Dynamical Optimal Transport on Discrete Surfaces

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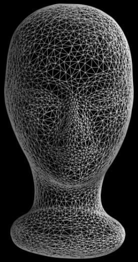
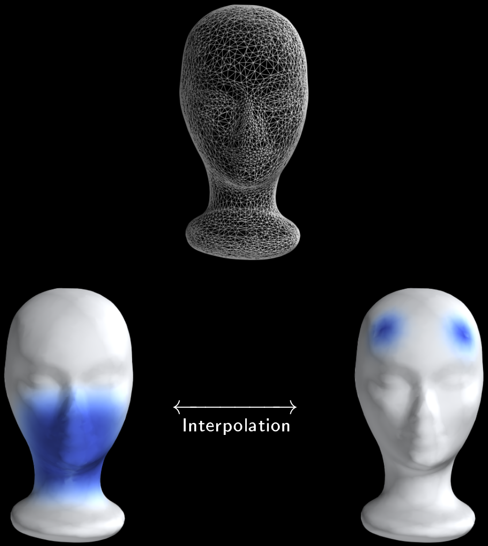
SIGGRAPH Asia 2018



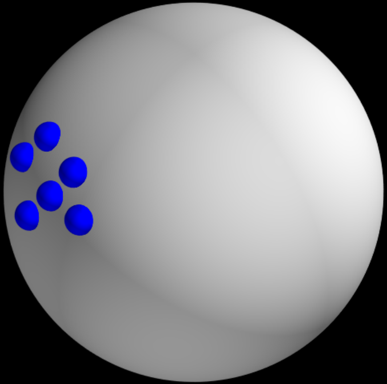
Fixed surface $\mathcal{M}$.
Given by a **triangle mesh**.

Initial $(t = 0)$ $\bar{\mu}_0$

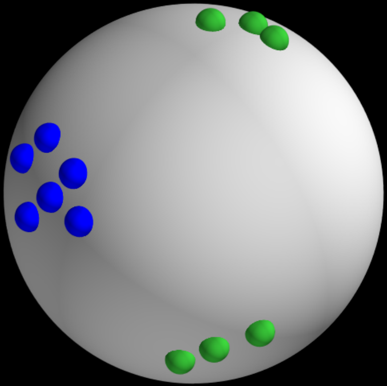
Final $(t = 1)$ $\bar{\mu}_1$



Initial $(t = 0)$		Final $(t = 1)$
$\bar{\mu}_0$	μ_t	$\bar{\mu}_1$



$$\bar{\mu}_0 = \sum_i a_i \delta_{x_i},$$



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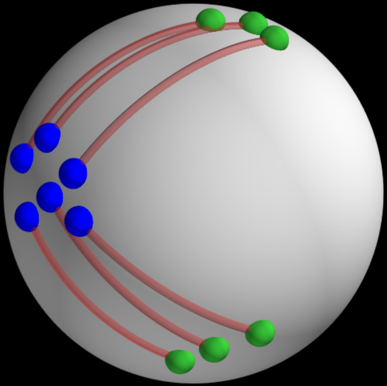


$$\bar{\mu}_0 = \sum_i a_i \delta_{x_i}, \;\; \bar{\mu}_1 = \sum_j b_j \delta_{y_j}$$

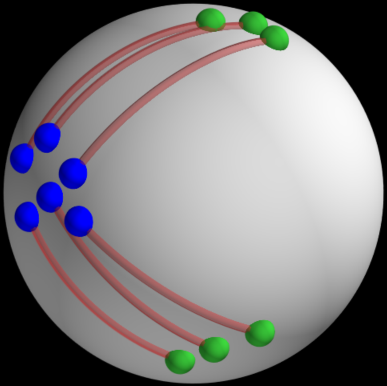
Solve the **Linear Programming problem**

$$\min_{\pi} \sum_{i,j} \pi_{ij} \; d(x_i, y_j)^2$$

with conservation of mass constraints

$$\begin{cases} \sum_j \pi_{ij} = a_i, \\ \sum_i \pi_{ij} = b_j, \end{cases}$$

Villani, Cédric. *Topics in optimal transportation*. 2003.



