Supplementary Material

for NIPS submission 'New Outer Bounds on the Marginal Polytope'

1 Mapping from $\mathcal{M}_{\{0,1\}}$ to cut polytope

Given a graph $G = (V, E)$ and $S \subseteq V$, let $\delta(S)$ denote the vector of $\mathbb{R}^E$ defined for $(i, j) \in E$ by,

$$\delta(S)_{ij} = 1 \text{ if } |S \cap \{i, j\}| = 1, \text{ and 0 otherwise.} \tag{1}$$

In other words, the set S gives the cut in G which separates the nodes in S from the nodes in $V \setminus S$; $\delta(S)_{ij} = 1$ when i and j have different assignments. The *cut polytope* projected onto G is the convex hull of the above cut vectors:

$$\text{CUT}^{\square}(G) = \Big\{ \sum_{S \subseteq V_n} \lambda_S \delta(S) \mid \sum_{S \subseteq V_n} \lambda_S = 1 \text{ and } \lambda_S \geq 0 \text{ for all } S \subseteq V_n \Big\}. \tag{2}$$

The cut polytope for the complete graph on n nodes is denoted simply by $\text{CUT}^{\square}_n$. Let $\mathcal{M}_{\{0,1\}}$ denote the marginal polytope for Ising models, which we will call the *binary marginal polytope*:

$$\mathcal{M}_{\{0,1\}} := \big\{ \mu \in \mathbb{R}^d \mid \exists p(\mathbf{x}) \text{ s.t. } \mu_i = E_p[x_i], \mu_{ij} = E_p[x_i x_j] \big\} \tag{3}$$

Suppose that we are given a MRF defined on the graph $G = (V, E)$. To give the mapping between the cut polytope and the binary marginal polytope we need to construct the *suspension graph* of G, denoted ∇G. Let $\nabla G = (V', E')$, where $V' = V \cup \{n+1\}$ and $E' = E \cup \{(i, n+1) \mid i \in V\}$. The suspension graph is necessary because a cut vector $\delta(S)$ does not uniquely define an assignment to the vertices in G – the vertices in S could be assigned either 0 or 1. Adding the extra node allows us to remove this symmetry.

Definition 1. The linear bijection ξ from $\mu \in \mathcal{M}_{\{0,1\}}$ to $x \in \text{CUT}^{\square}(\nabla G)$ is given by $x_{i,n+1} = \mu_i$ for $i \in V$ and $x_{ij} = \mu_i + \mu_j - 2\mu_{ij}$ for $(i, j) \in E$.

Thus, the marginal polytope for binary pairwise MRFs[1] is isomorphic to the cut polytope [2, 1, 3].

2 Projection graphs

Theorem. *The projection Ψ_π given by the* single projection graph G_π *is surjective.*

Proof. Since Ψ_π is a linear map, it suffices to show that, for every extreme point $\mu' \in \mathcal{M}_{\{0,1\}}$, there exists some $\mu \in \mathcal{M}$ such that $\Psi_\pi(\mu) = \mu'$. The extreme points of $\mathcal{M}$ and $\mathcal{M}_{\{0,1\}}$ correspond one-to-one with assignments $\mathbf{x} \in \chi^n$ and $\{0, 1\}^n$, respectively. Given an extreme point $\mu' \in \mathcal{M}_{\{0,1\}}$, let $\mathbf{x}'(\mu')$ be its corresponding assignment. For each variable i, choose some $s \in \chi_i$ such that $\pi_i(s) = \mathbf{x}'(\mu')_i$, and assign $\mathbf{x}_i(\mu') = s$. The existence of such s is guaranteed by our construction of π (surjective). Defining $\mu = E[\phi(\mathbf{x}(\mu'))] \in \mathcal{M}$, we have that $\Psi_\pi(\mu) = \mu'$. $\square$

2.1 Example

Consider the *single projection graph* shown in Figure 3 and the corresponding cycle inequality (see eqns. 7-9 in paper), where F is illustrated by cut edges. The following is an example of an extreme point of LOCAL(G) which is violated by this cycle inequality:

$$\begin{array}{llll} \mu_{i;0} = \mu_{i;3} = .5, & \mu_{j;1} = \mu_{j;2} = .5, & \mu_{m;1} = \mu_{m;3} = .5, & \mu_{k;2} = \mu_{k;3} = .5 \\ & \mu_{ij;02} = \mu_{ij;31} = .5, & \mu_{im;01} = \mu_{im;33} = .5 & \\ & \mu_{jk;13} = \mu_{jk;22} = .5, & \mu_{mk;13} = \mu_{mk;32} = .5 & \end{array} \tag{4}$$

[1] In the literature on cuts and metrics (e.g. [3]), this is called the *correlation polytope*, denoted by $\text{COR}^{\square}_n$.

