Fast Planar Harmonic Deformations with Alternating Tangential Projections

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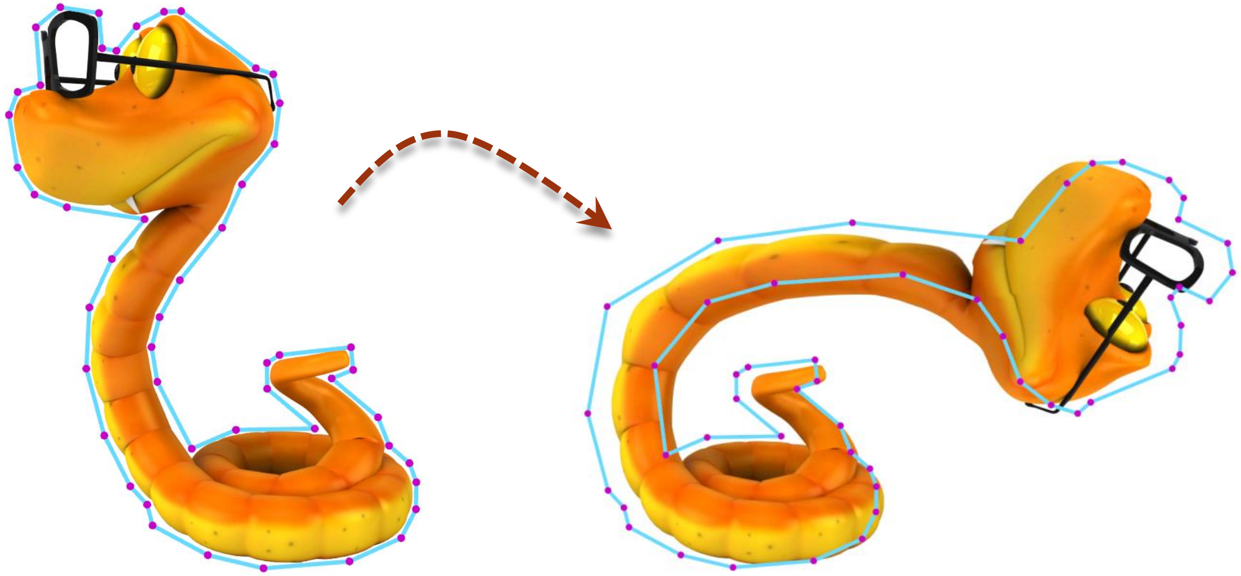
The Mapping Problem

$$f: \Omega \rightarrow \mathbb{R}^2$$

- Desirable properties:
  - Locally-injective
  - Bounded conformal distortion
  - Bounded isometric distortion
  - Real-time



Previous Work

- **Cage based methods (barycentric coords):**

  [Hormann and Floater 2006]

  [Joshi et al.2007]

  [Lipman et al. 2007]

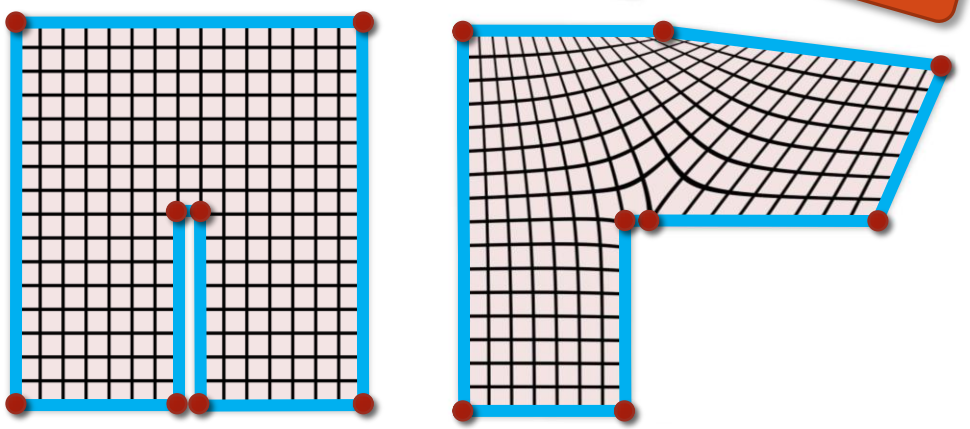
  [Weber et al. 2011]

  [Weber et al. 2009]

  …

  ✓ Fast

  ✗ Not BD



- **Bounded distortion:**

  [Lipman 2012]

  [Kovalsky at al. 2015]

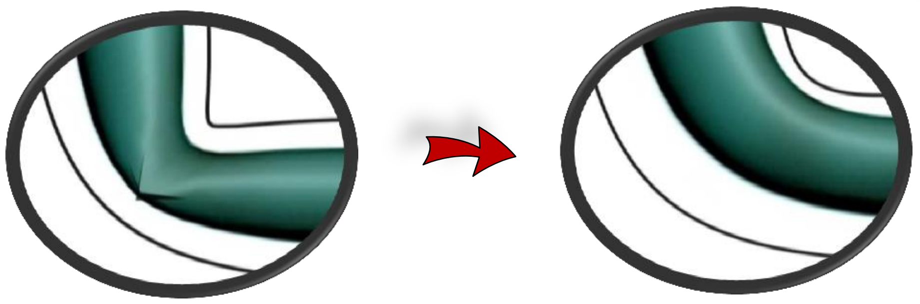
  [Chen and Weber 2015]

  [Levi and Weber 2016]

  …

  ✗ Slow

  ✓ BD



Notations

- **Planar mapping:** $f : \Omega \to \mathbb{R}^2$
- **Jacobian:**

$$J_f = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$$

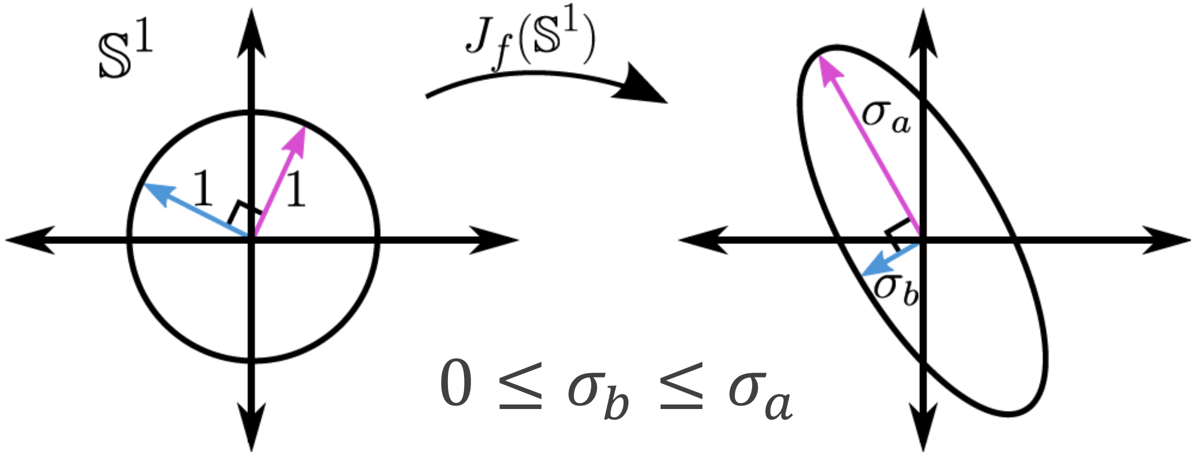
Similarity *Anti-similarity*

- **Complex Wirtinger derivatives:**

$$f_z = a + ib$$

$$f_{\bar{z}} = c + id$$

- **Distortion measures:**
  - Singular values of J_f



$$0 \le \sigma_b \le \sigma_a$$

$$\sigma_a = |f_z| + |f_{\bar{z}}|$$

$$\sigma_b = |f_z| - |f_{\bar{z}}|$$

Bounded Distortion Harmonic Mappings

- **The BD space:** $\forall z \in \Omega$

conformal $\quad k(z) = \frac{\sigma_a - \sigma_b}{\sigma_a + \sigma_b} = \frac{|f_{\bar{z}}|}{|f_z|} \leq C_k$