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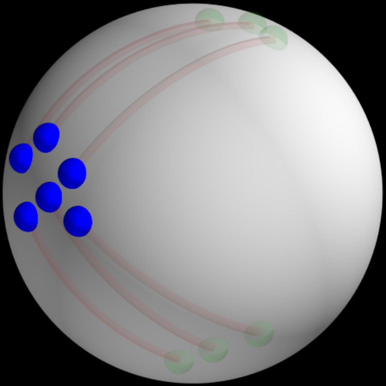

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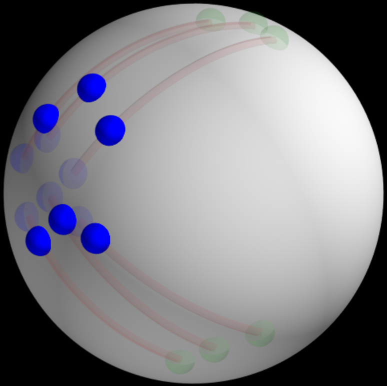

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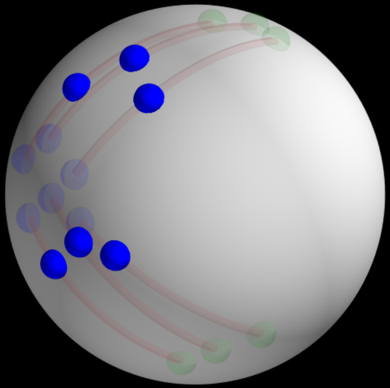

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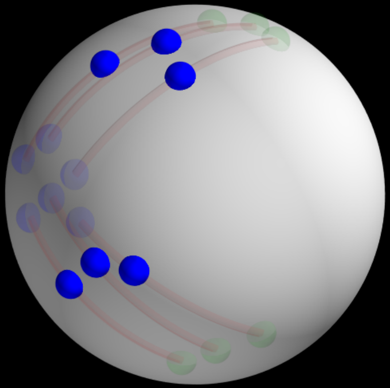

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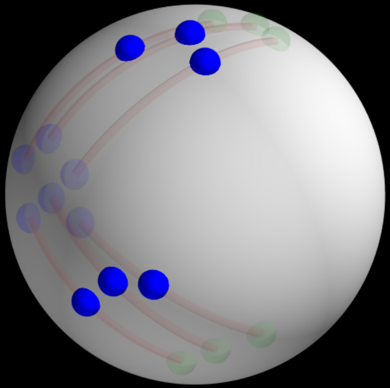

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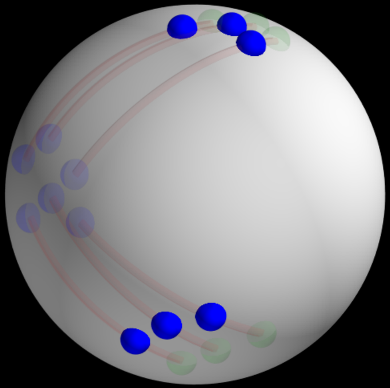

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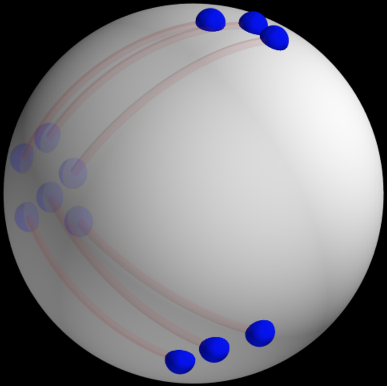

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We compute the whole interpolation in one single convex optimization problem

Primal Problem

Unknown : $\mu : \underbrace{[0,1]}_{\text{time}} \times \underbrace{\mathcal{M}}_{\text{space}} \to \mathbb{R}_+$

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where $\mathbf{m} = \mu \mathbf{v}$ is the momentum, under the constraints

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Dual Problem

Unknown : $\varphi : [0,1] \times \mathcal{M} \to \mathbb{R}$

$$\max_{\varphi} \left\{ \int_{\mathcal{M}} \varphi(1,\cdot)\bar{\mu}_1 - \int_{\mathcal{M}} \varphi(0,\cdot)\bar{\mu}_0 \right\}$$

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$$\partial_t \varphi + \frac{1}{2} \left| \nabla \varphi \right|^2 \leqslant 0.$$

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In the continuous world

Static OT $=$ Dynamical OT

Maas, Jan. *Gradient flows of the entropy for finite Markov chains*. 2011.

