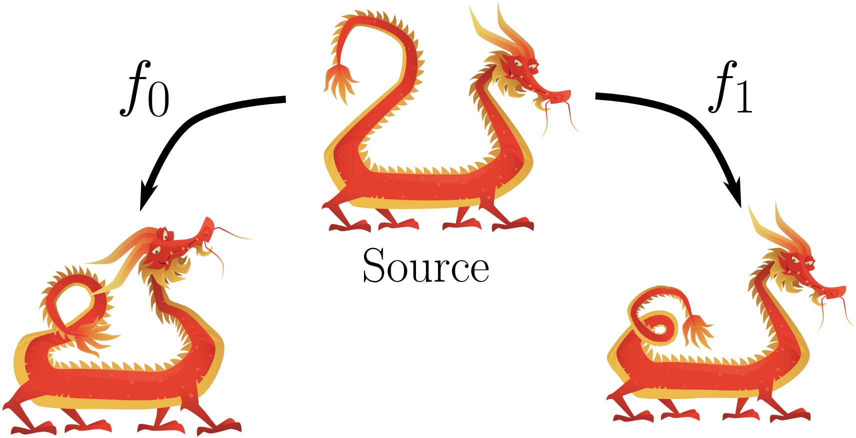
Bounded Distortion Harmonic Shape Interpolation

Edward Chien, Renjie Chen, Ofir Weber

SIGGRAPH 2016

Interpolation in Animation: Step 1

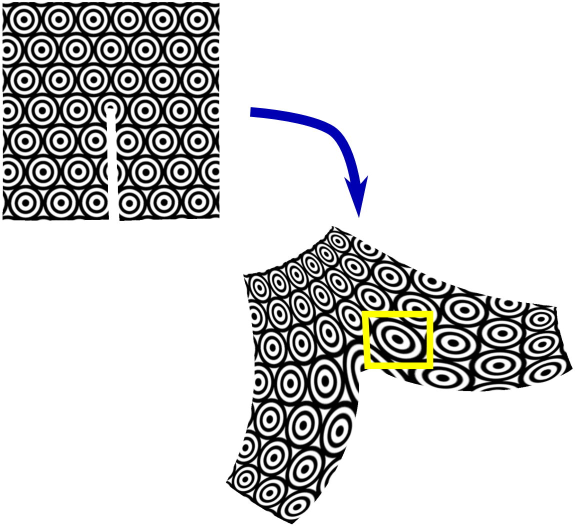


Deform source shape to obtain keyframes

Interpolation in Animation: Step 2

Interpolate keyframes for motion

Goal: control geometric distortion



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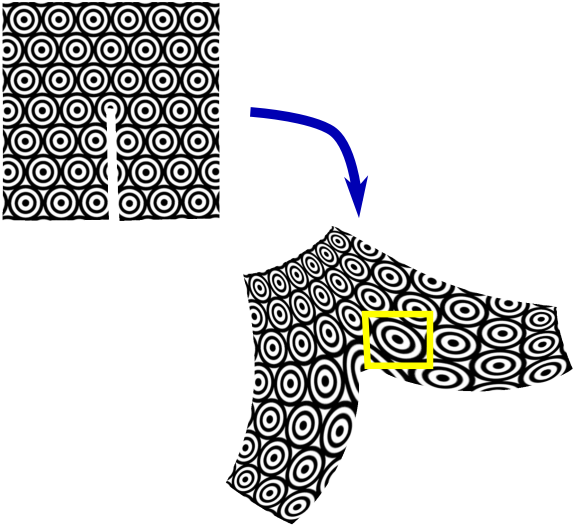
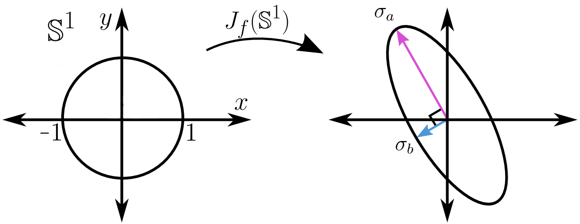


Distortion is local stretching and shearing

Previous works on bounded distortion maps:

- [Lipman '12]
- [Weber et al. '12]
- [Aigerman & Lipman '13]
- [Schuller et al. '13]
- [Kovalsky et al. '15]
- [Chen & Weber '15]
- and many, many more...