isometric $\quad \left.\begin{array}{l} \sigma_a(z) = |f_z| + |f_{\bar{z}}| \leq C_a \\ \sigma_b(z) = |f_z| - |f_{\bar{z}}| \geq C_b \end{array}\right\} \boldsymbol{\tau = \max\left(\sigma_a, \frac{1}{\sigma_b}\right)}$

- Non-convex space

- Harmonic mapping ➡ enforce bounds only on $\partial\Omega$ [Chen and Weber 2015]

The $\mathcal{L}_\nu$ Space [Levi and Weber 2016]

- Change of variables:

BD ➡ $\mathcal{L}_\nu$

$$l = log(f_z)$$

$$\nu = \frac{\overline{f_{\bar{z}}}}{f_z}$$

$\mathcal{L}_\nu$ ➡ BD

$$f_z = e^l$$

$$f_{\bar{z}} = \overline{\nu e^l}$$

- BD homeomorphic to $\mathcal{L}_\nu$

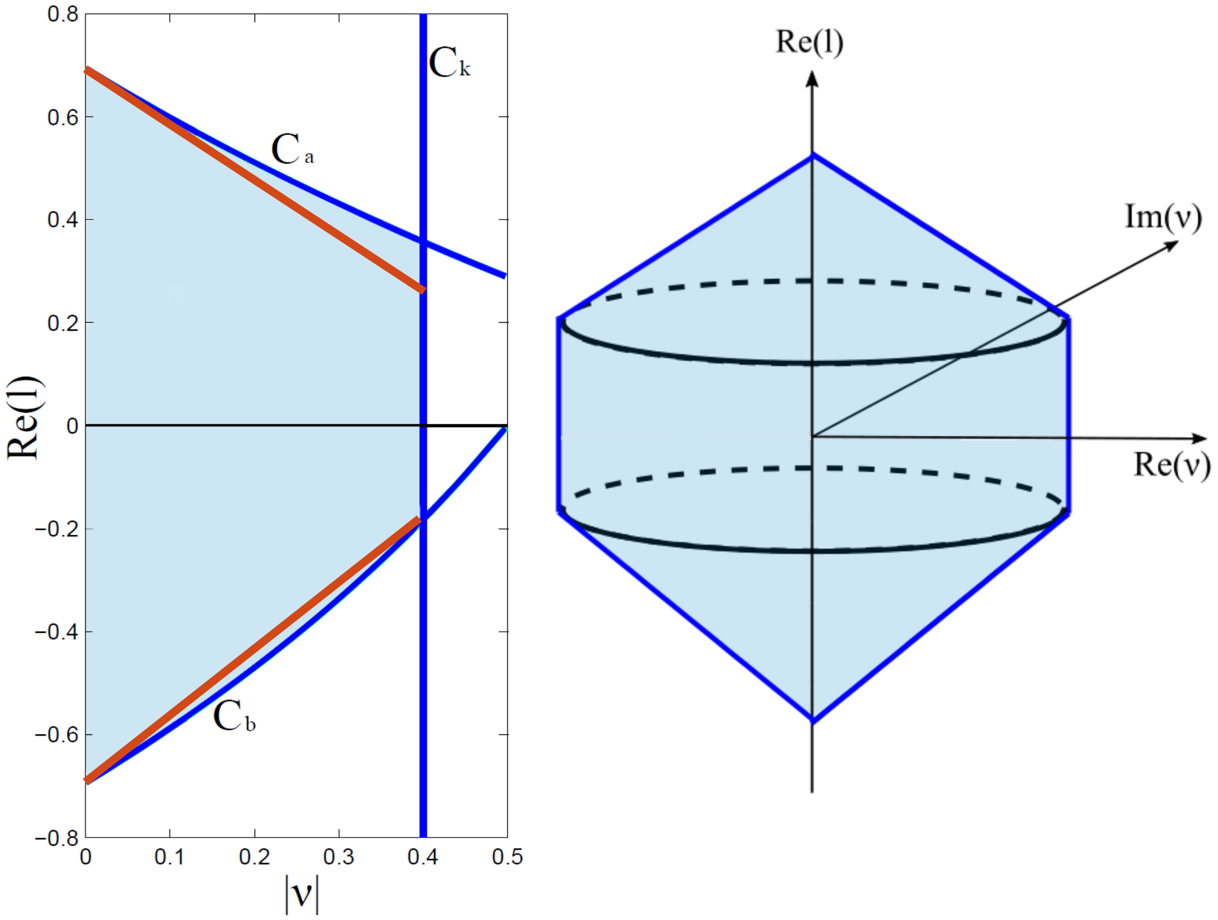
The $\mathcal{L}_\nu$ Space

- Near convex space

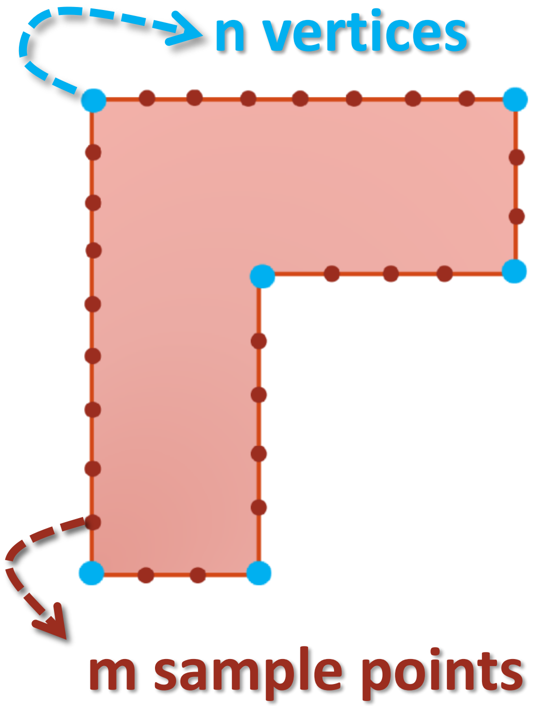
$$\forall w \in \partial\Omega$$

$$k(w) = |\nu(w)| \le C_k$$
$$\sigma_a(w) = e^{Re(l(w))}(1 + |\nu(w)|) \le C_a$$
$$\sigma_b(w) = e^{Re(l(w))}(1 - |\nu(w)|) \ge C_b$$

