1 - x, AB : x, BC : 0$. Now $A + BC \geq AB + AC$ gives $AC = 0$, i.e. $A - C$ is locking down.
3. $A - B$ is locking down, $B - C$ is locking down. $A : 1 - x, B : x, C : 1 - x, AB : 0, BC : 0$. Now $AB + BC + AC \geq A + B + C - 1$ gives $AC = 1 - x$, i.e. $A - C$ is locking up.

We have shown that all variables in the locking component are adjacent via locking edges, and that no triangle is strong frustrated. To demonstrate that there are no strong frustrated cycles (of any length): Suppose toward contradiction that there exists such a cycle, and let us pick one with minimum length composed of variables $v_1, v_2, \ldots, v_n$, so $n \geq 4$ is minimal. Now 'break' the cycle into two pieces: $\{v_1, v_2, \ldots, v_{n-1}\}$ and $\{v_{n-1}, v_n, v_1\}$. Since the second piece is a triangle, by the above it is not strong frustrated, i.e. the number of strong down edges in it is $0 \mod 2$. Edge $v_1 - v_{n-1}$ is either strong up or strong down, either way, twice the number of its strong down edges is $0 \mod 2$. Let r be the number of strong down edges in cycle $v_1, v_2, \ldots, v_{n-1} \mod 2$, then we have $r + 0 = 1 \mod 2$, contradiction since n was minimal. □

9.1.1 Edge marginals from locking components

In TRI, suppose i and j are any two variables in a locking component, and k is any other variable.

Lemma 15. *Given q_{ik}, q_{jk} is uniquely known. If one moves symmetrically, so too does the other. Specifically, if i and j are locked up then $q_{jk} = q_{ik}$; if i and j are locked down then $q_{jk} = q_k - q_{ik}$.*

Proof. This follows by applying the TRI inequalities to the triangle i, j, k. We show the case where i and j are locked up. Let $x = q_i = q_j$. Let $y = q_{ik}$ and $r = q_{jk}$. The singleton and edge marginals are shown in Figure 8. We must show that $r = y$.

First, $q_i + q_{jk} \geq q_{ij} + q_{ik}$, i.e. $x + r \geq x + y$, hence $r \geq y$. Next, $q_j + q_{ik} \geq q_{ij} + q_{jk}$, i.e. $x + y \geq x + r$, hence $r \leq y$. □

9.1.2 A problem triangle cannot have more than one variable in a specific locking component

This follows directly from the relevant definitions (see §4) and Lemma 14 since a problem triangle has no locking edges.

9.2 0 or 1 Singleton Marginals

We consider any variable X_i with singleton marginal $q_i \in \{0, 1\}$.

Lemma 16. *If a variable has singleton marginal 0 or 1, then its incident edge marginals are forced and will move symmetrically (on LOC or TRI). For any triplet containing the variable, all TRI inequalities are always satisfied for any (LOC valid) opposite edge marginal.*

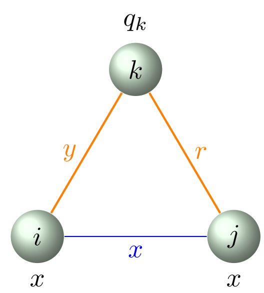


Figure 8: Marginals for variables i and j that are locked up in TRI. We show that $r = y$, see §9.1.1.

Proof. If variable X_i has singleton marginal $q_i = 0$, then for any incident edge (i, j), by the LOC constraint $q_{ij} \leq q_i$, we have $q_{ij} = 0$. If instead X_i has singleton marginal $q_i = 1$, then for any incident edge (i, j), by the LOC constraint $q_{ij} \geq q_i + q_j - 1$, we have $q_{ij} = q_j$.

Consider any triplet formed by X_i together with any variables X_j and X_k, which have singleton marginals q_j and q_k. Let q_{jk} be the LOC-valid edge marginal for the edge $X_j - X_k$ (i.e. q_{jk}, q_j, q_k satisfy (3)). It is straightforward to check that all TRI constraints (given by (9)-(10)) are satisfied. We demonstrate this for the case $q_i = 0$:

$$\begin{aligned}
q_i + q_{jk} - q_{ij} - q_{ik} &= 0 + q_{jk} - 0 - 0 && = q_{jk} \geq 0 \\
q_j + q_{ik} - q_{ij} - q_{jk} &= q_j + 0 - 0 - q_{jk} && = q_j - q_{jk} \geq 0 \\
q_k + q_{ij} - q_{ik} - q_{jk} &= q_k + 0 - 0 - q_{jk} && = q_k - q_{jk} \geq 0 \\
q_{ij} + q_{jk} + q_{ik} - q_i - q_j - q_k + 1 &= 0 + q_{jk} + 0 - 0 - q_j - q_k + 1 && = q_{jk} - (q_j + q_k - 1) \geq 0
\end{aligned}$$

□

10 Results on the Structure of Weak and Strong Down Edges in an Almost Attractive Model

Throughout this Section, we assume an almost attractive model, where edge marginals have been optimized over TRI given singleton marginals. Further, as justified by §9, we assume no locking edges or variables that have singleton marginal 0 or 1.

Lemma 17. *In every triplet of variables, at most one triplet constraint is tight.*

Proof. If any two triplet constraints hold, it is easily seen that this implies a locking edge. □

Lemma 18. *Any weak edge uv must be tight in some triplet constraint, that is there must exist some variable w s.t. there is a tight triplet constraint in u, v, w.*

Proof. If not, then the edge marginal uv may be perturbed up and down by a sufficiently small ϵ without violating any LOC or TRI constraints, hence we cannot be at a vertex. □

Lemma 19 (When 2 strong edges in a triangle force the 3rd edge to be strong)**.** *Consider triangle abc where edges ab and ac are strong. The following cases force the edge bc (all cases may be regarded as flippings of the first case):*

(i) *ab, ac up and $a \in [b, c]$ (a is in the middle) $\Rightarrow bc = \min(b, c)$ is strong up;*

(ii) *ab, ac down where one is 0 and the other is $> 0 \Rightarrow bc$ is strong up (with marginal equal to the end of the 0 edge from a);*

(iii) *ab up with $a > b$, and ac down with $ac = 0 \Rightarrow bc = 0$ strong down;*

(iv) *ab up with $a < b$, and ac down with $ac > 0 \Rightarrow bc = b + c - 1$ strong down.*

Proof. These are easily shown by applying TRI constraints to abc. We demonstrate the first case by applying the inequality $a + bc \geq ab + ac$: if $b \leq a \leq c$ then the inequality is $a + bc \geq b + a \Rightarrow bc = b$; similarly, $c \leq a \leq b \Rightarrow bc = c$. □

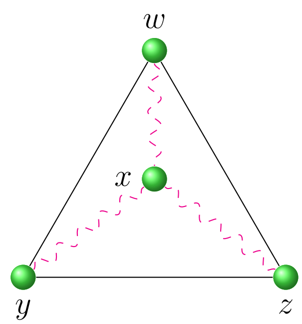


Figure 9: An illustration of the situation considered in Lemma 21. If $wx \to xz \to xy$ is a thistle, then so too is $wx \to xy$. Broken wavy edges indicate edges which are either strong down or weak (but not strong up), i.e. they are dw edges.

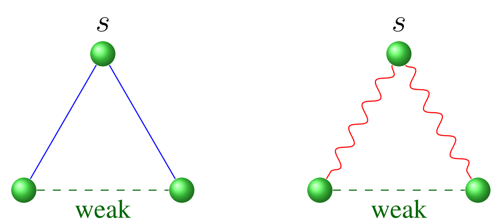


Figure 10: An illustration of the 2 structures which cannot occur in an almost attractive model if edge marginals are optimized; see Theorem 23. Solid blue edges are strong up, wavy red edges are strong down, and dashed green edges are weak. The right structure is equivalent to the left by a flipping of s.

We say that any edge which is strong down or weak is a *dw edge*. Thus, any edge which is not strong up is dw.

In an almost attractive model, any dw edge xy not incident to s must be being held down by some TRI constraint, say in triplet wxy. This must have one of two forms, either (i) $x + wy = wx + xy$, or (ii) $y + wx = wy + xy$. (The other 2 TRI inequalities, if tight, would hold up xy.) In case (i), we say that wy is *holding down* xy and write $wy \to xy$. In case (ii), wx is holding down xy and we write $wx \to xy$.

Note that $wy \to xy$ is equivalent to $wy \to wx$; both mean that $x + wy = wx + xy$.

Definition 20. A *thistle* from edge e^1 to edge e^k of length k is a sequence of edges $e_1^1 - e_2^1 \to e_1^2 - e_2^2 \to \ldots \to e_1^k - e_2^k$ where there is one variable in common between successive edges, that is $|\{e_1^i, e_2^i\} \cap \{e_1^{i+1}, e_2^{i+1}\}| = 1$ and each edge is holding down the next for all $i = 1, \ldots, k-1$.

An example thistle might be of the form $uv \to vw \to wx \to xy$. Note though that in general, a thistle may not be a direct path. For example, a thistle could take the form $uv \to vw \to vx \to xy$. In this example, we think of vw as a 'thorn' that sticks out to the side, which is why we call these structures thistles. We next provide two Lemmas which show that thistles of length 3 can be 'contracted' to length 2.

