Bounded Distortion Harmonic Shape Interpolation

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Interpolation in Animation

Step 1:
deform source
shape for
keyframes.

Step 2:
interpolate
deformations
for motion.

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Fewer works have focused on such methods for step 2. For comparison here, we consider four other methods:

- Alexa et al. '00 [ARAP] uses the polar decomposition of the Jacobian, interpolates the parts separately, and then reconstructs the map by finding integrable Jacobians that are close. No guarantees on distortion bounds.

Previous Work (cont.)

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- Chen/Weber '15 [Chen/Weber 15] computes bounded distortion harmonic mappings with positional constraints. Interpolation of handles offers an easy extension to interpolation.

The Complex Derivatives

A useful decomposition for the Jacobian J_f of a C^1 planar map $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$J_f = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$$

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Letting $z = x + \mathrm{i}y$, $f_z = a + \mathrm{i}b$, and $f_{\bar{z}} = c + \mathrm{i}d$, we get $J_f(x \;\; y)^T$ in $\mathbb{C}$:

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Formulae for the complex derivatives: $f_z := (f_x - \mathrm{i}f_y)/2$ & $f_{\bar{z}} := (f_x + \mathrm{i}f_y)/2$.

Holomorphic & Anti-holomorphic Mappings

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Complex conjugation switches back and forth between the two classes of mappings.

Harmonic Planar Mappings

Harmonic mappings $f = (u, v)$ have components that satisfy the Laplace equation:

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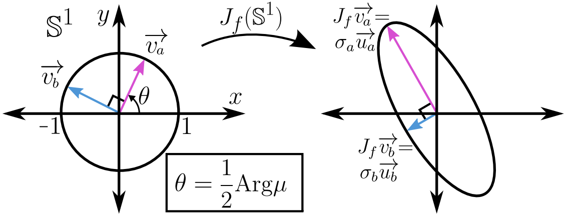
The converse is true as well with the sum of a holomorphic and anti-holomorphic mapping being harmonic.

Local Geometric Quantities

SVD: $J_f = U\Sigma V^T$



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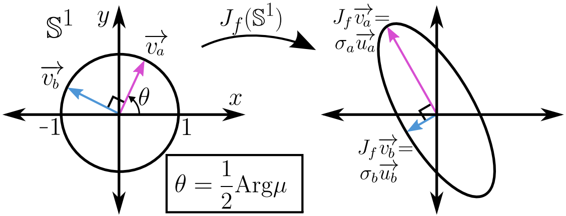

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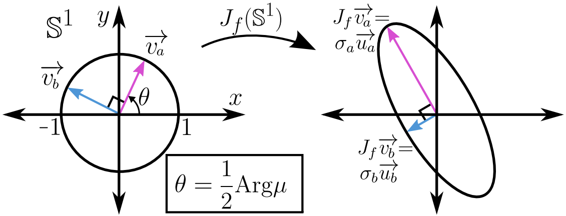

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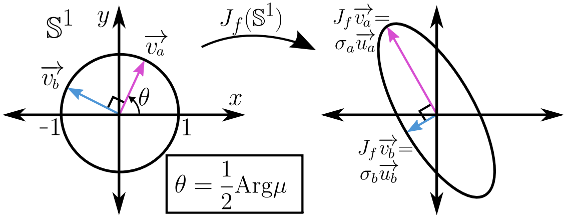

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- other isometric distortion measures: $\tau := \max(\sigma_a, \frac{1}{\sigma_b})$, $\sigma_a + \frac{1}{\sigma_b}$

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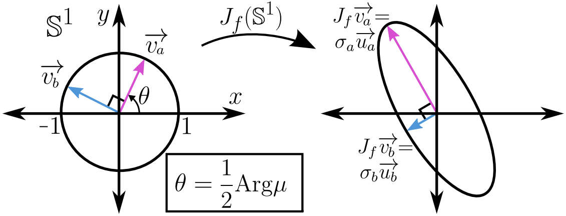

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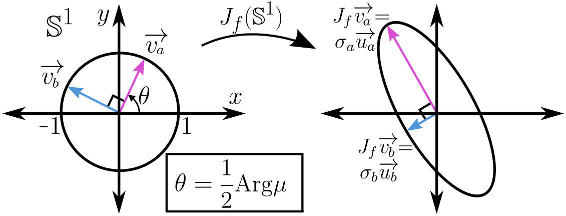