Convex



Discretization



- Enforce distortion constraints on m densely sampled points
- Use Cauchy complex barycentric coordinate :

$$l(z) = \sum_{j=1}^{n} s_j C_j(z) \quad \& \quad \nu(z) = \sum_{j=1}^{n} t_j C_j(z) \qquad s_j, t_j \in \mathbb{C}$$

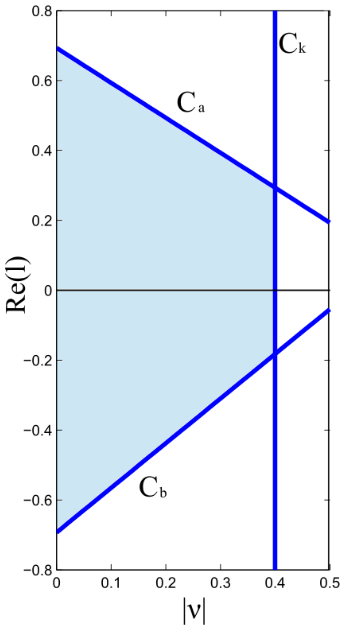
- Subspace of holomorphic functions
- 4n-dimensional

Affine

Our problem

Convex

4m-dimensional



Bounded distortion

$\cap$

Affine

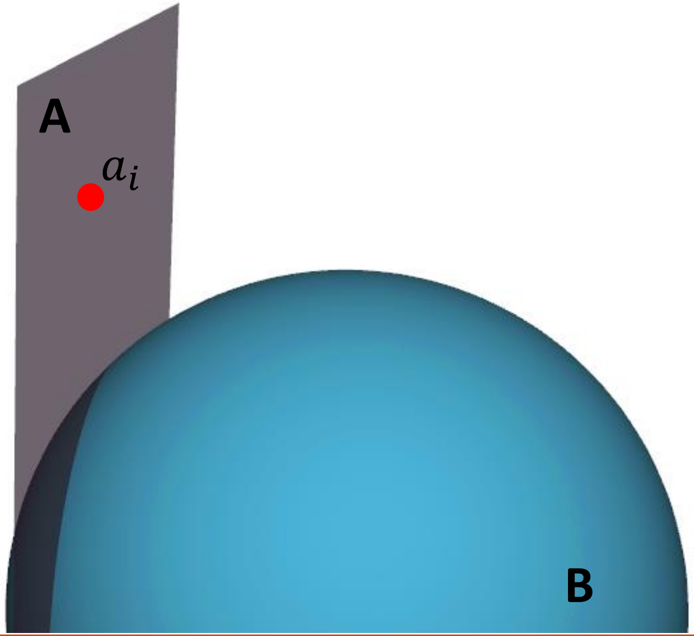
4n-dimensional

Harmonic mapping

convex subspace of $\mathbb{R}^{4m}$

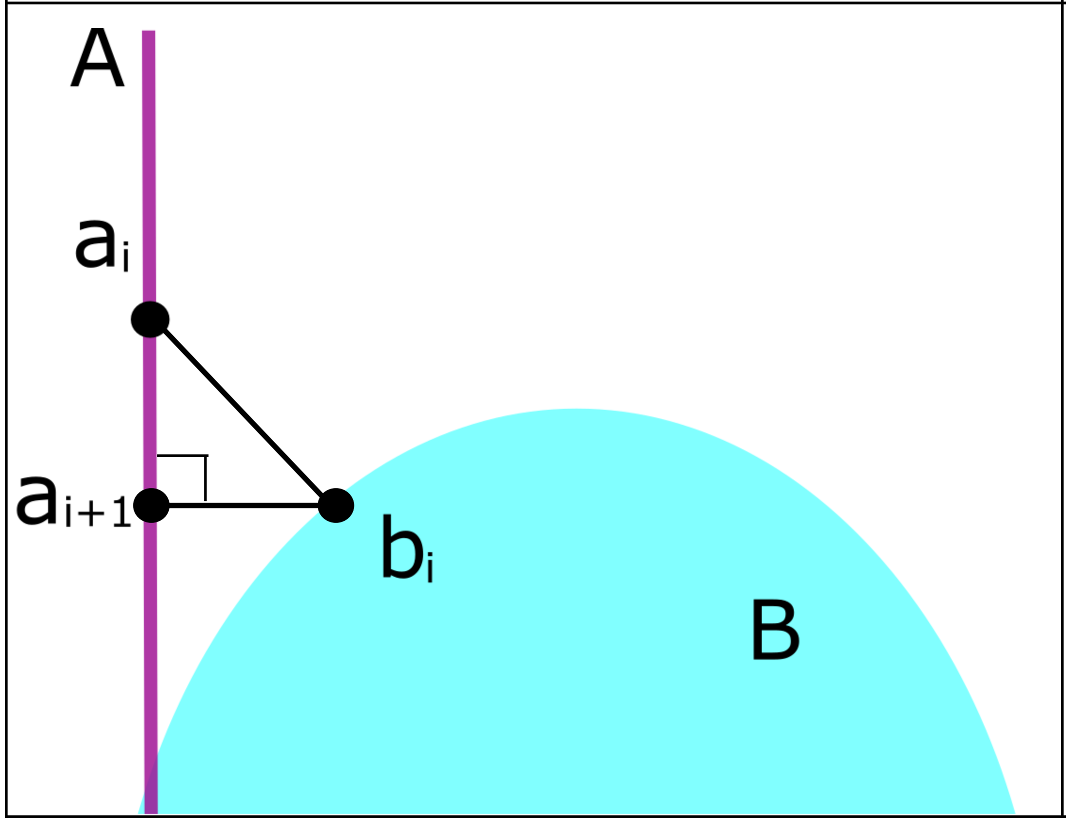
Our problem

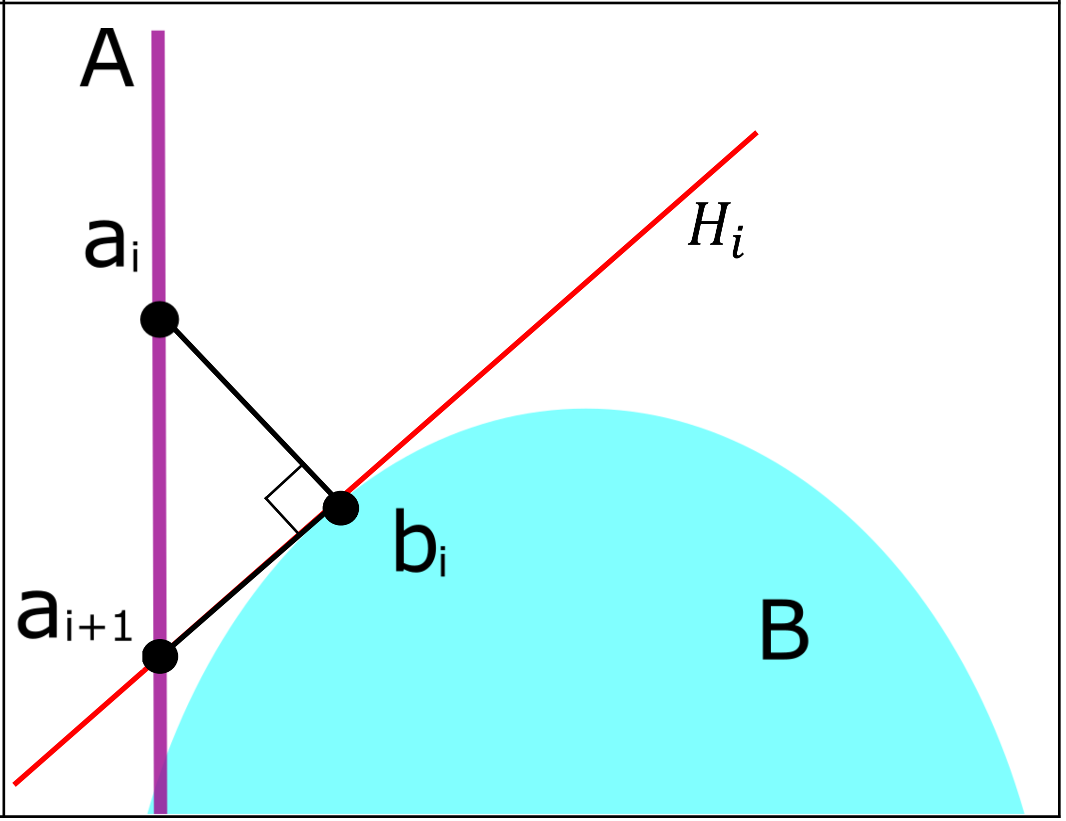
- **Input**: l and v values from cage data
- Find the closest point in the intersection of an affine space and a convex space