Lemma 21. *If $xw \to xz \to xy$ is a thistle (note this has a thorn), then so too is $xw \to xy$.*

Proof. Consider Figure 9. We know that xw is holding down xz and xz is holding down xy. Further we have an inequality for triangle wxy. Hence we have

$$z + xw = xz + wz \tag{12}$$
$$y + xz = xy + yz \tag{13}$$
$$y + xw \geq wy + xy \tag{14}$$

Now (12) + (13) gives $xw = xy + yz + wz - y - z$. Substituting into (14) gives $yz + wz \geq z + wy$. But now observe that we have $z + wy \geq wz + yz$ as a triplet constraint in wyz, hence (14) must hold with equality, which proves the result. $\square$

Lemma 22. *If $wx \to xy \to yz$ is a thistle (note this follows a path with no thorn), then so too is $wz \to yz$.*

Proof. The proof is similar to that of Lemma 21. We have that wx is holding down xy, and xy is holding down yz. Further, we use an inequality for the triangle wyz:

$$y + wx = wy + xy \tag{15}$$
$$z + xy = xz + yz \tag{16}$$
$$y + wz \geq wy + yz \tag{17}$$

Now (15) + (16) yields $yz = y + z + wx - wy - xz$. Substituting into (17) and rearranging gives $wz + xz \geq z + wx$. But we have the TRI inequality $z + wx \geq wz + xz$, so equality must be attained in $wz + xz \geq z + wx$, and so we must have equality in 17, which yields the result. □

Notice that in both Lemmas 21 and 22, the w variable in the first edge features exactly once in the conditions of the Lemmas, and then again features as one of the ends of the edge holding down the other in the conclusion of the result.

Using these earlier Lemmas, we show the following key structural result on dw edges.

Theorem 23 (dw edges away from s)**.** *Every dw edge xy which is not incident to s is pulled down by an edge incident to s, i.e. either $sx \to xy$ or $sy \to xy$.*

Proof. Any dw edge xy not incident to s is attractive, hence must be held down by another edge (i.e. xy must be in a triplet where there is a binding TRI constraint which upper bounds xy), which WLOG we may assume is ux for some u. If $u = s$ then we are done. Otherwise ux is attractive, and must be dw (since if ux were strong up, it is easily checked that it could not hold down xy, i.e. $y + ux \geq uy + xy$ will always hold, even if xy is strong up) and we may keep repeating the argument to grow a thistle back from xy: $\ldots \to ux \to xy$. As we work back, since the graph is finite, one of the following two cases must occur:

1. We eventually hit an edge incident to s. The result then follows by repeatedly applying Lemmas 21 or 22.
2. We have a sub-thistle, the edges of which form a chordless cycle in the graph of length $k \geq 3$, $a_1a_2 \to a_2a_3 \to \cdots a_ka_1$. Now repeatedly apply Lemmas 21 or 22 alternately to the sub-thistle given by the first three edges until we obtain either: $a_1a_k \to a_{k-1}a_k \to a_ka_1$ (if k is even) or $a_1a_{k-1} \to a_{k-1}a_k \to a_ka_1$ (if k is odd). In either case, this implies two tight triangle inequalities in $a_1a_{k-1}a_k$ (this follows directly from the definition above of the $\to$ notation; for example, $a_1a_k \to a_{k-1}a_k \to a_ka_1$ means $a_{k-1} + a_1a_k = a_1a_{k-1} + a_{k-1}a_k$ (from $a_1a_k \to a_{k-1}a_k$) and also $a_1 + a_{k-1}a_k = a_1a_{k-1} + a_1a_k$ (from $a_{k-1}a_k \to a_ka_1$)), which is a contradiction by Lemma 17.

Note that as a consequence of this Theorem, the two configurations shown in Figure 10 cannot occur.

□

We show a strengthening of the result if the dw edge is strong down.

Lemma 24 (Strong down edges away from s)**.** *If $xy = 0$ is a strong down edge with $s \notin \{x, y\}$, then either: $sx = x$ is strong up and $sy = 0$ is strong down; or $sx = 0$ is strong down and $sy = y$ is strong up.*
If $xy > 0$ is a strong down edge with $s \notin \{x, y\}$, then either: $sx = s$ is strong up and $sy > 0$ is strong down; or $sx > 0$ is strong down and $sy = s$ is strong up.

Proof. By Theorem 23, we have $sx \to xy$ or $sy \to xy$. The remainder of the statement of the proof follows as a straightforward application of the relevant TRI constraint. We show the case $xy = 0$ and $sx \to xy$: We have $y + sx = sy + xy = sy$. Rewrite this as $(y - sy) + sx = 0$. Both terms are ≥ 0 hence must both be exactly zero. □

11 Specification of Complete Symmetric Perturbation (including all edges)

Throughout this Section, we assume an almost attractive model with special variable s, where edge marginals have been optimized over TRI given singleton marginals. Further, as justified by §9, we assume no locking edges or variables that have singleton marginal 0 or 1.

We shall specify a perturbation for all singleton and all edge marginals with a number which is -1, 0 or 1 for each marginal. The perturbation up is formed by taking the vector of all these numbers and multiplying by a small ϵ. The perturbation down is exactly the negative of the perturbation up. ϵ is to be chosen sufficiently small s.t. any constraint (this includes all TRI constraints, all LOC constraints, and all constraints on a marginal being ≥ 0 and ≤ 1) which was not tight initially, remains so after either perturbation. In order for both perturbations to remain in TRI, we shall demonstrate that all tight TRI constraints (and also all LOC constraints, see §11.2.1) are exactly maintained in all cases.

11.1 Rule for Singleton Marginals

The perturbation for the singleton marginal of the variable s is 0. For any other variable $v \in \mathcal{V} \setminus \{s\}$, its perturbation depends on its edge marginal to s, i.e. sv, according to the following exhaustive options (recall that we are assuming no locking edges):

$$\begin{cases} v \text{ moves by } +1 & \text{if } v \text{ is strong up to } s \text{ and } v > s, \quad \text{or } v \text{ is strong down to } s \text{ and } v + s < 1 \\ v \text{ moves by } -1 & \text{if } v \text{ is strong up to } s \text{ and } v < s, \quad \text{or } v \text{ is strong down to } s \text{ and } v + s > 1 \\ v \text{ moves by } \ 0 & \text{if } v \text{ has a weak edge to } s. \end{cases} \tag{18}$$

We remark that this perturbation has the appealing property that it maps to itself (actually it maps to the negative of itself, but that is equivalent since we perturb up and down) under a flipping of s (if a perturbation works for all almost attractive models, then the version obtained from it by flipping s must also work for all almost attractive models, since flipping s is a bijection from the set of all almost attractive models to itself).

11.2 Rule for Edge Marginals

Given the changes in (18) for singleton marginals, we now show the perturbation for edge marginals.

11.2.1 Strong Edges

If an edge is strong (i.e. a LOC constraint is tight), we may immediately determine the perturbation required in order that LOC constraints are respected for both perturbations up and down. Specifically:

$$\begin{cases} uv \text{ moves with } \min(u, v) & \text{if } uv \text{ is strong up} \\ uv \text{ moves with } \max(0, u + v - 1) & \text{if } uv \text{ is strong down.} \end{cases} \tag{19}$$

11.2.2 Note on Consistency, Remaining within TRI

The above rules clearly ensure that our perturbed marginals remain in LOC. Note that for any edge that had a tight LOC constraint, i.e. was strong, the above rules exactly maintain this constraint when perturbed. We adopt this idea to ensure that we shall also remain in TRI. That is, for our perturbed marginals to be in TRI, it is clearly sufficient if we ensure that every TRI constraint that was tight, is exactly maintained for the perturbed marginals. In order to demonstrate that our perturbation satisfies this condition, we shall explicitly prescribe all perturbations for all weak edges, and show that our prescribed perturbation exactly maintains all TRI constraints that are tight. In order to do this, we shall have to demonstrate that our prescribed changes for edges are *consistent* with the change that is necessary in all other triplets to preserve tight TRI constraints. This is what we mean by consistency, which we explore fully in §12.

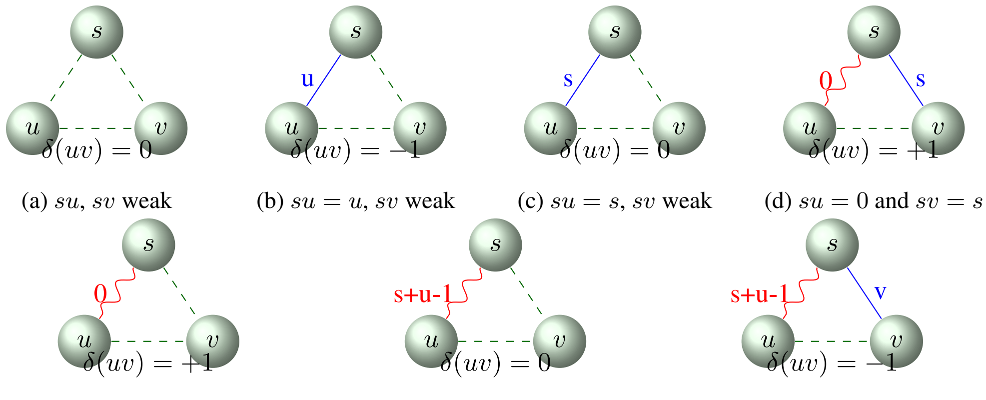
11.2.3 Weak Edges Incident to s

The perturbation for a weak edge sw incident to s is -1. This is chosen since it is necessary to ensure consistency for any TRI constraint involving the weak edge and any 2 strong edges, as we show in §12.2.1.