Geometric distortion quantified



Dimensions of image ellipse quantify geometric distortion.

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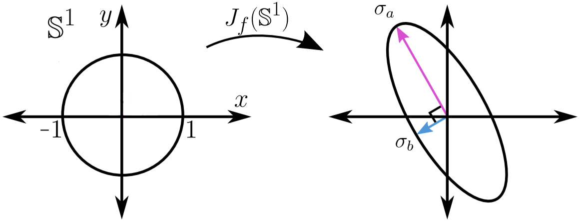

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Conformal disortion (shear):

$$K = \frac{\sigma_a}{\sigma_b} \in [1, \infty)$$

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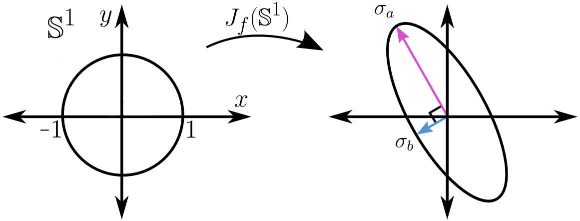

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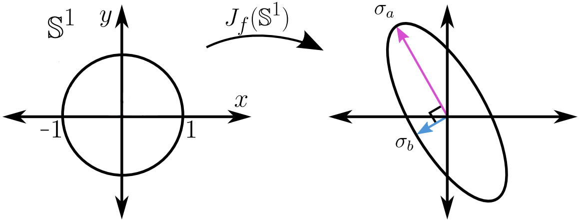

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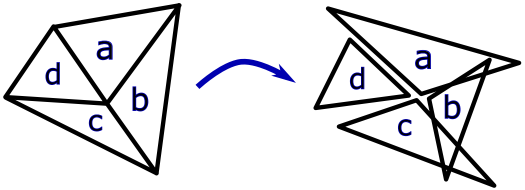
Differential blending:



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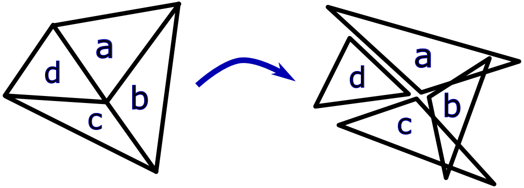

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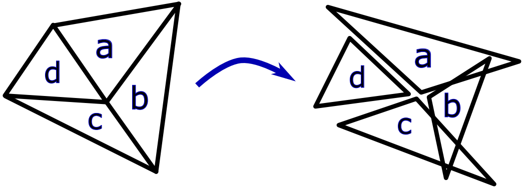

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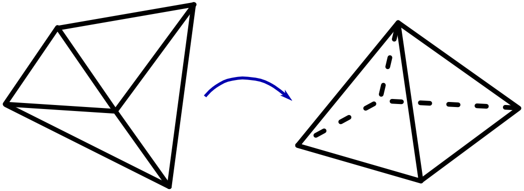

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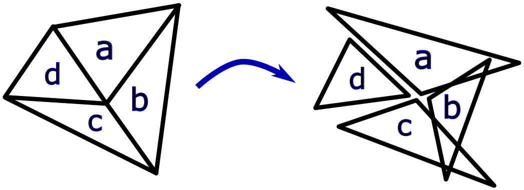

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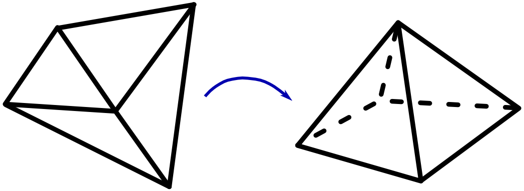

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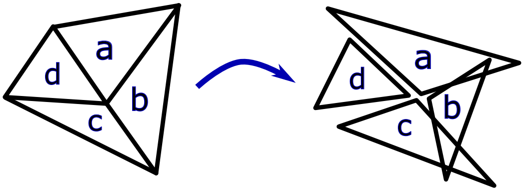

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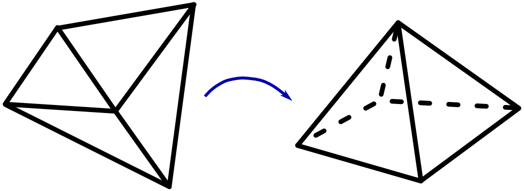