Alternating Projections

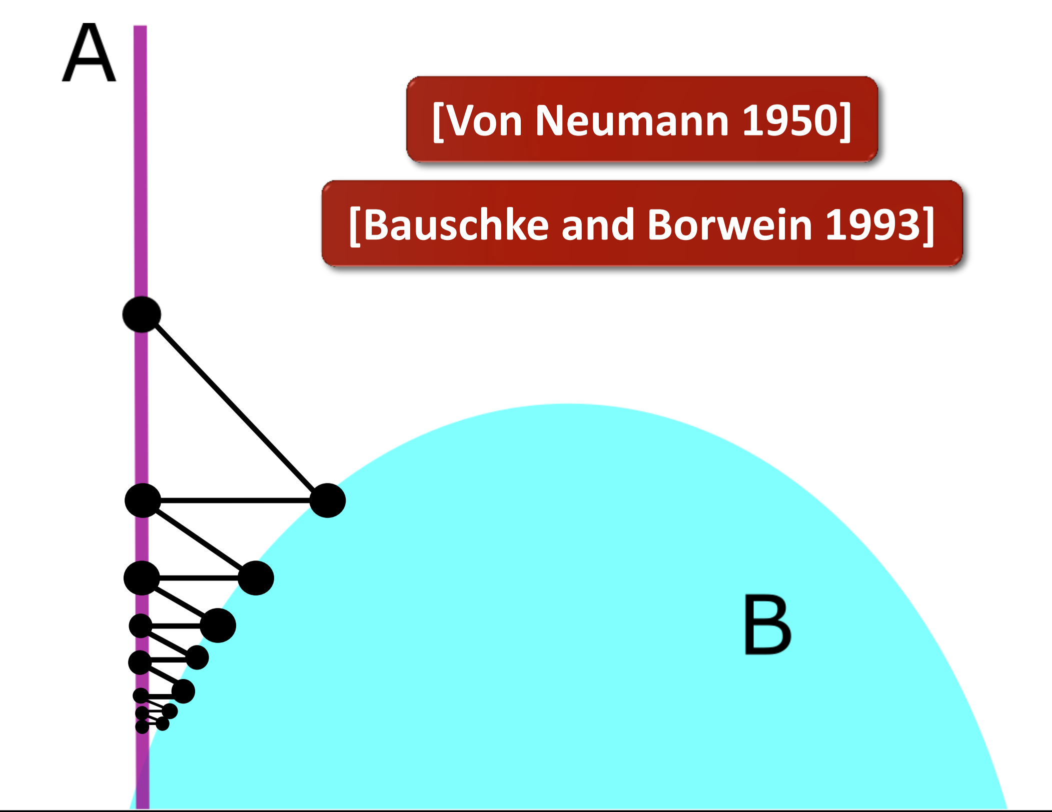
MAP	ATP



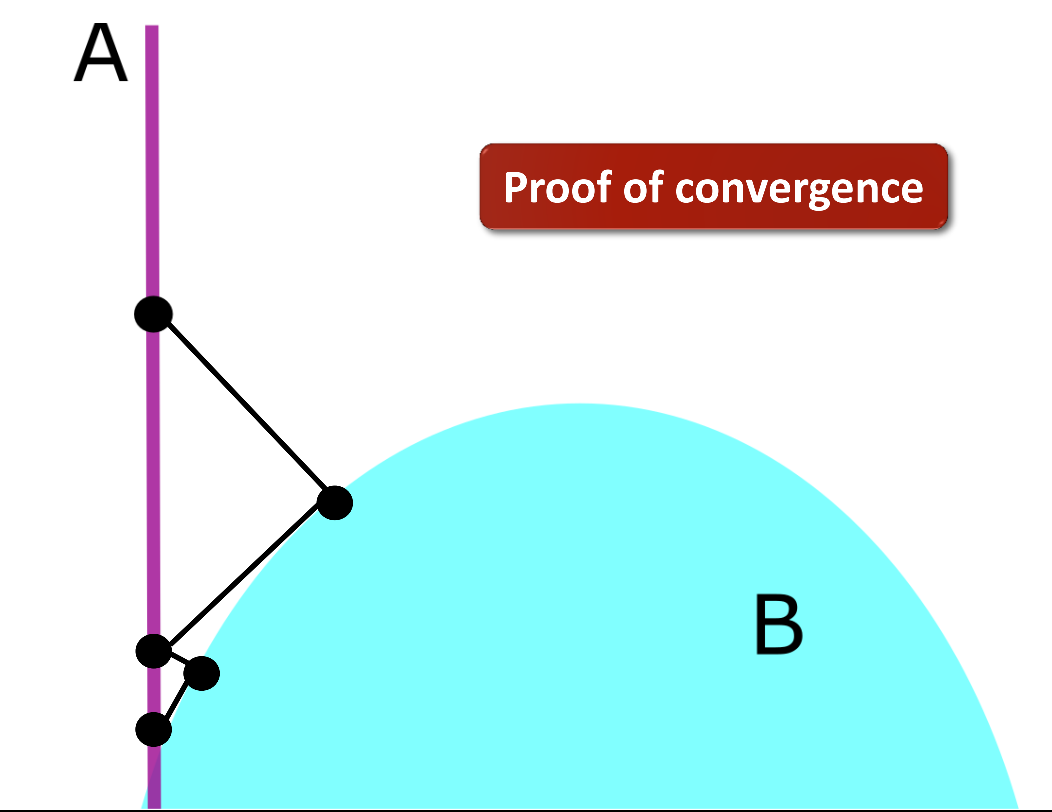


Alternating Projections

MAP



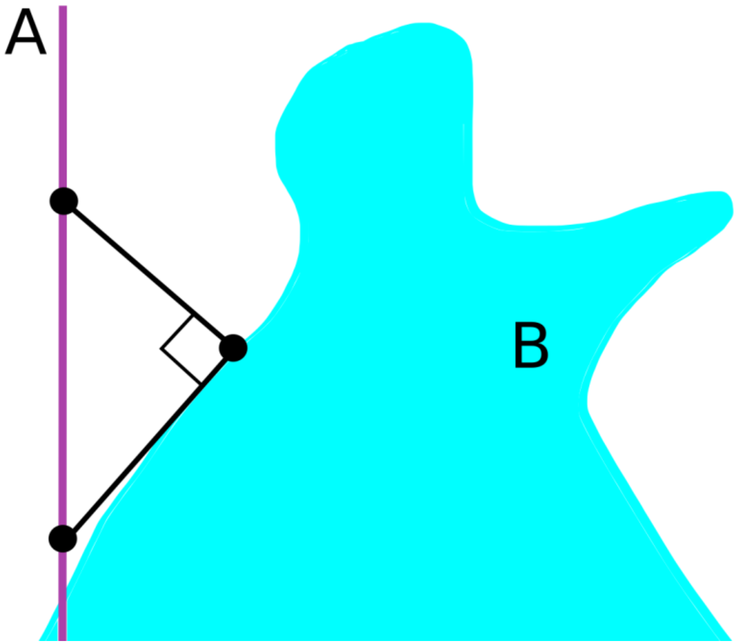
ATP



Large-Scale Bounded Distortion Mappings

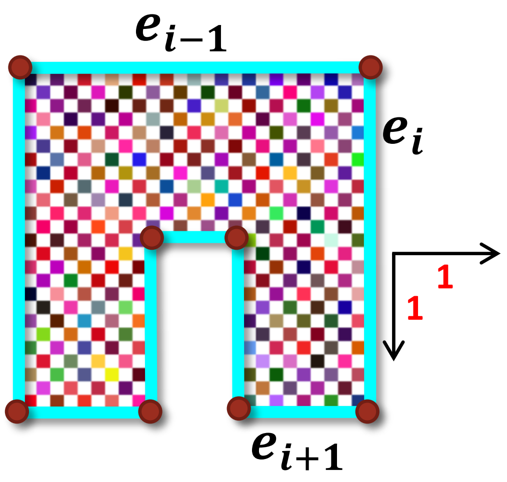
[Kovalsky et al. 2015]

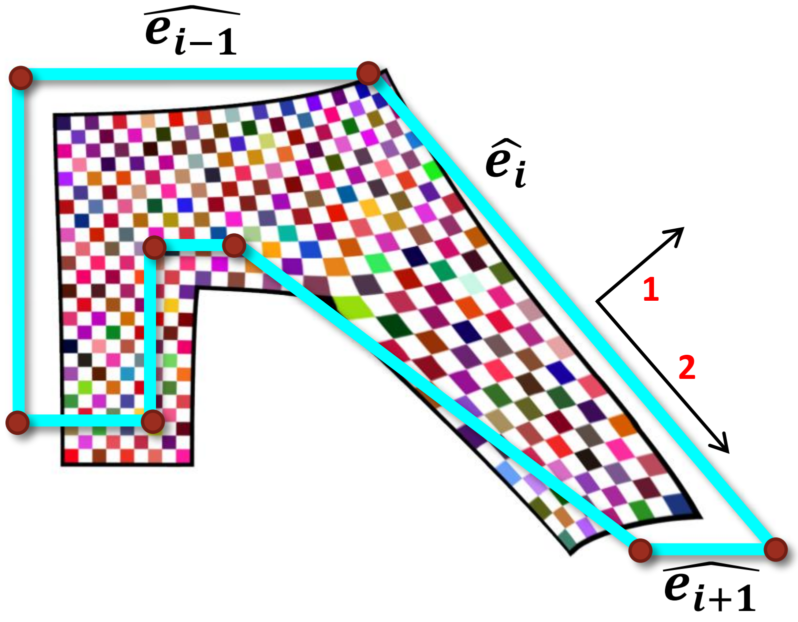
- Alternating Projections between an affine space and non-convex space
- No convergence guarantees
- Upon convergence, not necessarily locally injective