Note that we have now specified all edges (weak and strong) incident to s.

11.2.4 Weak Edges Not Incident to s, δ Notation

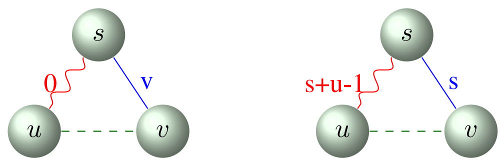
Given the above specifications, we may now use Theorem 23 to prescribe the change necessary for any weak edge uv not incident to s in order to maintain consistency. We adopt the notation $\delta(v) \in \{-1, 0, +1\}$ for the perturbation of a singleton marginal v, and $\delta(uv) \in \{-1, 0, +1\}$ for the perturbation of an edge marginal uv. There are exactly 7 possible cases to consider. In each case, it is straightforward to compute the required perturbation for the weak edge not incident to s, as shown in Figure 11. We provide more detail below.



(a) su, sv weak (b) $su = u$, sv weak (c) $su = s$, sv weak (d) $su = 0$ and $sv = s$

(e) $su = 0$, sv weak (f) $su = s + u - 1$, sv weak (g) $su = s + u - 1$ and $sv = v$

Figure 11: Cases where a weak edge uv is not incident to s. By Theorem 23, there must be a tight TRI constraint in suv. Here we show the possible forms with the implied perturbation for the weak edge. Note that the forms in the lower row may each by obtained from an appropriate form in the upper row by flipping s, so need not be considered separately. Specifically, under flipping s we have $a \leftrightarrow a, b \leftrightarrow e, c \leftrightarrow f, d \leftrightarrow g$.



(a) $su = 0$ and $sv = v$ (b) $su = s+u-1$ and $sv = s$

Figure 12: Cases which are not possible when suv has a tight TRI constraint since each implies that uv is strong down.

If a weak edge uv is not incident to s, then by Theorem 23 it lies in a tight triangle with s, and as described in §11.2.4, we may deduce its necessary edge perturbation by considering the tight TRI constraint in the triangle suv. The 7 possible cases are shown in Figure 11(a)-(g).

Note that by Lemma 19, the two configurations shown in Figure 12 cannot have a tight TRI constraint without contradicting the weakness of uv. Thus, these are omitted from Figure 11 and may be excluded from further analysis. Observe that a configuration of the form given in Figure 11g may be obtained by flipping the variable s in Figure 11d, and the configurations shown in Figures 11e and 11f may similarly be obtained from those in Figures 11b and 11c by flipping s. We may therefore exclude these cases from our analysis too, and need only show here that the perturbations defined for the weak edges in Figures 11a, 11b, 11c, and 11d are consistent.

The perturbations for the weak edge uv that are indicated in the various configurations of Figure 11 may be derived straightforwardly by considering the tight TRI constraint in each case, using the prescribed perturbation for the other edges as given by §11.1 and §11.2.1, and observing what perturbation of the weak edge is implied in order to maintain tightness of the relevant TRI constraint. We go through cases:

- In Figure 11a, the tight TRI constraint must be either $u + sv = su + uv$ or $v + su = uv + sv$. In either case, by noting that $\delta(u) = \delta(v) = 0$ and $\delta(su) = \delta(sv) = -1$, as prescribed in §11.2.3, it follows that to maintain tightness of the TRI constraint, we must have $\delta(uv) = 0$.

- In Figure 11b, the tight TRI constraint must be $u + sv = su + uv$. Noting that $\delta(u) = -1$, $\delta(sv) = -1$, and $\delta(su) = \delta(u) = -1$, we must have $\delta(uv) = -1$.

- In Figure 11c, the tight TRI constraint must be $u + sv = su + sv$. Noting that $\delta(u) = 1$, $\delta(sv) = -1$ and $\delta(su) = \delta(s) = 0$, we must have $\delta(uv) = 0$.

- In Figure 11d, the tight TRI constraint must be $v + su = sv + uv$. Noting that $\delta(v) = 1$, $\delta(su) = 0$, and $\delta(sv) = \delta(s) = 0$, we must have $\delta(uv) = 1$.

12 Demonstrating Consistency

We shall show that the perturbation prescribed in §11.2 maintains all tight TRI constraints, which is sufficient for us to stay within TRI after perturbing both up and down.

We must consider all cases of a triplet with a tight TRI constraint. We divide the cases up into 4 exhaustive classes:

(i) The triplet contains 0 weak edges (hence 3 strong edges), we call this *0-wedge consistency*. See §12.1.

(ii) The triplet contains 1 weak edge (hence 2 strong edges), we call this *1-wedge consistency*. See §12.2.

(iii) The triplet contains 2 weak edges (hence 1 strong edge), we call this *2-wedge consistency*. See §12.3.

(iv) The triplet contains 3 weak edges (hence 0 strong edges), we call this *3-wedge consistency*. See §12.4.

12.1 0-wedge consistency

In this Section we consider a triangle with 3 strong edges. Recall that by construction, our perturbation maintains the nature of all strong edges (strong up stay strong up, strong down stay strong down). We make the following observation.

Lemma 25. *In a triangle with three strong edges including an even number of strong down edges (so the triangle is not strong frustrated), all TRI constraints are always satisfied.*

Proof. This follows by straightforward checking of the TRI constraints (9)-(10). We demonstrate one case. Suppose abc is a triangle with 3 strong up edges. We shall show that $a + bc \geq ab + ac$. Consider $f = a + bc - ab - ac$, we shall show $f \geq 0$. We have $f = a + \min(b, c) - \min(a, b) - \min(a, c)$, clearly symmetric in b and c, thus we may consider just 3 subcases:

$$\begin{aligned} a \leq b \leq c \quad &\Rightarrow \quad f = a + b - a - a = b - a \geq 0 \\ b \leq a \leq c \quad &\Rightarrow \quad f = a + b - b - a = 0 \\ b \leq c \leq a \quad &\Rightarrow \quad f = a + b - b - c = a - c \geq 0. \end{aligned}$$

□

Hence we need consider only triangles that are strong frustrated. We may rule out 3 strong down edges.

Lemma 26. *A triangle with 3 strong down edges cannot occur.*

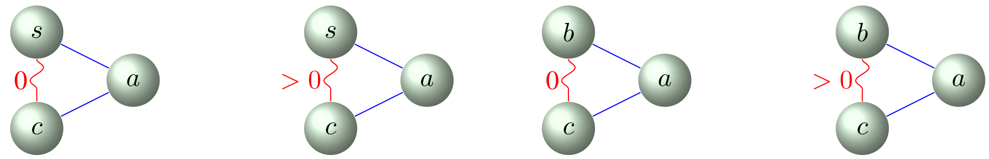
Proof. Lemma 24 shows that s cannot be in such a triangle. Now applying Lemma 24 to each edge in turn around the triangle yields a contradiction: we must alternate between strong up and strong down edges to s, yet this is not possible since we have an odd number of edges (if an edge is both strong up and strong down, one end must have marginal of 0 or 1, which we are assuming cannot occur). □

Thus we need consider only the case of a strong frustrated triangle abc that has 1 strong down edge bc and 2 strong up edges ab, ac, where a TRI constraint is tight. By Lemma 19, we must have $a < b, c$ or $a > b, c$. It is easily checked that the only TRI constraint of concern is where $a + bc = ab + ac$, hence we may assume that this holds. These are called problem triangles of type (i) and (ii) in the main paper §4. Note that s could be b or c (in which case, we assume b WLOG) but not a by Lemma 24. See Figure 13 for illustrations of the four possibilities.

Considering first the cases where s is in the triangle. If $a > s, c$ then we have $\delta(a) = +1, \delta(sc) = 0, \delta(sa) = \delta(s) = 0, \delta(ac) = \delta(c) = +1 \Rightarrow \delta(a + sc - sa - ac) = 1 + 0 - 0 - 1 = 0$ so we have consistency. If $a < s, c$ then $\delta(a) = -1, \delta(sc) = \delta(c) = -1, \delta(sa) = \delta(a) = -1, \delta(ac) = \delta(a) = -1 \Rightarrow \delta(a + sc - sa - ac) = -1 - 1 + 1 + 1 = 0$ as required.

If s is not in the triangle then we may use Lemma 24 to give the edges from s to b and c, which determine their perturbations. WLOG we shall assume that sb is strong down and sc is strong up. It remains to determine the perturbation change to a, which we shall do by considering the edge sa.

If $a > s, c$ (case c in Figure 13) then we have $sb = 0, sc = c, ac = c$. Also $a + bc = ab + ac \Rightarrow a = b + c$. From triangle sca we have $c + sa \geq sc + ac \Rightarrow sa \geq c + c - c = c$ while from sba we have $a + bs \geq ab + sa \Rightarrow sa \leq a + 0 - b = c$. Hence $sa = c$ a weak edge. Now $\delta(a + bc - ab - ac) = 0 + 0 - 1 + 1 = 0$ as required.