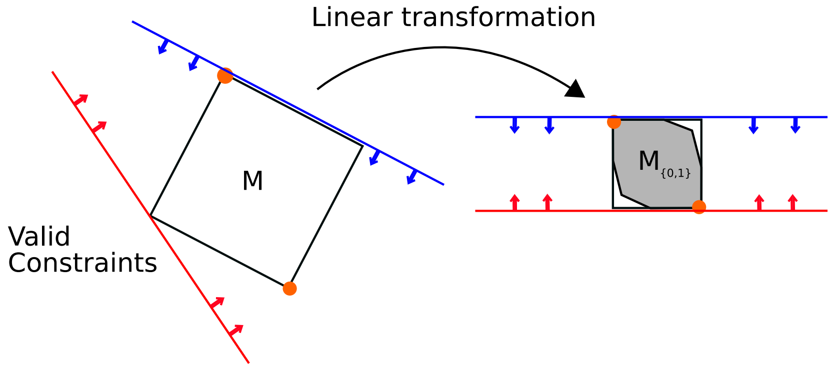

Figure 1: Illustration of projection from the marginal polytope of a non-binary MRF to the binary marginal polytope of a different graph. All valid inequalities for the binary marginal polytope yield valid inequalities for the marginal polytope, though not all will be facets. These projections map vertices to vertices, but the map will not always be onto.

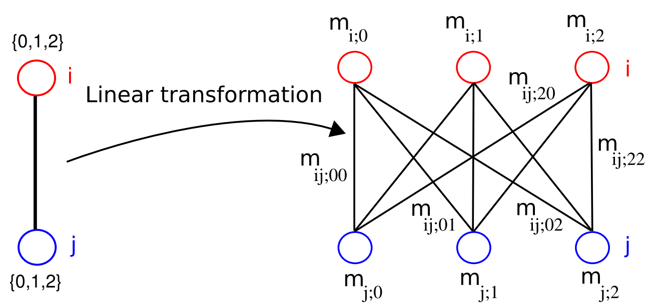

Figure 2: Illustration of the *$k-$projection graph* for one edge $(i,j) \in E$, where $\chi_i = \{0,1,2\}$. The nodes and (some of) the edges are labeled with the values given to them by the linear mapping, e.g. $\mu_{i;0}$ or $\mu_{ij;02}$.

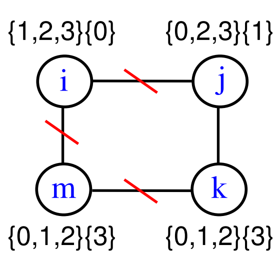

Figure 3: Illustration of the *single projection graph* G_π for a square graph, where all variables have states $\{0,1,2,3\}$. The three oblique lines indicate an invalid cut, since every cycle must be cut an even number of times.

3 Complexity

A natural question that is raised in this work is whether it is possible to efficiently test whether a point is in the marginal polytope.

Theorem. *The following decision problem is NP-hard: given a vector* $\mu \in \mathbb{R}_+^{V_n \cup E_n}$*, decide if* $\mu \in \mathcal{M}$*.*

Proof. Using the linear bijection ξ, the problem for $\mathcal{M}_{\{0,1\}}$ is equivalent to the decision problem for $\mathrm{CUT}_n^{\square}$ (the same as ℓ_1-embeddability). The latter is shown to be NP-complete in [3]. Membership in $\mathcal{M}_{\{0,1\}}$ trivially reduces to membership in $\mathcal{M}$. $\square$

References

[1] F. Barahona. On cuts and matchings in planar graphs. *Mathematical Programming*, 60:53–68, 1993.

[2] F. Barahona and A. R. Mahjoub. On the cut polytope. *Mathematical Programming*, 36:157–173, 1986.

[3] M. M. Deza and M. Laurent. *Geometry of Cuts and Metrics*, volume 15 of *Algorithms and Combinatorics*. Springer, 1997.