- Only bounds the conformal distortion and not isometric



Gathering Input Data

- Extract l and v values from cage data
- Linear transformations $e_i \longmapsto \widehat{e_i}$ that preserves the unit normal

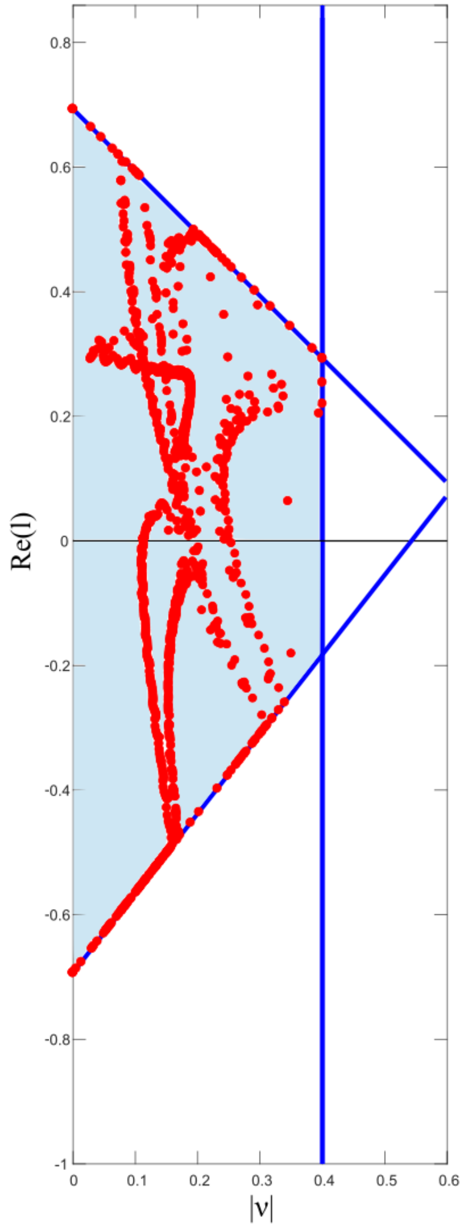




Implementation

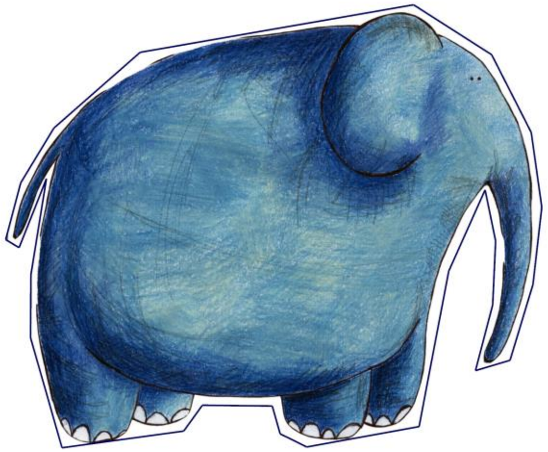
- **Local**:
  - Project each sample point to the bounded distortion space
  - GPU kernel

- **Global**:
  - Linear + fixed left hand side
  - GPU - Matrix-Vector products using cuBLAS