(a) $a > s, c$ (b) $a < s, c$ (c) $a > b, c$ (d) $a < b, c$

Figure 13: The four possible types of triangles abc with one strong down edge bc and two strong up edges to consider for 0-wedge consistency. We have $a + bc = ab + ac$, either $a > bc$ with $bc = 0$, or $a < b, c$ with $bc > 0$. On the left we have the cases where s is in the triangle. See §12.1.

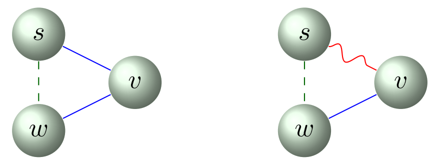


Figure 14: The two possible types of triangles with one weak edge incident to s

If $a < s, c$ (case d in Figure 13) then we have $sb > 0, sc = s, ac = ab = a$. Also $a + bc = ab + ac \Rightarrow a = b + c - 1$. Applying the same TRI inequalities as in the last case, we obtain $sa = s + b - 1$, again a weak edge. Now $\delta(a + bc - ab - ac) = 0 + 0 - 0 + 0 = 0$ as required.

12.2 1-wedge consistency

We split the 1-wedge class into subclasses. We first consider in §12.2.1 the case that the 1 weak edge is incident to s. Then in the following Sections, we demonstrate consistency exhaustively for all possible configurations of weak edges that are not incident to s. These are illustrated in Figure 11. We need consider only cases shown in 11a to 11d, since the remaining cases may be obtained from these by flipping s.

12.2.1 Perturbation of a weak edge incident to s consistent with a TRI including 2 strong edges

As in §11.2.3, let sw be a weak edge incident to s. Recall that we prescribed $\delta(sw) = -1$. Here we shall consider any possible triangle involving a third variable v with sv and vw strong, and demonstrate consistency.

By Lemma 24, vw cannot be a strong down edge, hence vw must be strong up. There are therefore two cases to consider: (i) both sv and vw are strong up; and (ii) sv is strong down, and vw is strong up. See Figure 14.

We first consider case (ii). By Lemma 19, because sw is weak, we must have one of the 2 subcases shown in Figure 15. The only possible tight TRI constraint, which must therefore apply, is $w + sv = sw + vw$.

In order to maintain this TRI constraint through the perturbations, we must have

$$\delta(w) + \delta(sv) = \delta(sw) + \delta(vw).$$

Using our rules for perturbation from §11.1 and §11.2.1, in both subcases this gives $\delta(sw) = -1$ which is consistent with our rule in §11.2.3.

We now consider case (i) where sv and sw are both strong up. The only possible tight TRI constraint, which must hold, is $v + sw = sv + vw$. By Lemma 19, we must have either $v < s, w$ or $v > s, w$, see Figure 16. In either case, to preserve the

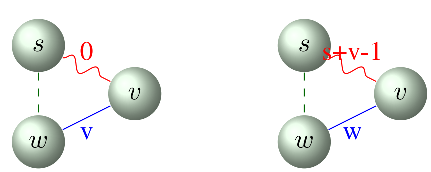


Figure 15: Possible cases where there is a weak edge incident to s in a triangle with one strong up and one strong down edge, with a tight TRI constraint

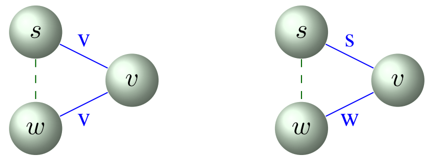


Figure 16: Possible cases where there is a weak edge incident to s in a triangle with two strong up edges, with a tight TRI constraint

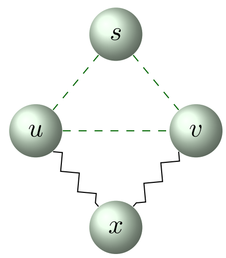


Figure 17: The consistency case to be analysed in this section. Black zigzag lines indicate generic strong edges.

tightness of the TRI constraint, following our rules for perturbation from §11.1 and §11.2.1, we must have $\delta(sw) = -1$, which is consistent with our rule in §11.2.3.

12.2.2 1-wedge Consistency of Weak Edge Perturbations Defined in Figure 11a

Here we prove that for any weak edge uv of the form appearing in Figure 11a, if there exists another variable x such that uvx is a triangle with a tight TRI constraint, and ux, vx are strong edges, then the tight TRI constraint is maintained the prescribed perturbation for uv. This scenario is illustrated in Figure 17.

First note that by Lemma 24, ux and vx cannot be strong down. Therefore the only scenario to consider in this case is when ux and vx are strong up. Recall from Lemma 19, we must have either $x < u, v$ or $x > u, v$. Note also that we have the prescribed perturbations $\delta(u) = \delta(v) = \delta(uv) = 0$. Since the only possible tight TRI constraint in uvx that doesn't contradict the weakness of uv is $x + uv = ux + vx$, this equality must hold. By considering Figure 18, note that in each case, we must prove that $\delta(x) = 0$ in order for tightness of this constraint to be maintained. Thus, it is sufficient in each case to prove that sx is weak.

First, we consider $x > u, v$ - see Figure 18a. In the tight triangle suv, one of the TRI constraints $u + sv \geq su + uv$ and $v + su \geq sv + uv$ must be tight; without loss of generality, we take $u + sv = su + uv$. In the tight triangle uvx, it must be the case that the tight TRI constraint is $x + uv = ux + vx$. From these two equations, we obtain $x = v - sv + su$. Now considering TRI inequalities in svx, we note $v + sx \geq sv + vx$, so $v + sx \geq sv + v$, and so $sx \geq sv$. We also have $x + sv \geq sx + vx$, which leads to $su \geq sx$ (by using the fact that $x = v + su - sv$). So we obtain

$$\min(s, x) \geq \min(s, u) > su \geq sx \geq sv > 0$$

Therefore if we can show that $sx \neq s + x - 1$, we have that sx is weak, so that $\delta(x) = 0$, and so the tight TRI constraint $x + uv = ux + vx$ is maintained under the perturbation, as we set out to show. To show this, suppose $sx = s + x - 1$, and consider the TRI constraint $u + sx \geq su + ux$. Substituting in our expression for x, we obtain $s + v - 1 \geq sv$, contradicting weakness of sv. Therefore sx is weak, and $\delta(x) = 0$, as required.

Next, we consider $x < u, v$, as in Figure 18b. Again, for the tight TRI constraint in suv we may assume without loss of generality that $u + sv = su + uv$. The only TRI constraint that can be tight in uvx (without contradicting weakness of uv) is $x + uv = ux + vx$, which implies $uv = x$, so $x = u + sv - su$. Considering the TRI constraint $u + sx \geq su + ux$ gives $sx \geq sv$, and considering the TRI constraint $x + sv \geq sx + vx$ gives $sv \geq sx$. Therefore we have $sv = sx$, and so immediately we have $sx > 0$ and $sx < s$. We now just need to rule out $sx = s + x - 1$ and $sx = x$. If $sx = s + x - 1$, then by considering the TRI constraint $v + sx \geq vx + sv$, we obtain $sv \leq s + v - 1$, contradicting weakness of sv. If $sx = x$, then we obtain $su = u$ is strong up, a contradiction.

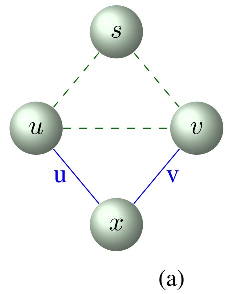


Figure 18: The possible configurations of the model shown in Figure 17

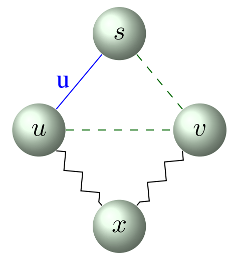


Figure 19: The consistency case to be analysed in this section. Black zigzag lines indicate generic strong edges.

12.2.3 1-wedge Consistency of Weak Edge Perturbations Defined in Figure 11b

Here we prove that for any weak edge uv of the form appearing in Figure 11b, if there exists another variable x such that uvx is a triangle with a tight TRI constraint, and ux, vx are strong edges, then the tight TRI constraint is maintained the prescribed perturbation for uv. This scenario is illustrated in Figure 19.

First, we note that xv cannot be strong down, by Lemma 24. Therefore we take xv strong up. Note also that since $u < s$, if ux is strong down, then it has edge marginal 0 and sx is strong down with edge marginal 0 too. Recall also that if ux is strong up, then by Lemma 19 we have $x > u, v$ or $x < u, v$. Figure 20 illustrates these cases.

In Figure 20a, note that the tight TRI constraint in uvx must be $v + ux = uv + vx$. Noting that in this case, we have $\delta(v) = 0$, $\delta(ux) = 0$, $\delta(uv) = -1$, if $v > x$, then $\delta(vx) = \delta(x) = -1$ (so the tightness of the TRI constraint is maintained). If $v < x$, then the TRI constraint $v + sx \geq xv + sv$, implies $sv = 0$, a contradiction.

In Figure 20b, note that the only possible tight TRI constraint in uvx is $x + uv = ux + vx$ (all other contradict weakness of uv). Note also that $u + sv = su + uv$ is the only possible tight TRI constraint in suv, so $sv = uv$. Lastly, we have the TRI constraint $x + sv \geq sx + vx$. But $x + sv = u + v$, so $u \geq sx$. But considering the TRI constraint $u + sx \geq su + ux$ gives $sx \geq u$. So $sx = u$ is weak. So we have $\delta(x) = 0$, $\delta(u) = -1$, $\delta(v) = 0$, and $\delta(uv) = -1$, so the tightness of $x + uv = ux + vx$ is maintained.