On discrete surfaces

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Our contribution : discretization and implementation of dynamical OT

- ∇, $\nabla\cdot$ on a curved surface ;
- Average to go from faces ($\mathbf{m}$) to vertices (μ) to compute $\displaystyle\iint \frac{|\mathbf{m}|^2}{2\mu}$;
- Preserving the Riemannian structure of the Wasserstein space.

Code available at `https://github.com/HugoLav/DynamicalOTSurfaces`

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We have a single finite-dimensional convex (SOCP) optimization problem :

- Size $\sim N \times M$ (N temporal grid, M number of vertices).
- Alternating Direction Method of Multipliers (only non local step : space-time fixed Poisson problem)
- $N = 30$, 5000 vertices : ~ 5 minutes.

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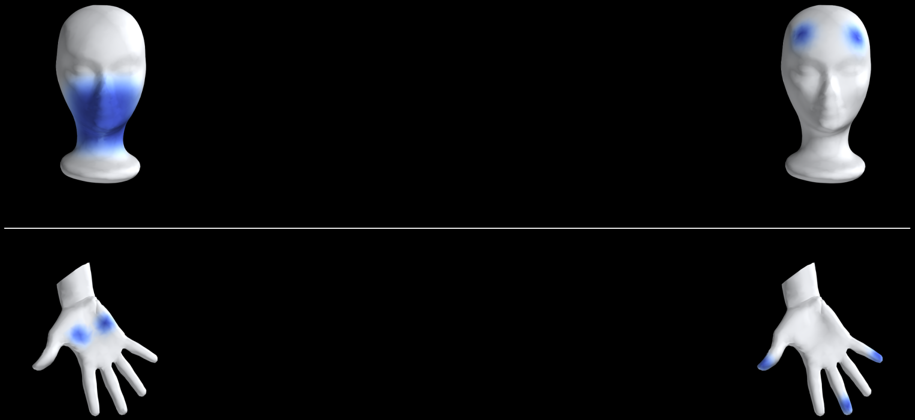

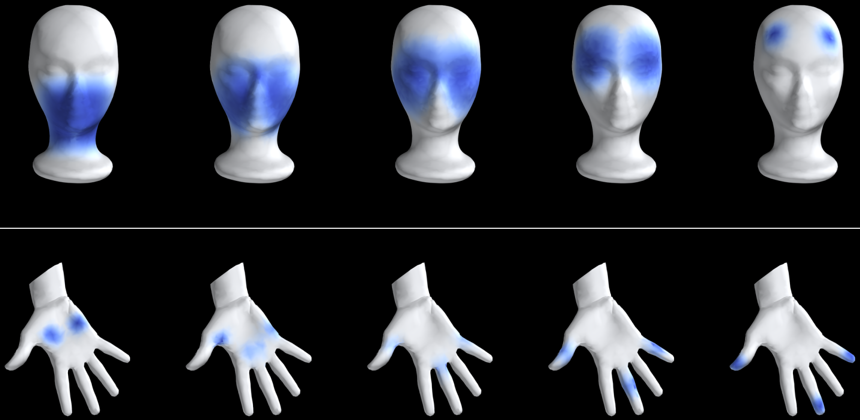

Positivity and mass preservation are enforced **automatically**

Adding $+\frac{\alpha}{2}\int_0^1 \int_{\mathcal{M}} \mu_t^2 \,\mathrm{d}t$ in the (primal) objective functional.