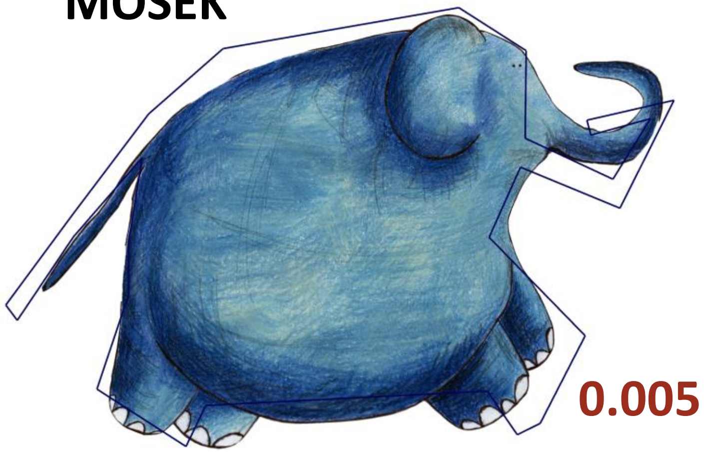
Results

Near-optimality of alternating projection methods



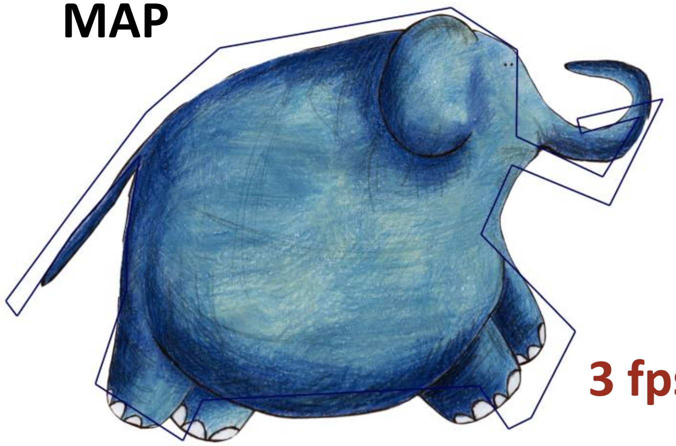
source

MOSEK



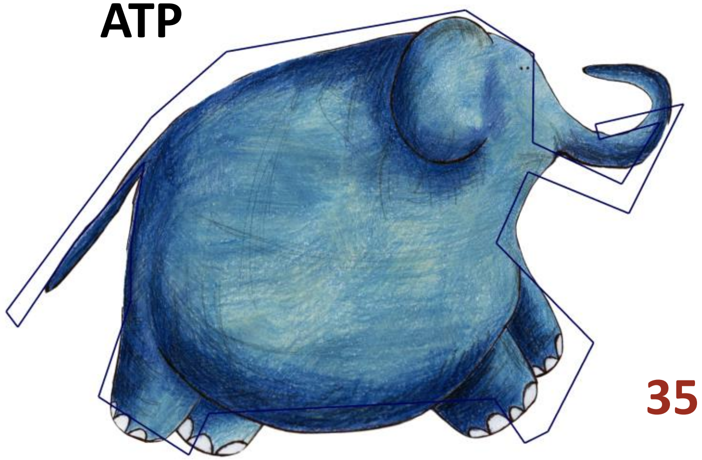
0.005 fps