In Figure 20c, note that by considering the TRI constraint $u + sx \geq ux + su$, we obtain $sx = x$. We then note that the tight TRI constraint in uvx must be $x + uv = ux + vx$ (all others contradict the weakness of uv). Note that we have $\delta(x) = -1$, $\delta(uv) = -1$, $\delta(ux) = -1$, and $\delta(vx) = -1$, so the tightness of the TRI constraint is maintained.

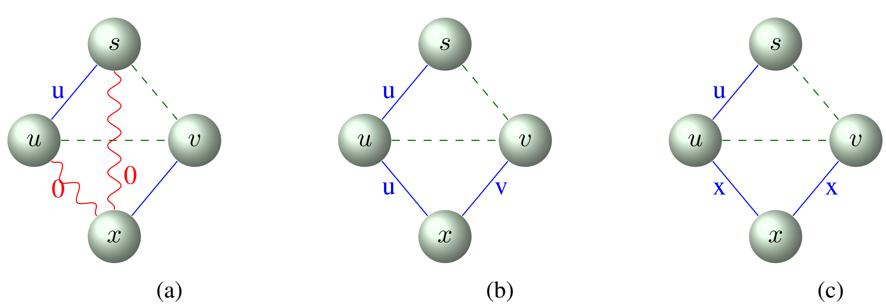


Figure 20: The possible configurations of the model shown in Figure 19

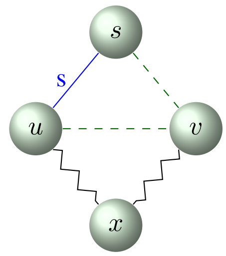


Figure 21: The consistency case to be analysed in this section. Black zigzag lines indicate generic strong edges.

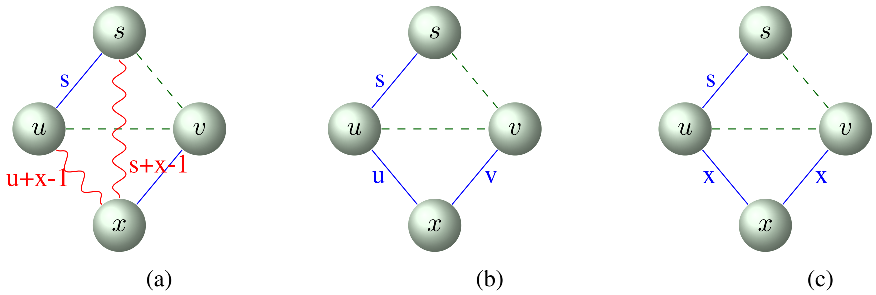


Figure 22: The possible configurations of the model shown in Figure 21

12.2.4 1-wedge Consistency of Weak Edge Perturbations Defined in Figure 11c

Here we prove that for any weak edge uv of the form appearing in Figure 11c, if there exists another variable x such that uvx is a triangle with a tight TRI constraint, and ux, vx are strong edges, then the tight TRI constraint is maintained by the prescribed perturbation for uv. This scenario is illustrated in Figure 21.

First, we note that xv cannot be strong down, by Lemma 24. Therefore we take xv strong up. Note also that since $u < s$, if ux is strong down, then it has edge marginal $u + x - 1$ and sx is strong down with edge marginal 0 too. Recall also that if ux is strong up, then by Lemma 19 we have $x > u, v$ or $x < u, v$. Figure 22 illustrates these cases.

In Figure 22a, note that if $vx = x$, then by considering the TRI constraint $v + sx \geq sv + xv$ implies that sv is strong down, a contradiction. So $vx = v$. Note that the only possible tight TRI constraint in uvx is $v + ux = uv + vx$, and we have $\delta(v) = 0$, $\delta(uv) = 0$, $\delta(vx) = 0$, and $\delta(ux) = 0$, so the TRI constraint remains tight.

In Figure 22b, note that from the TRI constraint $u + sx \geq su + ux$, we obtain $sx \geq s$, and so $sx = s$. The only possible tight TRI constraint in uvx is $x+uv = ux+vx$ (all others contradict the weakness of uv). But then note we have $\delta(x) = 1$, $\delta(ux) = 1$, $\delta(vx) = 0$ and $\delta(uv) = 0$, so the TRI constraint remains tight.

In Figure 22c, note that the only possible tight TRI constraint in uvx is $x + uv = ux + vx$ (all others contradict the weakness of uv), so we obtain $uv = x$. Note that $u + sx \geq su + ux$, so $sx \geq s + x - u$. Also, $x + sv \geq sx + vx$, so $sv \geq sx$. But the only possible tight TRI constraint in suv is $u + sv = su + uv$, and this yields $s + x - u \geq sx$, so $sx = s + x - u$. Note this quantity is less than s and x, so sx not strong up; it is greater than $s + x - 1$, and if it is equal to 0, then we have $sv = 0$, a contradiction. Therefore sx is weak. From this, note that $\delta(x) = 0$, $\delta(ux) = 0$, $\delta(vx) = 0$ and $\delta(uv) = 0$, so the TRI constraint in uvx remains tight.

12.2.5 1-wedge Consistency of Weak Edge Perturbations Defined in Figure 11d

In this section, we prove that for any weak edge uv of the form appearing in Figure 11d, if there exists another variable x such that uvx is a triangle with a tight TRI constraint, and ux, vx are strong edges, then the tight TRI constraint is maintained the prescribed perturbation for uv. This scenario is illustrated in Figure 23.

There are three separate realisations of the scenario in Figure 23 to consider; see Figure 24.

Firstly, the case where ux is strong down - this is illustrated in Figure 24a. Since ux is incident to us, which has edge marginal 0, $ux = 0$ too, by Lemma 24. Again by Lemma 24, sx is strong up and equal to x. Applying the TRI constraint $s + xv \geq sx + sv$ yields $vx \geq x$, so $vx = x$ and is strong up. Finally, checking which TRI constraint can hold in the tight

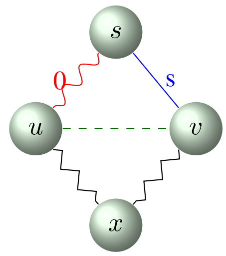


Figure 23: The consistency case to be analysed in this section. Black zigzag lines indicate generic strong edges.

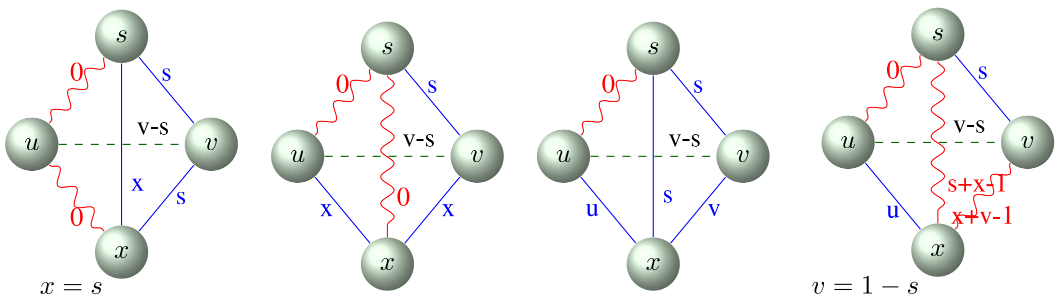


Figure 24: The possible configurations of the model shown in Figure 23

triangle uvx, we note that the only possibility not contradicting the weakness of uv is $v + ux = uv + vx$, which leads to $s = x$, so we have a locking component and need not consider this example further.

Secondly, we consider ux, vx strong up; see Figures 24b-24c. Recall from Lemma 19 that we need only consider $x < u, v$ and $x > u, v$. First consider $x < u, v$. The only TRI constraint in uvx that can be tight without contradicting the weakness of uv is $x + uv = xu + xv$, so $uv = x$. But from the tight TRI constraint in usv, we get $uv = v - s$, so $x = v - s$. Now considering the TRI constraint $v + sx \geq vs + vx$ gives $sx = 0$. Therefore we have $\delta(x) = 1$, $\delta(uv) = 1$, $\delta(xv) = 1$, $\delta(xu) = 1$, and verify that the constraint $x + uv = xu + xv$ remains tight under the perturbation. If $x > u, v$, then considering $v + sx \geq vs + vx$ gives $sx = s$, so again we obtain $\delta(x) = 1$, $\delta(uv) = 1$, $\delta(xv) = 1$, $\delta(xu) = 1$, and verify that the TRI constraint remains tight under the perturbation.

Finally, we consider ux strong up and vx strong down; see Figure 24d. vx strong down implies that sx strong down, and $sv = s$ implies that both sx and vx are strong down with edge marginal greater than 0. The only TRI constraint in uvx that can be tight without contradicting the weakness of uv is $u + vx = uv + ux$. This implies $ux = x + (u + v - 1 - uv) < x$ (as uv not strong down), so $ux = u$, and so $uv = x + v - 1$. But note since $v + su = sv + uv$, we have $uv = v - s$, and so $s = 1 - x$. Thus we have a locking component and need not consider this case further.