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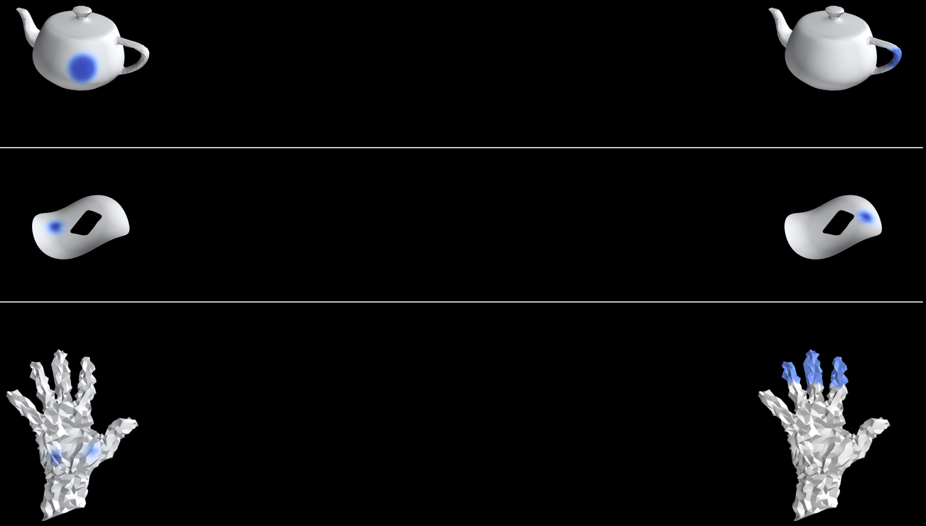
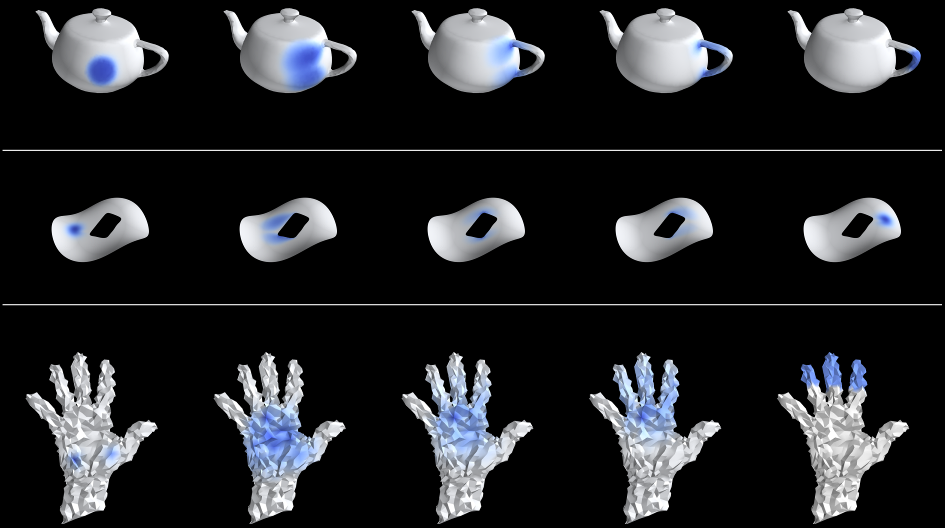
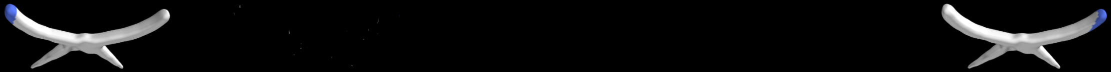
- Still convex, only a few lines of codes to add.
- No problem in taking $\alpha = 0$.

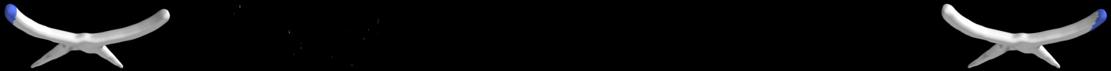




Comparison with entropic OT [Solomon et al, 2015]



Our method



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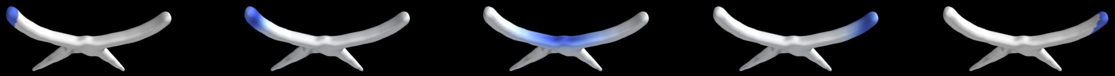

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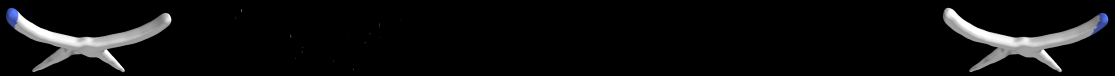

