Bounded Distortion Harmonic Shape Interpolation

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Interpolation in Animation

Step 1:
deform source shape for keyframes.

Step 2:
interpolate deformations for motion.
In recent years, many works have focused on bounded distortion methods for step 1.
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In recent years, many works have focused on bounded distortion methods for step 1. Key contributors: Lipman, Zorin, Weber, Chen, Schuller, Aigerman, Kovalksy, etc. Fewer works have focused on such methods for step 2. For comparison here, we consider four other methods:

- Alexa et al. ’00 [ARAP] uses the polar decomposition of the Jacobian, interpolates the parts separately, and then reconstructs the map by finding integrable Jacobians that are close. No guarantees on distortion bounds.
Previous Work (cont.)

- Kircher/Garland '08 [FFMP] use differential trihedron connection coordinates, requiring a two-step reconstruction process. Also no guarantees on bounded distortion.
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- Chen et al. '13 [Chen et al. 13] interpolate edge lengths squared of the mesh. Equivalent to linear interpolation of the metric tensor. Bounded conformal distortion.
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The Complex Derivatives

A useful decomposition for the Jacobian $J_f$ of a $C^1$ planar map $f : \mathbb{R}^2 \to \mathbb{R}^2$:

$$J_f = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} + \begin{pmatrix} c & d \\ d & -c \end{pmatrix}$$
Mathematical Background

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Letting $z = x + iy$, $f_z = a + ib$, and $f_{\bar{z}} = c + id$, we get $J_f(x \ y)^T$ in $\mathbb{C}$:

$$J_f(z) = f_z z + f_{\bar{z}} \bar{z}$$
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Formulae for the complex derivatives: $f_z := (f_x - if_y)/2$ & $f_{\bar{z}} := (f_x + if_y)/2$. 
Mathematical Background

Holomorphic & Anti-holomorphic Mappings

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Anti-holomorphic mappings $f$ are those for which $f_{\overline{z}} = 0$ everywhere, and they have analogous properties.

Complex conjugation switches back and forth between the two classes of mappings.
Harmonic Planar Mappings

Harmonic mappings $f = (u, v)$ have components that satisfy the Laplace equation:

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ 

The value of a harmonic mapping is intuitively the average of its surrounding values.
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If $f : \Omega \to \mathbb{R}^2$ has a simply-connected domain $\Omega$, then it may be represented as the sum of a holomorphic and anti-holomorphic mapping:

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The converse is true as well with the sum of a holomorphic and anti-holomorphic mapping being harmonic.
Mathematical Background

Local Geometric Quantities

**SVD:** $J_f = U \Sigma V^T$

- $\det J_f = |f_z|^2 - |f_\bar{z}|^2$

**Isometric distortion measures:**

- $\sigma_a = |f_z| + |f_\bar{z}|$
- $\sigma_b = |f_z| - |f_\bar{z}|$

- Other isometric distortion measures:

  - $\tau := \max(\sigma_a, 1/2 \sigma_b, \sigma_a + 1/2 \sigma_b)$

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Bounded Distortion Harmonic Shape Interpolation
Mathematical Background

Local Geometric Quantities

SVD: \( J_f = U \Sigma V^T \)

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Mathematical Background

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Bounded Distortion Harmonic Shape Interpolation
Mathematical Background

Local Geometric Quantities (cont.)

\[ SVD: \quad J_f = U \Sigma V^T \]

- \( \mu = \frac{f_x}{f_z} \), Beltrami coefficient

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Mathematical Background

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SVD: $J_f = U\Sigma V^T$

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- Conformal distortion measure:
  $k = |\mu| \in [0, 1)$
Mathematical Background

Local Geometric Quantities (cont.)

SVD: $J_f = U \Sigma V^T$

- $\mu = \frac{f_z}{f_x}$, Beltrami coefficient
- conformal distortion measure: $k = |\mu| \in [0, 1)$
- alternate conformal distortion measure: $K = \frac{\sigma_a}{\sigma_b} \in [1, \infty)$

$\theta = \frac{1}{2} \text{Arg} \mu$

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Mathematical Background

Local Geometric Quantities (cont.)