12.3 2-wedge consistency

12.3.1 The case where the 2 weak edges are both incident to s

This case is shown in Figure 25. The two possible tight TRI constraints are $s + sv = uv + su$ or $v + su = uv + sv$. In either case, it is clear that we obtain a consistent conclusion that $\delta(uv) = 0$ (consistent with $\delta(u) = \delta(v) = 0$ and hence the strong edge uv does not move).

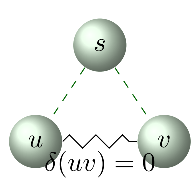


Figure 25: su, sv weak and uv strong (of any type)

12.3.2 All other cases of 2-wedge consistency

For all these cases, we consider a tight triangle xyz away from s, with xy, yz, weak and xz strong. We note that by earlier arguments, sxy and syz must be triangles from Figure 11. Therefore it is sufficient to consider all pairs of triangles sxy and syz from Figure 11, and show that the tight TRI constraint in xyz remains tight under the perturbation. A priori this gives $7 \times 7 = 49$ cases to check. However, by flipping s if necessary, sxy may always be taken to be one of a)-d) from Figure 11, reducing the burden to 28 cases. We further note that by symmetry we always take sxy to be a triangle listed no later in Figure 11 than triangle syz - this rules out a further 6 cases to check. The remaining cases are exhaustively examined below.

Note that the nature of the edge xz is not specified explicitly by the triangles sxy and syz. However, since in this section we consider triangles xyz with exactly two weak edges, we do not consider the cases where xz is weak - these are covered in §12.4.

In some cases, we will want to argue that certain combinations of tight TRI constraints and strong edges contradict our assumptions that we have no locking edges, and/or our assumptions about which edges are weak. It is possible, but laborious, to prove these contradictions of our assumptions algebraically; here we briefly explain a MATLAB script written to verify these contradictions automatically, in the context of its use in §12.3.4. In this case, we wish to show that it cannot be the case that $sz = z$, $xz = z$, all other edges weak, and $z + sy = sz + yz$, $z + xy = xz + yz$ and $y + sx = sy + xy$ without our assumptions of no locking edges, or the weakness of the other edges, being contradicted. To do this, we run the script shown at the top of Listing 1.

Listing 1: Example script used in this section

```
>> equalities = {'z=sz','z=xz','z+sy','z+xy','y+sx'};

% Test weak edges:
testWeakness(equalities, 'sx')
testWeakness(equalities, 'sy')
testWeakness(equalities, 'xy')
testWeakness(equalities, 'yz')
% Test whether strong edges lock:
testLocking(equalities, 'sz', 'up')
testLocking(equalities, 'sx', 'up')
```

The variables `equalities` is a cell containing strings, which code for which LOC and TRI constraints we would like to take to be tight. This gives rise to a new polytope, the restricted polytope given by intersecting TRI with all of these constraints. The function `testWeakness` examines a particular input edge uv in the graph to see whether it is always strong. This is implemented by checking whether any of the equations $uv = 0$, $uv = u + v - 1$, $uv = u$, $uv = v$ always hold in the restricted polytope. All four equations are checked in a similar way; for example, to check whether $uv = u$ at all points in the restricted polytope, two linear programs are set up to maximise and minimise the quantity $uv - u$ over the restricted polytope. If the maximum and minimum are both found to be 0 (in practice, we use a threshold of $1e - 6$), then we deduce that $uv = u$ at all points in the polytope, and so we deduce that edge uv is forced to be strong, given the set of constraints assumed in the `equalities` variable. Similarly, the function `testLocking` checks whether a particular edge is locking up or down, by checking whether the two incident edge marginals are always equal (in the case of locking up) or always sum to 1 (in the case of locking down) at all points in the restricted polytope; again, this is achieved by setting up two linear programs to maximise and minimise a particular objective, and checking whether the maximum and minimum attained are equal.

Listing 2: Output generated by Listing 1

```
Warning: The edge sy is actually strong up, with value y
> In testWeakness (line 26)
Warning: The edge xy is actually strong up, with value y
> In testWeakness (line 26)
Warning: The edge yz is actually strong up, with value y
> In testWeakness (line 21)
Warning: The edge yz is actually strong up, with value z
> In testWeakness (line 26)
```

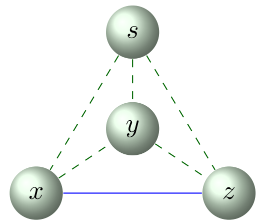


Figure 26: Model configuration for case a)-a)

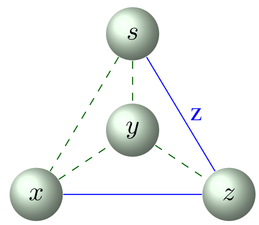


Figure 27: Model configuration for case a)-b)

The output (see Listing 2) to the script listed in Listing 1 indicates which tested edges the program found to be locking/strong. In particular, our assumption that sy is weak is shown to be contradicted by the set of TRI constraints we assumed to be tight; the program indicates that $sy = y$ at all points in the restricted polytope, and so sy is actually implied to be strong, a contradiction. This means we need not consider the case where our assumed set of TRI constraints holds. As a point of interest, note that the output states that yz is forced to be equal to y and z at all points in the restricted polytope; this implies that yz is locked up, and this can indeed be verified, as demonstrated in Listing 3.

Listing 3: Demonstration of an edge which is noted to be forced into being locked up

```
>> testLocking(equalities, 'yz', 'up')
Warning: y and z are locked up
> In testLocking (line 22)
```

This general approach allows us to deal efficiently with several of the checks described below. The code is available from the authors' websites.

We indicate where this approach has been used below with the comment (verified via MATLAB program).

12.3.3 Case a)-a)

Consider the case where sxy and syz are both triangles of type 11a; see Figure 26 for an illustration. Note that xz cannot be strong down, by Lemma 24. Therefore we may take xz strong up. Note that as sx, sy, sz are all weak, we have $\delta(x) = \delta(y) = \delta(z) = 0$. We also note from Figure 11a that $\delta(xy) = \delta(yz) = 0$. Finally, note also that $\delta(xz) = 0$, as it strong up and its incident variables do not move. Therefore whatever TRI constraint is tight in xyz, it remains tight after the perturbation, as all singleton and edge marginals do not move.

12.3.4 Case a)-b)

Consider the case where sxy is of type 11a and syz is of type 11b (so sz is the strong edge of the triangle syz); see Figure 27 for an illustration. xz can't be strong down by Lemma 24. So we may take xz strong up. We have $\delta(x) = 0$, $\delta(y) = 0$, $\delta(z) = -1$, and $\delta(xy) = 0$, $\delta(yz) = -1$.

If $z < x$, note that this implies $\delta(xz) = \delta(z) = -1$. There are two possible TRI inequalities that could be tight in xyz. If $x + yz = xy + xz$, then note that this TRI constraint remains tight. If $z + xy = xz + yz$, then note in sxy, either $x + sy \geq sx + xy$ is tight - but then $sx = x$ is strong up (verified via MATLAB program) - or $y + sx \geq sy + xy$ is tight - but then $sy = y$ is strong up (verified via MATLAB program), so we need not consider these cases further.

If $z > x$, then consider triangle szx: $x < z < s$ implies sx strong up by Lemma 19, so we need not consider this case further.

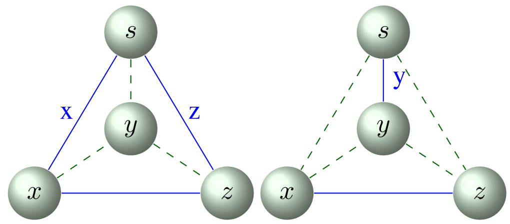


Figure 28: Model configurations for case b)-b)

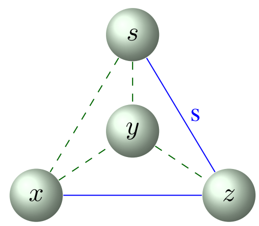


Figure 29: Model configurations for case a)-c)

12.3.5 Case b)-b)

There are two ways in which triangles sxy and syz may be of type 11b; they may share either a weak edge incident to s, or a strong edge incident to s.

For the former, we consider sxy of type 11b (with sx strong), and syz of type 11b (with sz strong). xz can't be strong down by Lemma 24, so we may take xz strong up. We have $\delta(x) = -1$, $\delta(y) = 0$, $\delta(z) = -1$, and $\delta(xy) = -1$, $\delta(yz) = -1$. Note that $\delta(xz) = -1$ whether $xz = x$ or $xz = z$. The two possible tight TRI inequalities in xyz are $x + yz = xy + xz$ and $z + yx = xz + yz$. By symmetry of this case in x and z, it suffices to consider one of these equations, and by substituting in the perturbations for each variable and edge, note that it remains tight.

For the latter, we consider sxy of type 11b and syz of type 11b, (with sy the strong edge in both triangles). xz cannot be strong down by Lemma 24, so we may take xz strong up. We have $\delta(x) = 0$, $\delta(y) = -1$, $\delta(z) = 0$, and $\delta(xy) = -1$, $\delta(yz) = -1$. Whether $xz = x$ or $xz = z$, we have $\delta(xz) = 0$. The two possible tight TRI inequalities in xyz are $x + yz = xy + xz$ and $z + yx = xz + yz$. By symmetry of this case in x and z, it suffices to consider one of these equations, and by substituting in the perturbations for each variable and edge, note that it remains tight.

