Stochastic Wasserstein Barycenters

Sebastian Claici, Edward Chien, Justin Solomon
Motivation

Summarizing data from multiple sources.

Weather Stations

Subsets of a Dataset

Sensor Network

[NASA]

[Srivastava et al., 2015]

[Sarma Vrudhula]
Optimal Transport

A *geometry* for probability measures.

**Intuition**: moving piles of sand.
Wasserstein Distance

The 2-Wasserstein distance between $\mu_1$ and $\mu_2$ is

$$W_2^2(\mu_1, \mu_2) = \left( \inf_{\gamma \in \Gamma(\mu_1, \mu_2)} \int_{X \times X} d(x, y)^2 \, d\gamma(x, y) \right)$$

Intuition: coupling tells you where to shovel.
Barycenters in Probability Space

Often want to *average* distributions.
Barycenters in Probability Space

Euclidean averaging doesn’t contain geometric information.

Expectation

Euclidean Average
Wasserstein Barycenters

The Wasserstein barycenter of measures $\mu_j$ is the measure $\nu$ that minimizes

$$F[\nu] = \frac{1}{N} \sum_{j=1}^{N} W_2^2(\nu, \mu_j)$$

The standard Euclidean mean is the analog of this.
Why Wasserstein Barycenters?

Barycenters under the Wasserstein distance are more intuitive.
Related Work

[Cuturi & Doucet, 2014]

[Solomon et al., 2015]

[Staib et al., 2017]
Pipeline

Approximate barycenter by a discrete uniform measure:

\[ \nu = \frac{1}{m} \sum_{i=1}^{m} \delta_{x_i} \]

Solve for positions of the points \( x_i \).

Make use of duality and an alternating optimization scheme.

Only assume sample access to input measures.
Simplifying

Barycenter problem is additive.

Assume that we only have one input measure.

What is the best discrete approximation to a measure?
Kantorovich Dual Problem

The transport problem admits a dual formulation:

$$\sup_f \int_X f(x) \, d\mu_1(x) + \int_X \inf_x \{ d(x, y)^2 - f(x) \} \, d\mu_2(y)$$

Dual problem has a supply/demand interpretation.
Kantorovich Dual Problem

Can simplify when one measure is discrete:

\[
F[f, x_1, \ldots, x_m] = \frac{1}{m} \sum_{i=1}^{m} f_i + \sum_{i=1}^{m} \int_{V_{x_i}} (d(x_i, y)^2 - f_i) \, d\mu_2(y)
\]
Optimization -- Weights

The dual problem is concave in the weights.

Can apply gradient ascent with gradient:

\[
\frac{\partial F}{\partial f_i} = \frac{1}{m} - \int_{V_{x_i}} d\mu_2(y)
\]
Optimization -- Points

The dual is non-convex in the point positions.

Gradient yields a simple fixed point iteration scheme.

\[
\frac{\partial F}{\partial x_i} = x_i \int_{V_{x_i}} d\mu_2(y) - \int_{V_{x_i}} y d\mu_2(y)
\]

barycenter of cell
Semidiscrete Problem

Map a continuous distribution to a discrete one.
Optimization -- Weights

Adjust weights
Optimization -- Points

Adjust weights

Move points

$\mathbf{x}_1 \quad \mathbf{x}_2$

$\mathbf{\mu}_2$

$f_i$

$\mathbf{x}_1 \quad \mathbf{x}_2$
Algorithm

Extension to barycenter as simple as summing gradients.

- Sample seed points for the barycenter.
- Repeat until converged:
  1. Solve for weights via gradient ascent.
  2. Shift points via fixed point iteration.
Results -- Known Barycenter

A few cases where we know what the true barycenter is
Results -- Sharp Features

Our method is inherently suited to distributions with sharp features.

[Solomon et al., 2015]

Ours -- 100 samples
Results -- Sharp Features

[Solomon et al., 2015]  [Staib et al., 2017]  Ours -- 50 samples
Applications -- Blue Noise Approximation

Original image

Blue noise approximation
Applications -- Super Samples

Uniform samples

Ours
Conclusions

Compute Wasserstein barycenters \textit{without} regularization.

Works with \textit{continuous} input distributions.

Easy to implement and parallelizable.

Come see our poster at stand #69!