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- conformal distortion measure: $k = |\mu| \in [0, 1)$

- alternate conformal distortion measure: $K = \frac{\sigma_a}{\sigma_b} \in [1, \infty)$

- stretch direction: $\theta = \frac{1}{2} \text{Arg} \mu \in [0, \pi)$

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Bounded Distortion Harmonic Shape Interpolation
Problem Statement

Given: \( f^0, f^1 : \Omega \rightarrow \mathbb{R}^2 \) locally injective, orientation-preserving, harmonic; \( \Omega \) simply-connected
Problem Statement & Basic Approach

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3. (loc. inj.) $f|_{\{t\} \times \Omega}$ is loc. inj. orientation-preserving $\forall t \in [0, 1]$
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3. (loc. inj.) \( f|_{\{t\} \times \Omega} \) is loc. inj. orientation-preserving \( \forall t \in [0, 1] \)
4. (smoothness) \( f|_{[0,1] \times \{z\}} \) is \( C^\infty \) for all \( z \in \Omega \)
Problem Statement & Basic Approach

Problem Statement (cont.)

\[ f^t := f_{\{t\} \times \Omega} \] (analogous superscript notation used for other quantities)

Additionally, we'd like \( f \) to be bounded distortion:

- **Conf. distortion**: \( k_t \leq \max(k_0, k_1) \) for all \( t \in [0, 1] \)
- **Max scaling**: \( \sigma_t \leq \max(\sigma_0, \sigma_1) \) for all \( t \in [0, 1] \)
- **Min scaling**: \( \sigma_t \geq \min(\sigma_0, \sigma_1) \) for all \( t \in [0, 1] \)

Note: Can consider these desired bounds as pointwise or global.

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Bounded Distortion Harmonic Shape Interpolation
Problem Statement (cont.)

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\[
\begin{align*}
\text{(conf. distortion)} &\quad k_t \leq \max(k_0, k_1) \quad \text{for all } t \in [0, 1] \\
\text{(max scaling)} &\quad \sigma_{ta} \leq \max(\sigma_{0a}, \sigma_{1a}) \quad \text{for all } t \in [0, 1] \\
\text{(min scaling)} &\quad \sigma_{tb} \geq \min(\sigma_{0b}, \sigma_{1b}) \quad \text{for all } t \in [0, 1]
\end{align*}
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Problem Statement & Basic Approach

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6. (max scaling) \( \sigma_a^t \leq \max(\sigma_a^0, \sigma_a^1) \) for all \( t \in [0, 1] \)

7. (min scaling) \( \sigma_b^t \geq \min(\sigma_b^0, \sigma_b^1) \) for all \( t \in [0, 1] \)
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7 (min scaling) \( \sigma^t_b \geq \min(\sigma^0_b, \sigma^1_b) \) for all \( t \in [0, 1] \)

Note: Can consider these desired bounds as pointwise or global.
Basic Approach

As with many other approaches, we aim to interpolate the Jacobians as these approximate the mapping locally.
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With harmonic input maps, the decomposition: $f(z) = \Phi(z) + \overline{\Psi}(z)$, suggests a useful approach. Note that $f_z = \Phi_z$ and $f_{\overline{z}} = \overline{\Psi}_z$. 
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Upon integration, we may sum the results and obtain a harmonic map.
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Upon integration, we may sum the results and obtain a harmonic map.

This approach basically interpolates the similarity and anti-similarity parts of the Jacobian separately.