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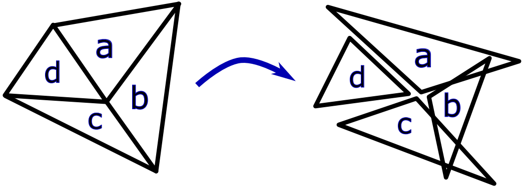

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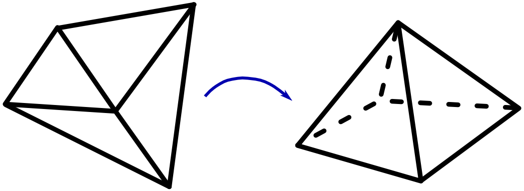

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(also, methods all mesh-based)

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Given: two locally-injective harmonic planar maps on a disc-like domain

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All done with automatic integrability, and a method that is embarrassingly parallel and meshless.

Background: Wirtinger Derivatives

For any C^1 planar map, there is a useful decomposition of the Jacobian:

$$J_f = \underbrace{\begin{pmatrix} a & -b \\ b & a \end{pmatrix}}_{\text{similarity}} + \underbrace{\begin{pmatrix} c & d \\ d & -c \end{pmatrix}}_{\text{anti-similarity}}$$

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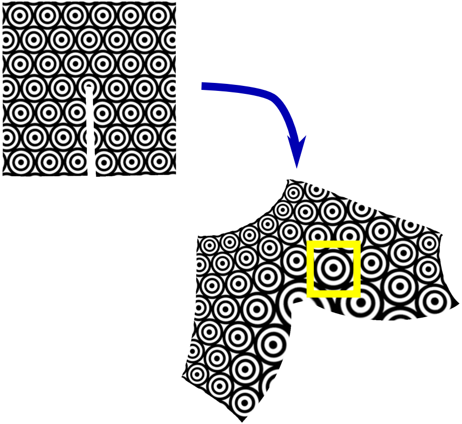
Gives geometric intuition for the Wirtinger derivatives:

$$f_z = a + \mathrm{i}b \qquad \& \qquad f_{\bar{z}} = c + \mathrm{i}d$$

Background: Holomorphic & Conformal Maps



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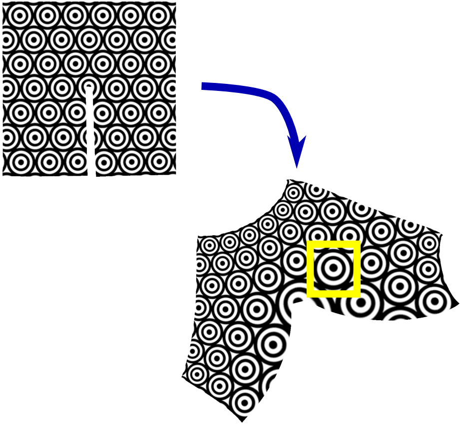

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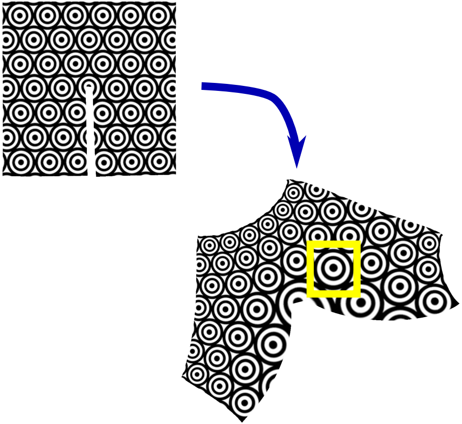

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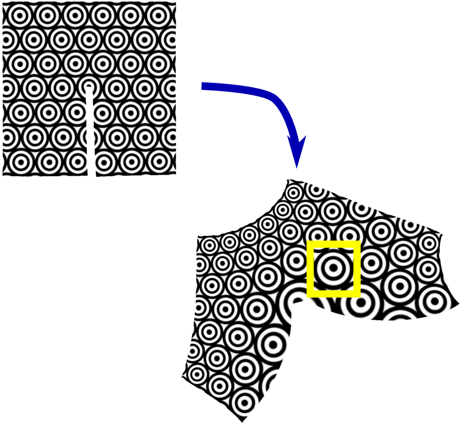

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Analogous definitions and facts hold for anti-holomorphic maps.