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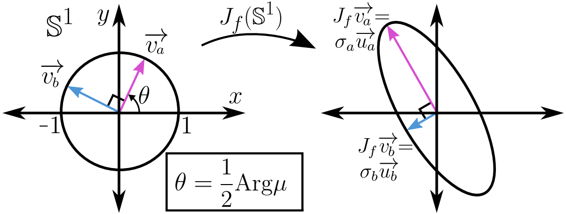

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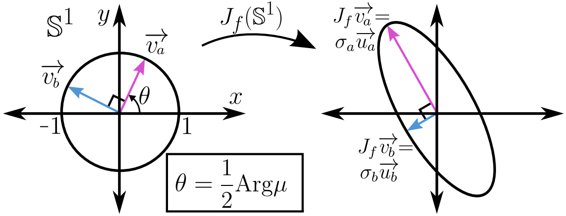

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- stretch direction: $\theta = \frac{1}{2}\mathrm{Arg}\,\mu \in [0, \pi)$

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4. (smoothness) $f|_{[0,1]\times\{z\}}$ is C^∞ for all $z \in \Omega$

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Note: Can consider these desired bounds as pointwise or global.

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Upon integration, we may sum the results and obtain a harmonic map.

This approach basically interpolates the similarity and anti-similarity parts of the Jacobian separately.

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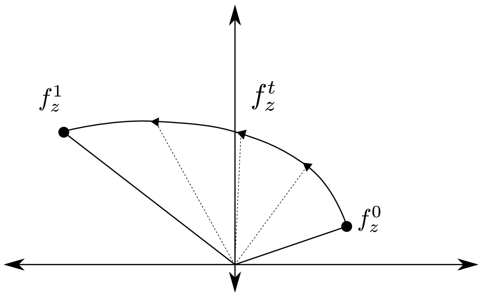
3. Local injectivity follows as long as we maintain $|f_z| > |f_{\bar{z}}|$ throughout interpolation.
4. Smoothness will result as long as f_z and $f_{\bar{z}}$ are smoothly interpolated with respect to time t.

Logarithmic Interpolation of f_z



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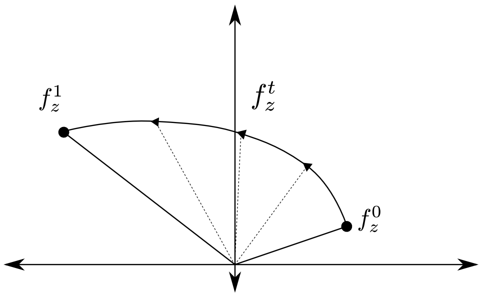

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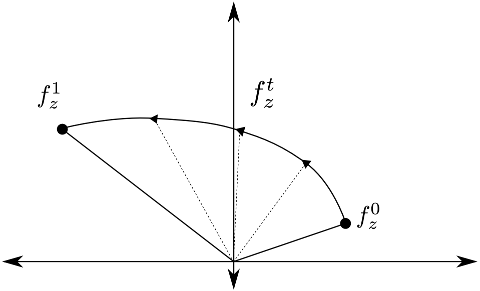

$$
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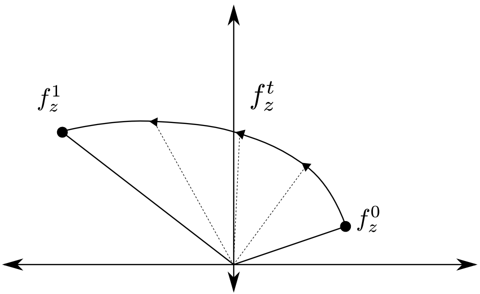

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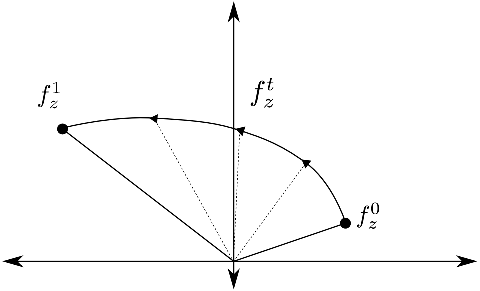

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Branches of logarithm need to be determined.