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Bounded Distortion Harmonic Shape Interpolation
Within this approach, we may see that most of the basic conditions are easily satisfied:

1. Interpolation is achieved with proper choices of integration constants.
2. Harmonicity is automatic in our approach.
3. Local injectivity follows as long as we maintain $|f_z| > |f_{\bar{z}}|$ throughout interpolation.
4. Smoothness will result as long as $f_z$ and $f_{\bar{z}}$ are smoothly interpolated with respect to time $t$. 
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Methods: Fully Parallel Variants

Logarithmic Interpolation of $f_z$

$$f_z^t = (f_z^0)^{1-t}(f_z^1)^t$$

Clearly holomorphic, and note $f_z^t \neq 0$. Branches of logarithm need to be determined.
Logarithmic Interpolation of $f_z$

Methods: Fully Parallel Variants

$$f_z^t = (f_z^0)^{1-t}(f_z^1)^t$$

$$= e^{(1-t) \log f_z^0} e^{t \log f_z^1}$$

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$$= e^{(1-t) \log f^0_z} e^{t \log f^1_z}$$

$$= \left| f_z \right|^{1-t} \left| f^1_z \right|^t e^{i \left( (1-t) \text{arg}(f^0_z) + t \text{arg}(f^1_z) \right)}$$

Clearly holomorphic, and note $f^t_z \neq 0$.

Branches of logarithm need to be determined.

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Bounded Distortion Harmonic Shape Interpolation
Methods: Fully Parallel Variants

*\( \nu \) variant

As a norm on any vector space is convex, we might try to determine \( f^t_z \) by linearly interpolating \( \mu \). This would preserve bounds on conformal distortion.

\[
\nu_t = (1 - t) \nu_0 + t \nu_1 = \Rightarrow f^t_z = \nu_t f_z
\]

As \( \nu_t \) is holomorphic, we see that we get an anti-holomorphic interpolation for \( f^t_z \).

Conformal distortion bounds are satisfied, as are bounds on \( \sigma_b \). Bounds on \( \sigma_a \) are nearly achieved.

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Bounded Distortion Harmonic Shape Interpolation
As a norm on any vector space is convex, we might try to determine $f^t_z$ by linearly interpolating $\mu$. This would preserve bounds on conformal distortion.

Unfortunately, this may not be done while maintaining anti-holomorphic interpolation of $f^t_z$. So we consider $\nu = \frac{(f^t_z)}{f_z}$, noting that $|\nu| = |\mu| = k$. 
**ν** variant

As a norm on any vector space is convex, we might try to determine $f_{\bar{z}}^t$ by linearly interpolating $\mu$. This would preserve bounds on conformal distortion.

Unfortunately, this may not be done while maintaining anti-holomorphic interpolation of $f_{\bar{z}}^t$. So we consider $\nu = \frac{(f_{\bar{z}})}{f_z}$, noting that $|\nu| = |\mu| = k$.

$$\nu^t = (1 - t)\nu^0 + t\nu^1 \implies f_{\bar{z}}^t = \overline{\nu^t f_{\bar{z}}^t}$$
**ν variant**

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As $\nu^t$ is holomorphic, we see that we get an anti-holomorphic interpolation for $f_z^t$. Conformal distortion bounds are satisfied, as are bounds on $\sigma_b$. Bounds on $\sigma_a$ are nearly achieved.
Methods: Fully Parallel Variants

\(\nu\) variant example

\(\nu\) variant example

\(\text{Source} \quad \text{ARAP} \quad \text{Ours}/\nu \quad \text{Ours}/\eta \quad \text{Ours}/\text{metric}\)

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Bounded Distortion Harmonic Shape Interpolation
Methods: Fully Parallel Variants

Stretch direction preservation

The $\nu$ variant doesn’t always produce intuitive behavior, partially because the map does not preserve stretch direction.
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The diagram illustrates the difference in behavior between $f^0(\Omega)$, $f^{0.8}(\Omega)$, and $f^1(\Omega)$, showing how the stretch direction is preserved in $f^0(\Omega)$ but not in the other two cases.
The $\nu$ variant doesn’t always produce intuitive behavior, partially because the map does not preserve stretch direction.
To preserve stretch direction, we introduce linear interpolation of $\eta = f_z^* \bar{f}_z$. It shares an argument with $\mu$ and is anti-holomorphic.
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$$\eta^t = (1 - t)\eta^0 + t\eta^1 \implies f_z^t = \frac{\eta^t}{f_z}$$
Methods: Fully Parallel Variants