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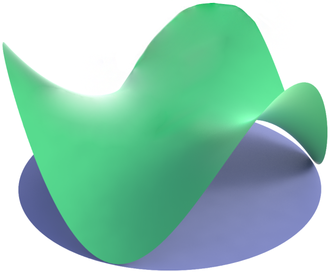

A harmonic function u satisfies Laplace's equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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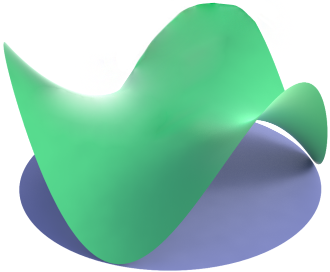

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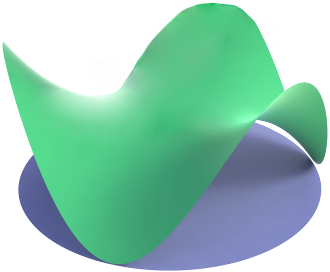

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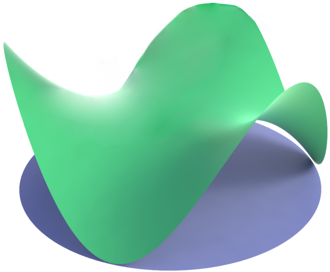

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On a disc-like domain,

$$f \text{ harmonic} \iff f = \Phi + \overline{\Psi}$$

where Φ, Ψ are holomorphic.

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They remain automatically integrable, and we may sum upon integration to get the final map.

The Metric Variant

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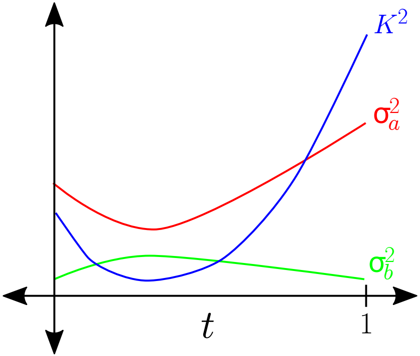

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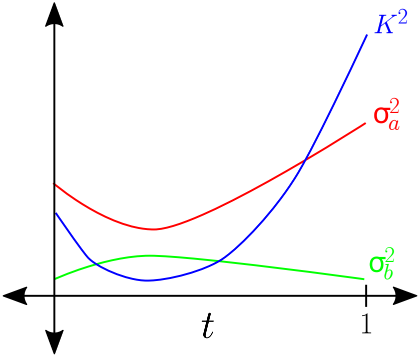

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$\Longrightarrow$ **bounded distortion!**

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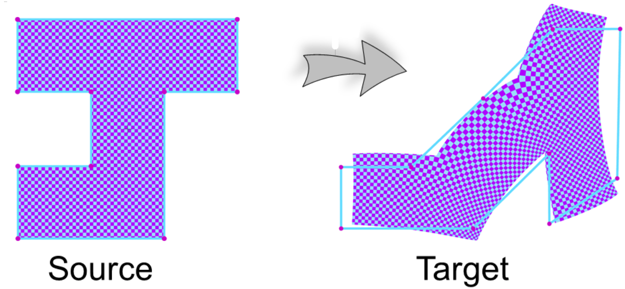

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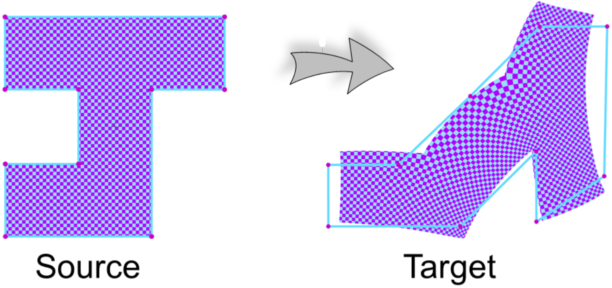