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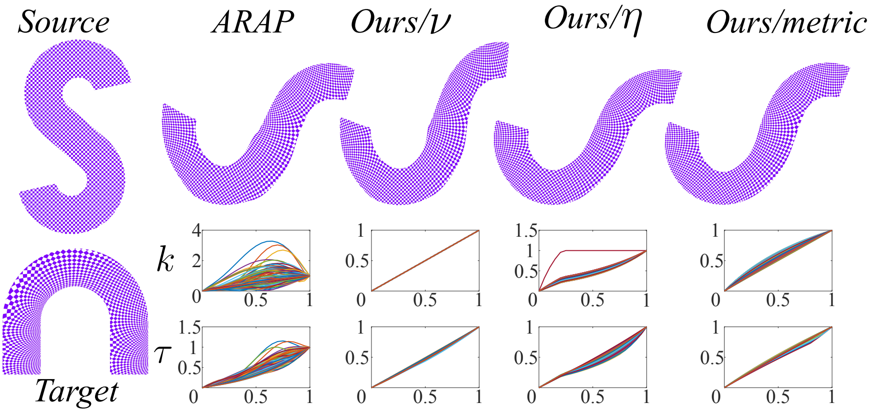
Conformal distortion bounds are satisfied, as are bounds on σ_b. Bounds on σ_a are nearly achieved.

ν variant example



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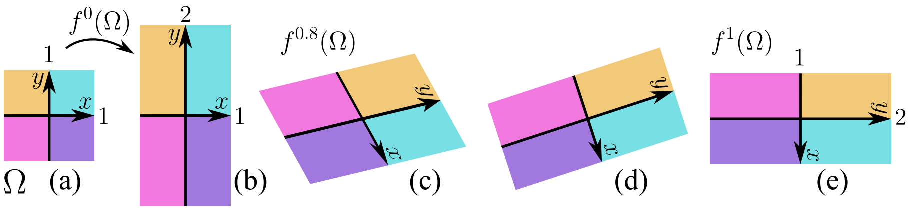

Stretch direction preservation

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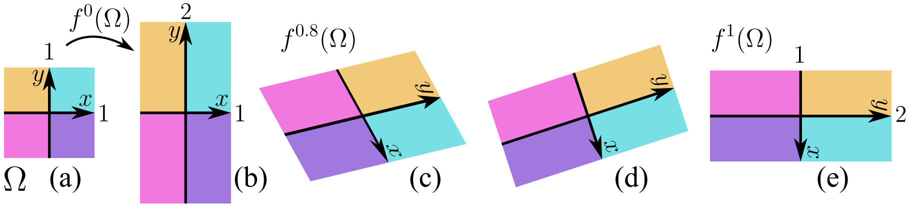

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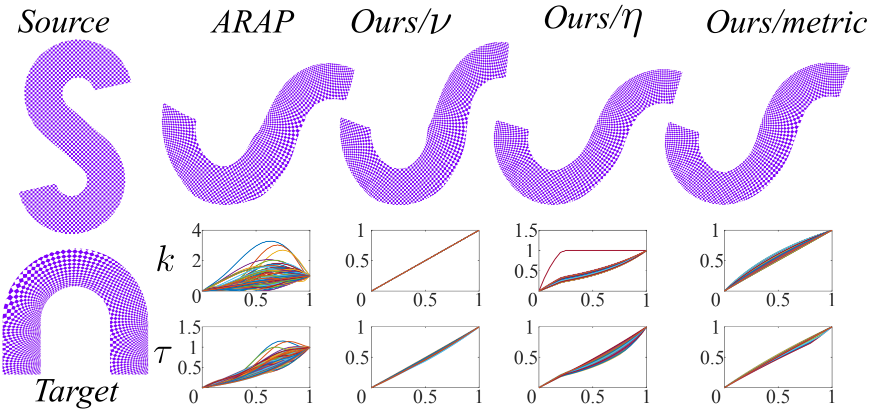
This scaling of the linear interpolation is applied globally.

η variant example



Metric background

For a planar mapping f, the metric tensor $M_f = J_f^T J_f$ is given by the following formula, where $\mathcal{A} := |f_z|^2 + |f_{\bar{z}}|^2$.

$$M_f = \begin{pmatrix} \mathcal{A} & 0 \\ 0 & \mathcal{A} \end{pmatrix} + 2 \begin{pmatrix} \mathrm{Re}\,(\eta) & \mathrm{Im}\,(\eta) \\ \mathrm{Im}\,(\eta) & -\mathrm{Re}\,(\eta) \end{pmatrix}.$$

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In terms of $\mathcal{A}$ and η, the distortion quantities are easily expressed:

$$\sigma_a^2 = \mathcal{A} + 2|\eta|, \quad \sigma_b^2 = \mathcal{A} - 2|\eta|, \quad K^2 = \frac{\sigma_a^2}{\sigma_b^2} = \frac{\mathcal{A} + 2|\eta|}{\mathcal{A} - 2|\eta|}$$

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The first two are convex in these variables, while the second is quasiconvex.