η variant

To preserve stretch direction, we introduce linear interpolation of $\eta = f_z \bar{f}_z$. It shares an argument with $\mu$ and is anti-holomorphic.

$$\eta^t = (1 - t)\eta^0 + t\eta^1 \implies f_z^t = \frac{\eta^t}{f_z}$$

In most cases, this is enough to achieve bounded distortion. However, when the input mappings differ greatly, the linear interpolation must be scaled in order to guarantee bounds.
Methods: Fully Parallel Variants

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This scaling of the linear interpolation is applied globally.
Methods: Fully Parallel Variants

\(\eta\) variant example

Source \hspace{2cm} ARAP \hspace{2cm} Ours/\nu \hspace{2cm} Ours/\eta \hspace{2cm} Ours/metric

Edward Chien∗, Renjie Chen†, Ofir Weber∗ ∗Bar Ilan University †Max Planck Institute for Informatics

Bounded Distortion Harmonic Shape Interpolation
For a planar mapping $f$, the metric tensor $M_f = J_f^T J_f$ is given by the following formula, where $A := |f_z|^2 + |f_{\bar{z}}|^2$.

$$M_f = \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} + 2 \begin{pmatrix} \text{Re}(\eta) & \text{Im}(\eta) \\ \text{Im}(\eta) & -\text{Re}(\eta) \end{pmatrix}.$$
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In terms of $\mathcal{A}$ and $\eta$, the distortion quantities are easily expressed:

$$\sigma_a^2 = \mathcal{A} + 2|\eta|, \quad \sigma_b^2 = \mathcal{A} - 2|\eta|, \quad K^2 = \frac{\sigma_a^2}{\sigma_b^2} = \frac{\mathcal{A} + 2|\eta|}{\mathcal{A} - 2|\eta|}.$$
Method: Metric variant

**Metric background**

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The first two are convex in these variables, while the second is quasiconvex.
Given these facts, we linearly blend the metric tensor on the boundary, which bounds the distortion pointwise on the boundary.
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Effectively, linear interpolation of the metric tensor determines the magnitude of $|f^t_z|$ via a quadratic. We then reconstruct $f^t_z$ on the domain with a Hilbert transform.
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Linear interpolation of $\eta$ then determines $f_t^Z$. This ensures preservation of stretch direction.
Methods: Metric variant

Metric variant example
Some Implementation Details

For results here, input generated with methods of [Chen/Weber 15], i.e., discretized with Cauchy barycentric coordinates.
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\Phi(z) = \sum_{j=1}^{n} C_j(z) \varphi_j, \quad \Psi(z) = \sum_{j=1}^{n} C_j(z) \psi_j
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The Hilbert transform also performed with Cauchy barycentric coordinates, requiring a multiplication by a small dense matrix. Otherwise, quantities are blended per vertex in parallel (fineness of mesh can be arbitrarily high), and the integration of \(f_z\) and \(f_{\bar{z}}\) is done numerically, which turns out to be quite accurate.
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Results

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Bounded Distortion Harmonic Shape Interpolation
And More Results

Edward Chien*, Renjie Chen†, Ofir Weber*  *Bar Ilan University  †Max Planck Institute for Informatics

Bounded Distortion Harmonic Shape Interpolation
Moar Results
Summary

Our methods interpolate bounded distortion harmonic input via holomorphic and anti-holomorphic interpolation of $f_z$ and $f_{\overline{z}}$. 

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Limitations & Future Work

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We are also limited to simply-connected domains and to planar mappings. Investigations on extensions beyond both these domains has begun as well (though collaboration would be welcomed!).
References


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Thank you for your attention!

Questions?