12.3.6 Case a)-c)

We consider sxy of type 11a, syz of type 11c (so sz is the strong edge of the triangle). We have $\delta(x) = 0$, $\delta(y) = 0$, $\delta(z) = 1$, and $\delta(xy) = 0$, $\delta(yz) = 0$. xz can't be strong down by Lemma 24, so take xz strong up.

If $xz = z$, then consider triangle sxz, and note that $s < z < x$, implying sx strong up by Lemma 19, so we don't need to consider this case further.

If $xz = x$, then $\delta(xz) = \delta(x) = 0$. There are two possible tight TRI constraints in xyz. If $x+yz = xy+xz$, then plugging in our singleton and edge perturbations immediately verifies this remains tight under the perturbation. If $z+yx = yz+xz$, then considering triangle sxy, we either have $x + sy = xy + sx$ - in which case sx is strong up (verified via MATLAB program) - or $y + sx = sy + xy$ - in which case sy is strong up (verified via MATLAB program), so we need not consider these cases further.

12.3.7 Case b)-c)

We consider sxy of type 11b (with sx strong) and syz of type 11c (with sz strong). We have $\delta(x) = -1$, $\delta(y) = 0$, $\delta(z) = 1$, and $\delta(xy) = -1$ and $\delta(yz) = 0$. Note that xz cannot be strong down by Lemma 24, so take xz strong up.

If $xz = z$, then $s >= x >= z >= s$, so sx is locked up be Lemma 19, so we need not consider this case further.

IF $xz = x$, then $\delta(xz) = \delta(x) = -1$. There are two possible tight TRI constraints in xyz. If $z + xy = yz + xz$, then s and x are locked up (verified via MATLAB program). If $x + yz = xy + xz$, then s and z are locked up (verified via MATLAB program), so we need not consider these cases further.

12.3.8 Case c)-c)

There are two ways in which triangles sxy and syz may be of type 11c; they may share either a weak edge incident to s, or a strong edge incident to s.

For the former, take sxy, syz of type 11c, with sx, sz strong. Note that xz cannot be strong down by Lemma 24, so take xz strong up. Either $xz = x$ or $xz = z$. By symmetry of the model in x and z, it suffices to deal with $xz = x$. Then we have $\delta(x) = 1$, $\delta(y) = 0$, $\delta(z) = 1$, and $\delta(xy) = 0$, $\delta(yz) = 0$, and $\delta(xz) = 1$. The tight TRI constraint in xyz is either $x + yz = xy + xz$ or $z + xy = xz + yz$, and in both cases the perturbation keeps the TRI constraint tight.

For the latter, take sxy, syz of type 11c, with $sy = s$ strong. Note that xz cannot be strong down by Lemma 24, so take xz strong up. Either $xz = x$ or $xz = z$. Again by symmetry of the problem in x and z, we need only consider $xz = x$. Then we have $\delta(x) = 0$, $\delta(y) = 1$, $\delta(z) = 0$, and $\delta(xy) = 0$, $\delta(yz) = 0$, and $\delta(xz) = 0$. There are two possible tight TRI constraints, $x + zy = xz + yz$ and $z + xy = xz + xz$ - in both cases, no terms are perturbed, so the constraints remain tight.

12.3.9 Case a)-d)

It is not possible for triangles of type 11a and 11d to share an edge incident to s, so we need not consider this case.

12.3.10 Case b)-d)

It is not possible for triangles of type 11b and 11d to share an edge incident to s, so we need not consider this case.

12.3.11 Case c)-d)

We consider sxy of type 11c (with $sy = s$ strong up) and syz of type 11d (with $sz = 0$ strong down). Note that xz cannot be strong down by Lemma 24, so take xz strong up.

If $xz = x$, then by considering the TRI constraint $x + sz \geq xz + sx$, we obtain $sx = 0$ strong down, a contradiction.

If $xz = z$, then note that we have $\delta(x) = 0$, $\delta(y) = 1$, $\delta(z) = 1$, and $\delta(xy) = 0$, $\delta(yz) = 1$, $\delta(xz) = 1$. There are two possible tight TRI constraints in xyz. If $x + yz = xy + xz$, then the above perturbation maintains the tightness of this constraint. If $z + xy = xz + yz$, this implies $sx = 0$ strong down (verified via MATLAB program), a contradiction.

12.3.12 Case d)-d)

There are two ways in which triangles sxy and syz may be of type 11d; they may share either a strong up edge incident to s, or a strong down incident to s.

For the former, we consider sxy, syz of type 11d (with sy the strong up edge). Note that xz cannot be strong down by Lemma 24, so take xz strong up. So $xz = x$ or $xz = z$; by symmetry in x and z, it suffices to consider $xz = x$. We have $\delta(x) = \delta(y) = \delta(z) = 1$, and $\delta(xy) = \delta(yz) = \delta(xz) = 1$, so immediately it follows that any tight TRI constraint in xyz remains tight after the perturbation.

For the latter, we consider sxy, syz of type 11d (with sy the strong down edge). Again, we must have xz strong up, and by symmetry in x and z, it suffices to consider $xz = x$. Note that we have $\delta(x) = \delta(y) = \delta(z) = 1$, and $\delta(xy) = \delta(yz) = \delta(xz) = 1$, so immediately it follows that any tight TRI constraint in xyz remains tight after the perturbation.

12.3.13 Case a)-e)

We consider sxy of type 11a, and syz of type 11e (with sz the strong down edge). Note that under a flipping of s, this case is the same as case a)-b).

12.3.14 Case b)-e)

We consider sxy of type 11b (with $sx = x$ strong up), and syz of type 11e (With $sz = 0$ strong down). By considering the TRI inequality $x + sz \geq sx + xz$, we note that $xz = 0$, and this TRI constraint is tight. The only possible tight TRI constraint in xyz is $y + xz = xy + yz$ ($x + zy = xy + xz$ implies xy strong, $z + xy = xz + yz$ implies yz strong, and $xy + xz + yz = x + y + z - 1$ implies s, x, z are locking (verified via MATLAB program)). We have $\delta(x) = -1$, $\delta(y) = 0$, $\delta(z) = 1$, and $\delta(xy) = -1$, $\delta(yz) = 1$, $\delta(xz) = 0$. Substituting these perturbations into the tight TRI constraint $y + xz = xy + yz$, we note tightness is maintained.

12.3.15 Case c)-e)

We consider sxy of type 11c (with $sx = s$ strong up) and syz of type 11e (with $sz = 0$ strong down). Note that xz cannot be strong down by Lemma 24, so take xz strong up.

If $xz = x$, then the TRI inequality $x + sz \geq xs + xz$ implies $s = 0$, so we need not deal with this case.

If $xz = z$, then the only possible tight TRI constraints are $x + yz = xy + xz$ and $z + xy = xz + yz$, which both imply that sy is strong (verified via MATLAB program), so we need not deal with this case.

12.3.16 Case d)-e)

We consider sxy of type 11d (with $sx = s$ strong up and $sy = 0$ strong down) and syz of type 11e (with $sz = 0$ strong down). Note that xz cannot be strong down by Lemma 24, so take xz strong up.

If $xz = x$, then $s < x < z$, so so be Lemma 19 $sx = s$ is strong up, so we need not consider this case further.

If $xz = z$, then we have $\delta(x) = 1$, $\delta(y) = 1$, $\delta(z) = 0$, and $\delta(xy) = 1$, $\delta(yz) = 1$, $\delta(xz) = 0$. There are two possible tight TRI constraints in xyz. If $z + xy = xz + yz$, then the perturbations described above maintain the tightness of the TRI constraint. If $x + yz = xy + xz$, this forces sz to be strong up (verified via MATLAB program), so we need not consider this case further.

12.3.17 Case a)-f)

We consider sxy of type 11a, and syz of type 11f (with sz the strong down edge). Note that under a flipping of s, this case is the same as case a)-c).

12.3.18 Case b)-f)

We consider sxy of type 11b (with sx strong up), and syz of type 11f (with sz the strong down edge). Note that under a flipping of s, this case is the same as case c)-e).

12.3.19 Case c)-f)

We consider sxy of type 11c (with $sx = s$ strong up), and syz of type 11f (with $sz = s + z - 1$ strong down). If $xz = x$, then TRI inequality $x + sz \geq sx + sz$ implies that sz is strong up, a contradiction. By Lemma 24, xz cannot be strong down and equal to 0. So the cases to consider are $xz = z$ and $xz = x + z - 1$.

If $xz = z$, then note $\delta(x) = 1$, $\delta(y) = 0$, $\delta(z) = -1$, and $\delta(xy) = 0$, $\delta(yz) = 0$, $\delta(xz) = -1$. There are two possible tight TRI constraints in xyz. If $z + xy = xz + yz$, then the perturbation described above maintains tightness of the constraint. If $x + yz = xy + xz$, then xy is implied to be strong up (verified via MATLAB program), a contradiction.