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Linear interpolation of η then determines $f_{\bar{z}}^t$. This ensures preservation of stretch direction.

Metric variant example

Some Implementation Details

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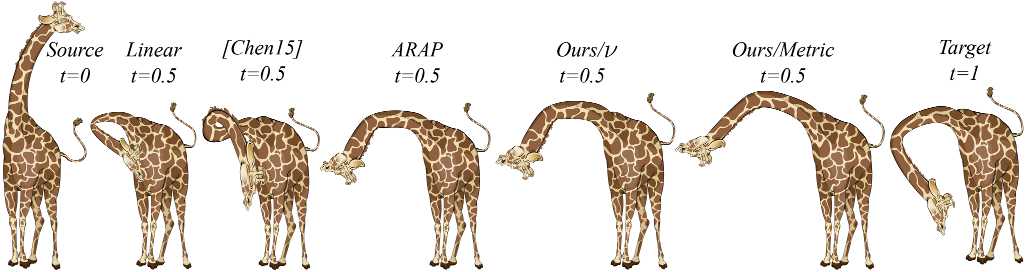
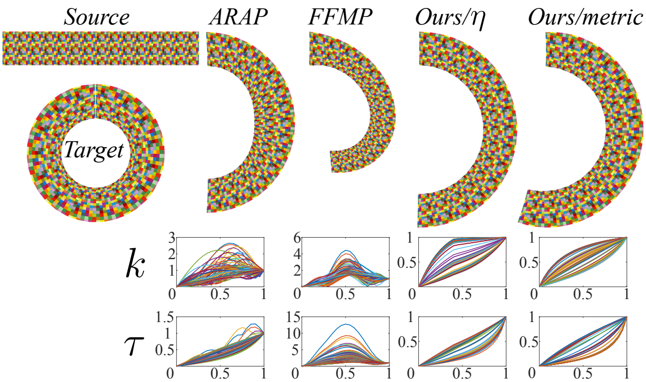
Otherwise, quantities are blended per vertex in parallel (fineness of mesh can be arbitrarily high), and the integration of f_z and $f_{\bar{z}}$ is done numerically, which turns out to be quite accurate.

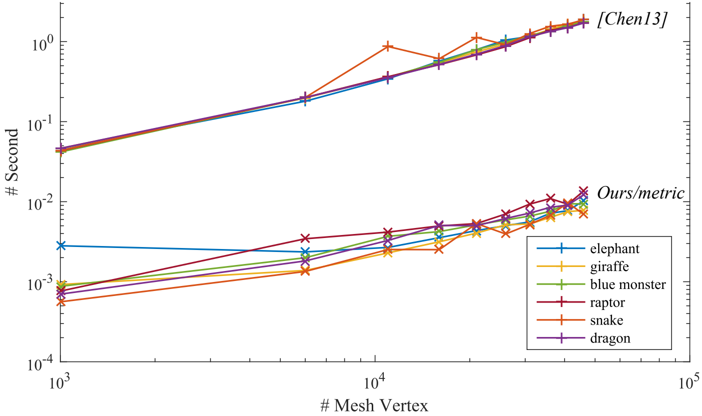
Results


Source
t=0
Linear
t=0.5
[Chen15]
t=0.5
ARAP
t=0.5
Ours/ν
t=0.5
Ours/Metric
t=0.5
Target
t=1

More Results



And More Results



Moar Results

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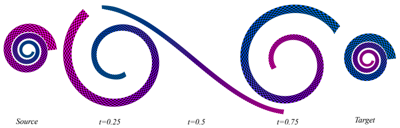

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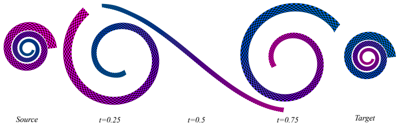

We are also limited to simply-connected domains and to planar mappings. Investigations on extensions beyond both these domains has begun as well (though collaboration would be welcomed!).

References

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- Chen, R.; Weber, O. 2015. Bounded distortion harmonic mappings in the plane. *ACM Transactions on Graphics (TOG) 34*, 4, 73.
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- Kircher, S.; Garland, M. 2008. Free-form motion processing. *ACM Transactions on Graphics (TOG) 27*, 2, 12.

Thank you!

Thank you for your attention!

Questions?