MAP



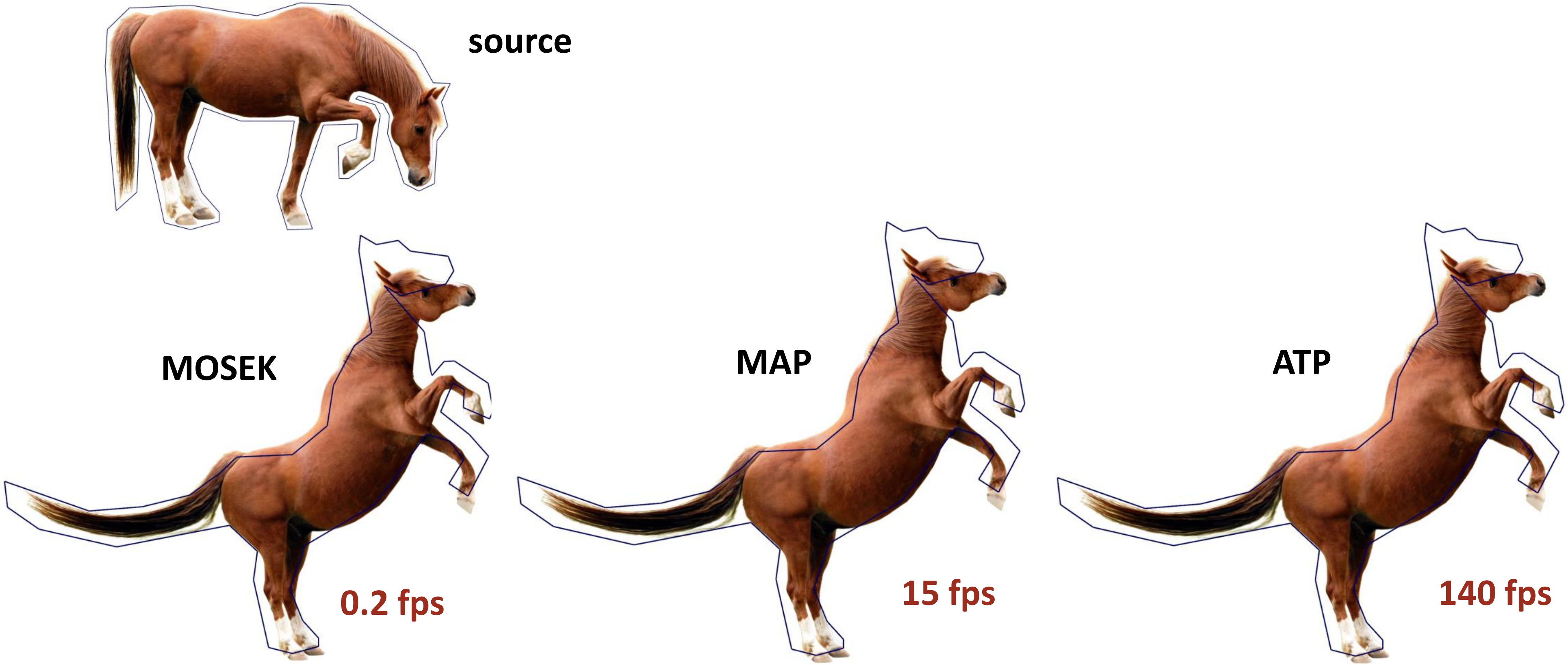
3 fps

ATP

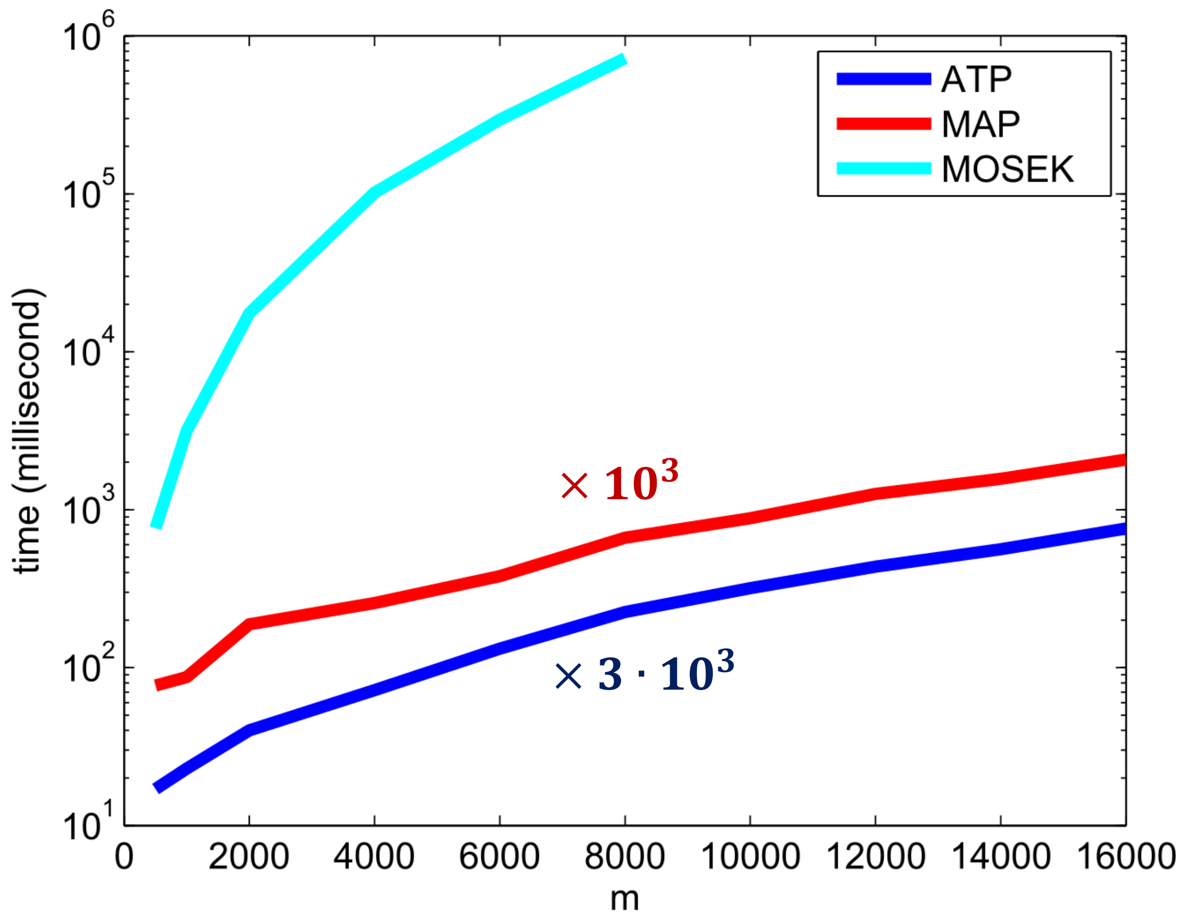


35 fps

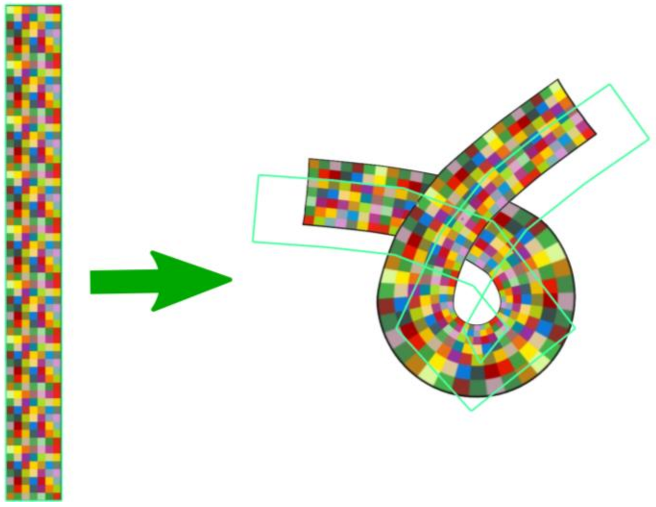
Near-optimality of alternating projection methods