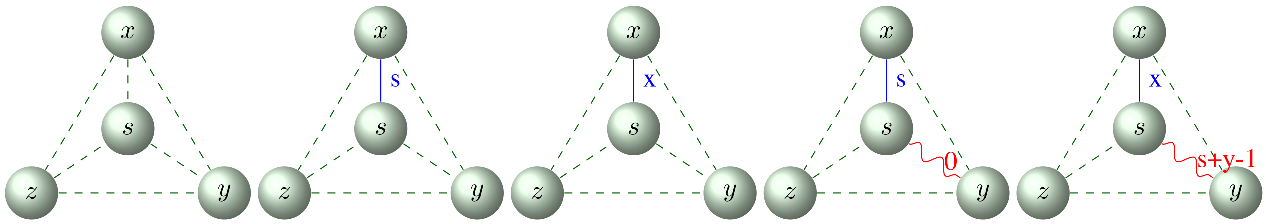
If $xz = x + z - 1$, then the only TRI constraint that can be tight in xyz is $y + xz = xy + yz$ (verified via MATLAB program). This is maintained by the perturbation described above.

12.3.20 Case d)-f)

It is not possible for triangles of type 11d and 11f to share an edge incident to s, so we need not consider this case.

12.3.21 Case a)-g)

It is not possible for triangles of type 11a and 11g to share an edge incident to s, so we need not consider this case.



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4 (e) Case 5

Figure 30: The five cases to consider

12.3.22 Case b)-g)

We consider sxy of type 11b (with $sy = y$ strong up), and syz of type 11g (with $sz = s + z - 1$ strong down). Note that $\delta(x) = 0$, $\delta(y) = -1$, $\delta(z) = -1$, and $\delta(xy) = -1$, $\delta(yz) = -1$. Note that xz cannot be strong down by Lemma 24, so take xz strong up.

If $xz = x$, then $\delta(xz) = \delta(x) = 0$. The two possible tight TRI constraints in xyz are $x + yz = xy + xz$ (for which it can be checked that the constraint remains tight with the perturbations specified above), and $z + xy = xz + yz$, which implies sx is strong down (verified via MATLAB program), a contradiction.

If $xz = z$, then $\delta(xz) = \delta(z) = -1$. The two possible tight TRI constraints in xyz are $z + xy = xy + xz$ (for which it can be checked that the constraint remains tight with the perturbations specified above), and $x + yz = xy + xz$, which implies sx is strong down (verified via MATLAB program), a contradiction.

12.3.23 Case c)-g)

It is not possible for triangles of type 11c and 11g to share an edge incident to s, so we need not consider this case.

12.3.24 Case d)-g)

It is not possible for triangles of type 11d and 11g to share an edge incident to s, so we need not consider this case.

12.4 3-wedge consistency

We now consider the case where a triangle xyz not incident to s has a tight TRI constraint, and demonstrate that this TRI constraint remains tight when all singleton and edge marginals are perturbed according to the description given in §11.

We begin by arguing that we need check only 5 cases. First, the case where all edges sx, sy, sz are weak (Case 1). If two of the edges sx, sy, sz are weak, then without loss of generality we make take sx strong. Note also that if sx strong down, this is obtained from a case where sx strong up by flipping s, so we only need to consider cases where sx is strong up (Cases 2 and 3). If exactly one of the edge sx, sy, sz are weak, then without loss of generality we may take sx, sy strong. It cannot be the case that both sx, sy are strong up or both strong down, as this would contradict Theorem 23, so without loss of generality we take sx strong up and sy strong down. As noted in §11.2.4, there are only two cases to consider; $sx = s$, $sy = 0$, and $sx = x$, $sy = s + y - 1$; these form Cases 4 and 5. It cannot be the case that all three edges sx, sy, sz are strong, since again this would contradict Theorem 23.

12.5 Case 1

All singleton and edge marginals have 0 perturbation, so any tight TRI constraint is preserved.

12.6 Case 2

We take $sx = s$, all other edges weak. We note that we have $x + sz = sx + xz$ and $x + sy = sx + xy$. There must also be a tight constraint in syz holding yz down, by symmetry in y and z we may take it to be $y + sz = sy + yz$. We then consider the four possible TRI constraints that could be tight in xyz. If $y + xz = xy + yz$, then the perturbation maintains the tightness of this constraint. The other three constraints lead to contradictions of tight (verified via MATLAB program).

12.7 Case 3

We take $sx = x$, all other edges weak. We note that we have $x + sz = sx + xz$ and $x + sy = sx + xy$. There must also be a tight constraint in syz holding yz down; by symmetry in y and z we may take it to be $y + sz = sy + yz$. There is also a tight constraint in xyz by assumption. If it is $y + xz = xy + yz$, then the perturbation given for the singletons and edge marginals maintains the tightness of this constraint. The other three constraints lead to contradictions (verified via MATLAB program).

12.8 Case 4

We take $sx = s$, $sy = 0$, and all other edges weak. We must have $x + sz = sx + xz$, $x + sy = xy + sx$, and $z + sy = sz + yz$. We consider the four possible TRI constraints that could be tight in xyz. If $z + xy = xz + yz$, then the perturbation given for the singletons and edge marginals maintains the tightness of this constraint. The other three constraints lead to contradictions (verified via MATLAB program).

12.9 Case 5

We take $sx = x$, $sy = s + y - 1$, and all other edges weak. We must have the TRI constrains $x + sz = sx + xz$, $x + sy = xy + sx$, and $z + sy = sz + yz$. We consider the four possible TRI constraints that could be tight in xyz. If $z + xy = xz + yz$, then the prescribed perturbation works. The other three constraints lead to contradictions (verified via MATLAB program).

13 Gathering Earlier Results to Provide Proofs of Theorems 8, 9 and 11

We gather together earlier results and use them to prove the following Theorems from the main paper.

Theorem 8. For an almost balanced model, LP+TRI (the LP relaxation over TRI) is tight.

Theorem 9. For an almost balanced model with special variable s, $F^s_{\text{TRI}}(x)$ is a linear function.

Theorem 11. In an almost balanced model with special variable s, if we fix $q_s = x \in [0, 1]$ and optimize in TRI over all other marginals, then an optimum is achieved with: $q_j \in \{0, x, 1 - x, 1\}$ $\forall j$; all edges (other than to variables which have 0 or 1 singleton marginal) are locking or anti-locking, with no strong frustrated cycles.

We shall first prove Theorem 11 then use it to derive Theorem 9, after which Theorem 8 will easily follow.

Proof of Theorem 11, uses another simple perturbation. As before, we assume an almost attractive model and as justified by §9, we assume no locking edges or variables that have singleton marginal 0 or 1. We shall prove the result by showing that, given these assumptions, the graph must have no variables other than s.

Given the results in §11-12, we have shown that if s is fixed while other marginals are optimized, then an optimum vertex cannot occur unless the perturbation defined in §11 does not exist, i.e. we know that all other variables have a weak edge to s.

Hence, at an optimum vertex, there are no strong edges incident to s. In particular, there are no strong down edges incident to s, and hence there are no strong down edges anywhere in the graph (by Lemma 24).

Since there are no strong down edges, it is now easily checked that the following perturbation (times a sufficiently small ϵ s.t. all constraints which were not tight remain so) up and down preserves all tight LOC and TRI constraints:

$$\begin{cases} s & 0 \\ v \in \mathcal{V} \setminus \{s\} & +1 \\ sv \text{ edge, with } v \in \mathcal{V} \setminus \{s\} & +\frac{1}{2} \\ uv \text{ edge, with } u, v \in \mathcal{V} \setminus \{s\} & +1. \end{cases}$$

Thus, it must be that at a vertex, all variables are either 0, 1 or in a locking component. This completes the proof of Theorem 11. □

Proof of Theorem 9. This is similar to the proof of Theorem 6. As there, we need only prove that $F^s_{\text{TRI}}(x)$ is convex, then linearity follows from Lemma 5.

For any $y \in [0,1]$, consider an $\arg\max$ of $F^i_{\text{TRI}}(y)$ as given by Theorem 11. Partition the variables into 4 exhaustive sets: $A_y = \{j : q_j = 0\}, B_y = \{j : q_j = y\}$, $C_y = \{j : q_j = 1-y\}$ and $D_y = \{j : q_j = 1\}$. Define the function $f_y : [0,1] \to \mathbb{R}$ given by $f_y(x) = f(q(x;y))$ where $q(x;y)$ is defined explicitly for singleton and edge marginals by:

$$q_j(x;y) = \begin{cases} 0 & j \in A_y \\ x & j \in B_y \\ 1-x & j \in C_y \\ 1 & j \in D_y \end{cases}, \qquad q_{ij}(x;y) = \begin{cases} 0 & i \in A_y \text{ or } j \in A_y \\ q_i & j \in D_y \\ q_j & i \in D_j \\ x & i,j \in B_y \\ 1-x & i,j \in C_y \\ 0 & i \in B_y \text{ and } j \in C_y; \text{ or } i \in C_y \text{ and } j \in B_y. \end{cases}$$

It is straightforward to check that always $q(x;y) \in$ TRI (all edges are strong and there are no strong frustrated cycles). Observe that $f_y(x)$ is the linear function achieved by holding fixed the partition of variables A_y, B_y, C_y, D_y that was determined for the $\arg\max$ of the constrained optimum at $q_i = y$. Now $F^i_{\text{TRI}}(x) = \sup_{y\in[0,1]} f_y(x)$, hence is convex. □

Proof of Theorem 8. Given Theorem 9, it must be the case that a global optimum occurs at $s = 0$ or $s = 1$. If we condition on either case, the remaining model is balanced, and the result follows from Theorem 2. □