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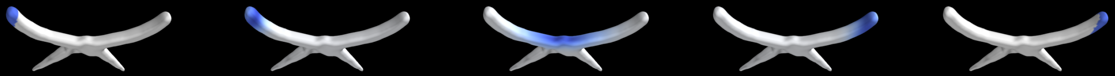

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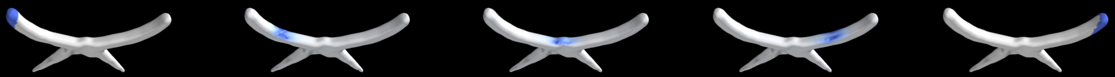

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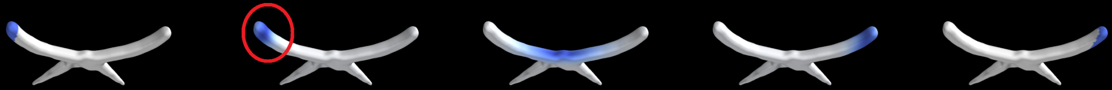

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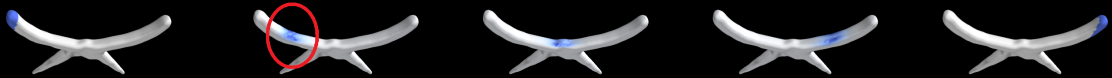

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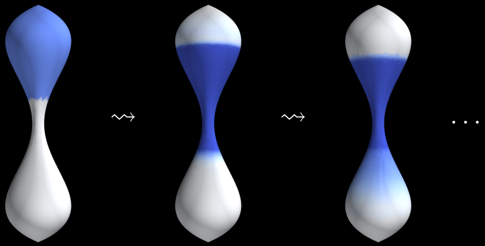
F functional on the space of densities, we want to compute the **gradient flow**

$$\dot{\mu} = -\nabla_W F(\mu).$$

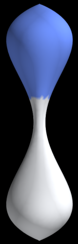
If μ^k is known, to compute μ^{k+1} we use the JKO scheme, same complexity as before.

$$\mu^{k+1} \text{ minimizes } \begin{cases} \iint \frac{|\mathbf{m}|^2}{2\mu} + \tau F(\mu^{k+1}) \\ \partial_t \mu + \nabla \cdot \mathbf{m} = 0 \\ \mu_0 = \mu^k \\ \mu_1 = \mu^{k+1} \end{cases}$$



Jordan, Richard, David Kinderlehrer, and Felix Otto. *The variational formulation of the Fokker–Planck equation.* 1998.

F is gravitational energy + constraint for the density to stay below a threshold :



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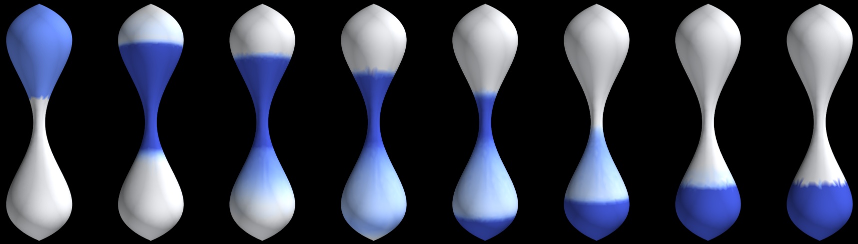

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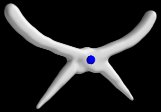

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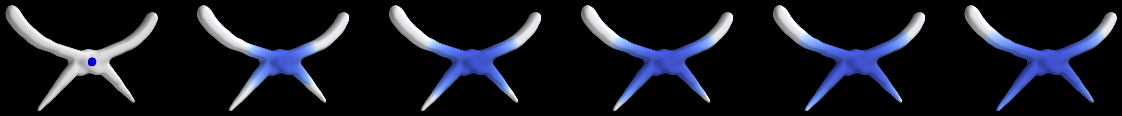

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- Only need to know how to compute ∇ on a surface.
- Yet complex geometries are handled.

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Thank you for your attention