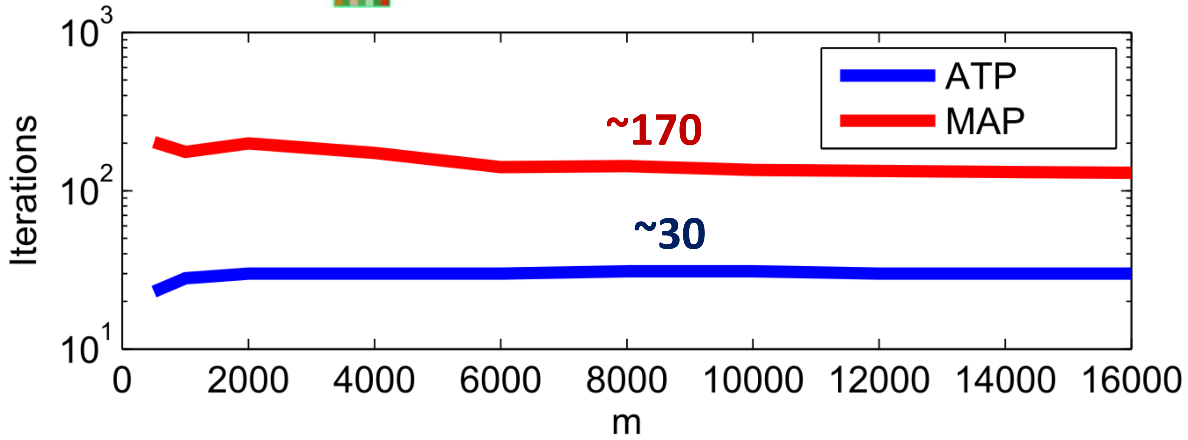


Speedup

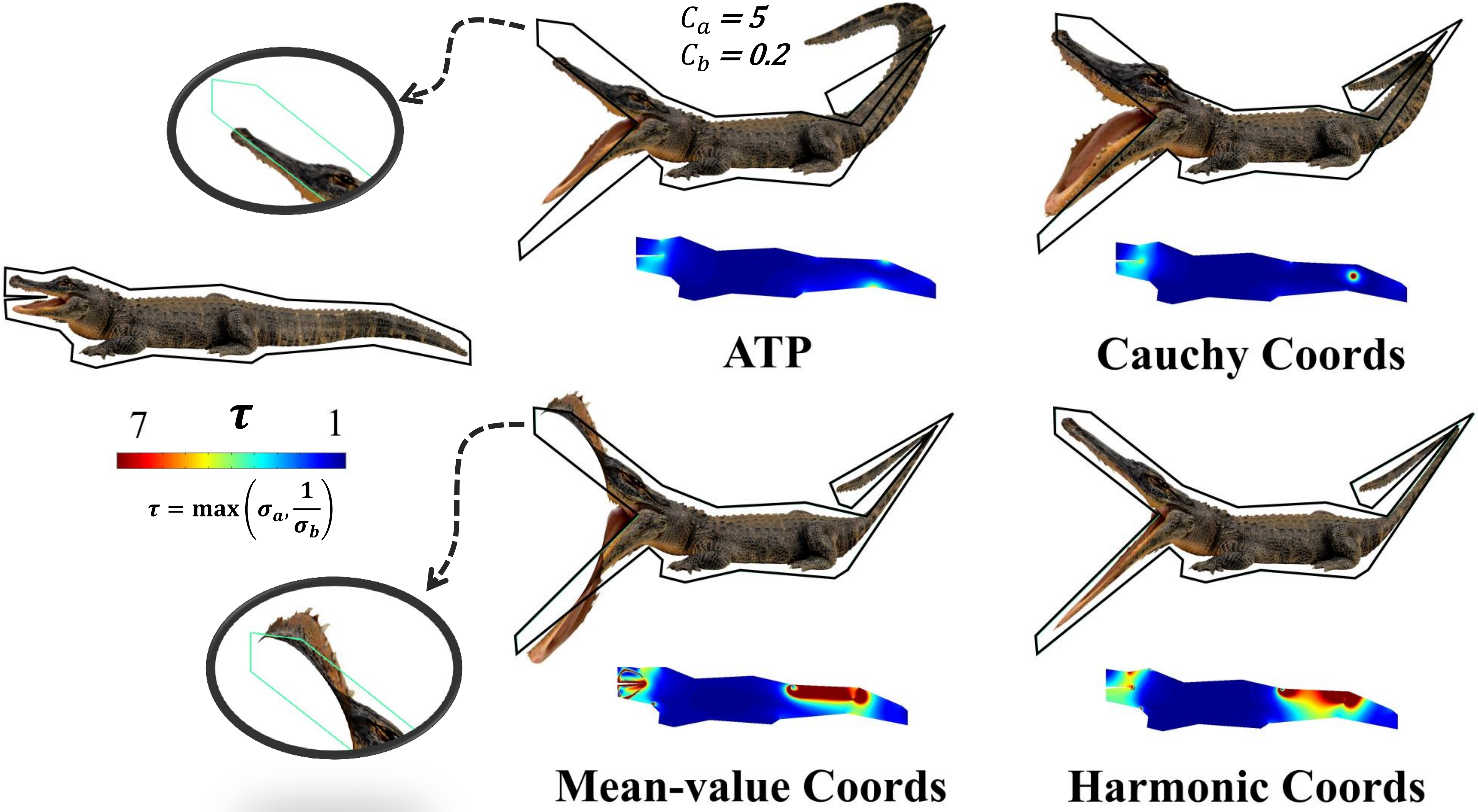


10^6
10^5
10^4
10^3
10^2
10^1
time (millisecond)
ATP
MAP
MOSEK
$\times 10^3$
$\times 3 \cdot 10^3$
0
2000
4000
6000
8000
10000
12000
14000
16000
m

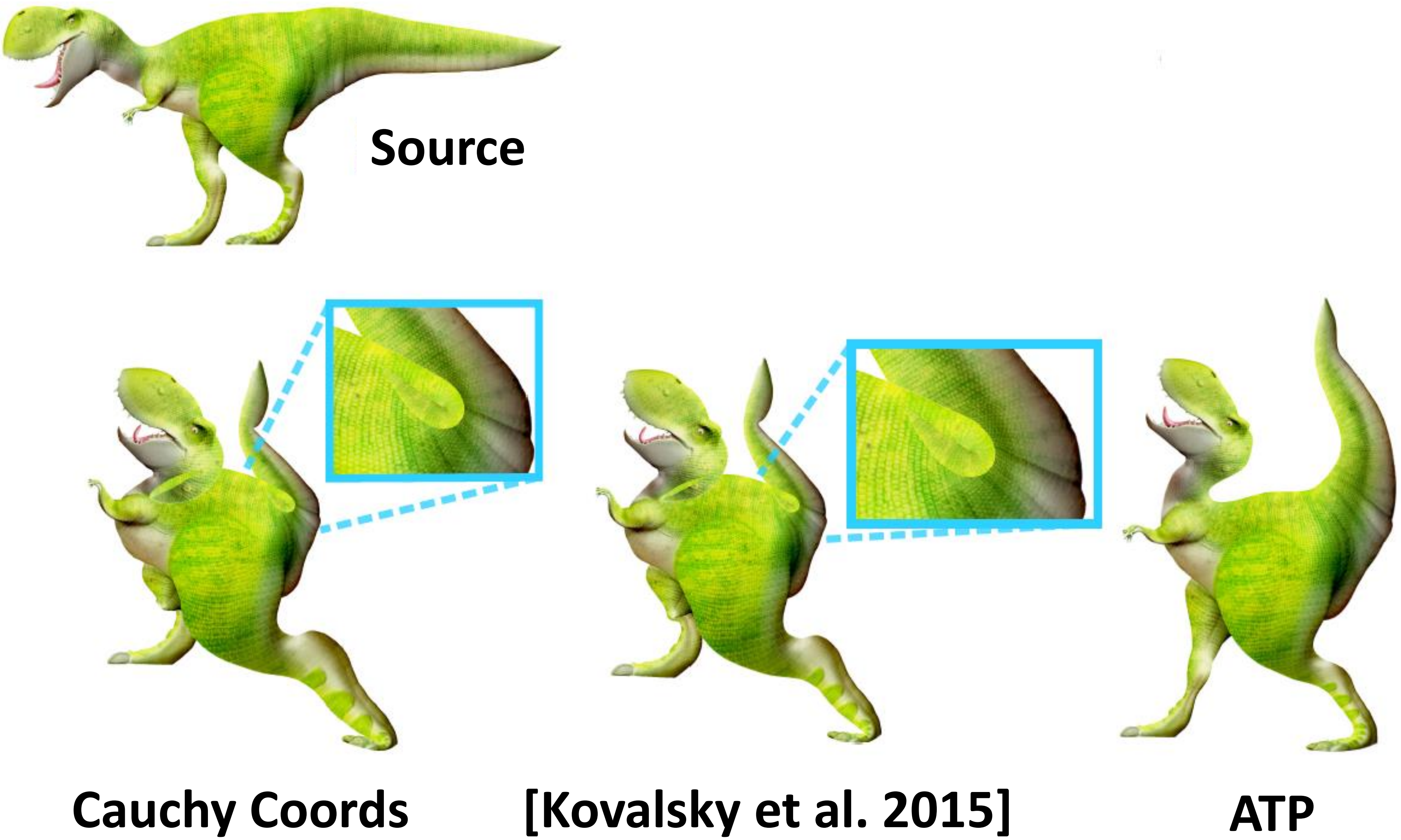




10^3
10^2
10^1
Iterations
ATP
MAP
~170
~30
0
2000
4000
6000
8000
10000
12000
14000
16000
m



$C_a = 5$
$C_b = 0.2$
ATP
Cauchy Coords
7
τ
1
$\tau = \max\left(\sigma_a, \frac{1}{\sigma_b}\right)$
Mean-value Coords
Harmonic Coords


Source
Cauchy Coords
[Kovalsky et al. 2015]
ATP

Summary

- Planar deformation
- GPU accelerated – speedup of 3×10^3
- Guaranteed local injectivity and bounded distortion
- Homeomorphism of BD and $\mathcal{L}_\nu$
- General proof of convergence

- **Future Work:**
  - Positional constraints
  - Extension to 3D / parametrization of surfaces