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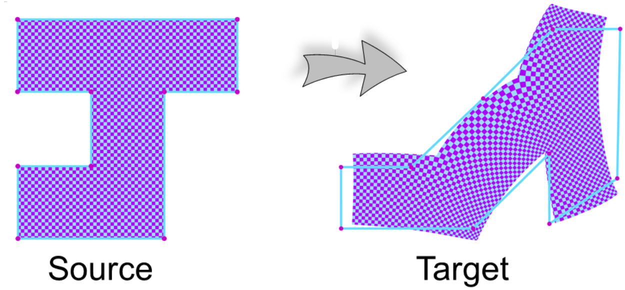

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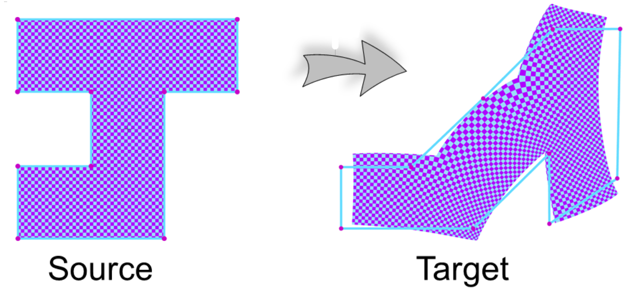

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A^{-1} is pre-computed and only multiplication by it is required.

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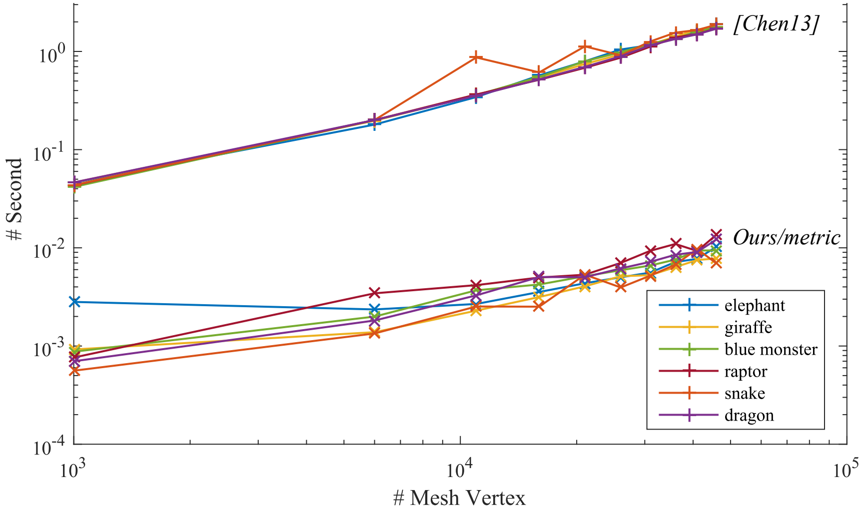
3. Integrate interpolated f_z and $f_{\bar{z}}$ numerically.

Running Time Data



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- integrability of Jacobians automatic
- guaranteed distortion bounds
- method is parallel
- method is smooth, meshless

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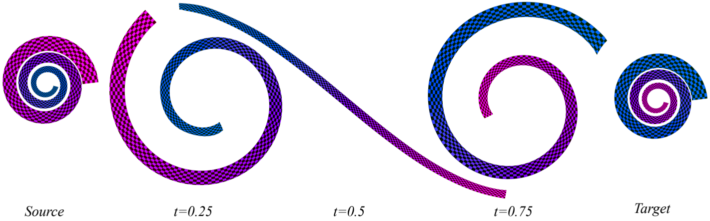

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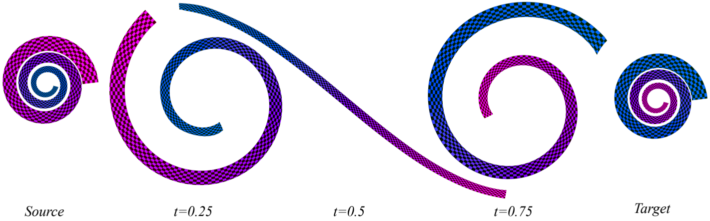

Q: Is it possible to extend the approach to multiply-connected domains?

Thank You!

Supported by:

- Israel Science Foundation
- Max Planck Center for Visual Computing and Comm.
- NVIDIA

I am looking for a postdoc position for AY '17/'18.

Code available at: http://people.mpi-inf.mpg.de/~chen/bdh.zip