THE BIDIMENSIONAL THEORY OF BOUNDED-GENUS GRAPHS*

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Abstract. Bidimensionality provides a tool for developing subexponential fixed-parameter algorithms for combinatorial optimization problems on graph families that exclude a minor. This paper extends the theory of bidimensionality for graphs of bounded genus (which is a minor-excluding family). Specifically we show that, for any problem whose solution value does not increase under contractions and whose solution value is large on a grid graph augmented by a bounded number of handles, the treewidth of any bounded-genus graph is at most a constant factor larger than the square root of the problem's solution value on that graph. Such bidimensional problems include vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, r-dominating set, connected dominating set, planar set cover, and diameter. On the algorithmic side, by showing that an augmented grid is the prototype bounded-genus graph, we generalize and simplify many existing algorithms for such problems in graph classes excluding a minor. On the combinatorial side, our result is a step toward a theory of graph contractions analogous to the seminal theory of graph minors by Robertson and Seymour.

Key words. Treewidth, grids, graphs on surfaces, graph minors, graph contractions

AMS subject classifications. 05C83, 05C85

DOI. 10.1137/040616929

1. Introduction. The recent theory of fixed-parameter algorithms and parameterized complexity [DF99] has attracted much attention in its less than 10 years of existence. In general the goal is to understand when NP-hard problems have algorithms that are exponential only in a parameter k of the problem instead of the problem size n. Fixed-parameter algorithms whose running time is polynomial for fixed parameter values—or more precisely $f(k) \cdot n^{O(1)}$ for some (superpolynomial) function $f(k)$—make these problems efficiently solvable whenever the parameter k is reasonably small.

In the last five years, several researchers have obtained exponential speedups in fixed-parameter algorithms for various problems on several classes of graphs. While most previous fixed-parameter algorithms have a running time of $2^{O(k)}n^{O(1)}$ or worse, the exponential speedups result in subexponential algorithms with typical running times of $2^{O(\sqrt{k})}n^{O(1)}$. For example, the first fixed-parameter algorithm for finding a dominating set of size k in planar graphs [AFF+01] had running time $O(8^k n)$; subsequently, a sequence of subexponential algorithms and improvements have been obtained, starting with running time $O(4^{6\sqrt{34k}}n)$ [ABF+02], then $O(2^{27\sqrt{k}}n)$ [KP02], and finally $O(2^{15.13\sqrt{k}}k + n^3 + k^4)$ [FT03]. Other subexponential algorithms for

*Received by the editors October 13, 2004; accepted for publication (in revised form) August 29, 2005; published electronically DATE. A preliminary version of this paper appeared in *Proceedings of the 29th International Symposium on Mathematical Foundations of Computer Science*, Prague, 2004, pp. 191–203.

http://www.siam.org/journals/sidma/x-x/61692.html

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other domination and covering problems on planar graphs have also been obtained [ABF+02, AFN04, CKL01, KLL02, GKLY05].

All subexponential fixed-parameter algorithms developed so far are based on showing a "treewidth-parameter bound": any graph whose optimal solution has value k has treewidth at most some function $f(k)$. In many cases, $f(k)$ is sublinear in k, often $O(\sqrt{k})$. Combined with algorithms that are singly exponential in treewidth and polynomial in problem size, such a bound immediately leads to subexponential fixed-parameter algorithms.

A series of papers [DFHT05, DFHT, DFHT04] introduce the notion of *bidimensionality* as a general approach for obtaining treewidth-parameter bounds and therefore subexponential algorithms. This theory captures essentially all subexponential algorithms obtained so far. Roughly speaking, a parameterized problem is *bidimensional* if the parameter is large in a "grid-like graph" (linear in the number of vertices) and either closed under contractions (*contraction-bidimensional*) or closed under minors (*minor-bidimensional*). Examples of bidimensional problems include vertex cover, feedback vertex set, minimum maximal matching, dominating set, edge dominating set, r-dominating set,[1] connected dominating set, planar set cover, and diameter. Diameter is a simple computational problem, but its bidimensionality has important consequences as it forms the basis of locally bounded treewidth for minor-closed graph families [DH04a].

Treewidth-parameter bounds have been established for all minor-bidimensional problems in H-minor-free graphs for any fixed graph H [DFHT, DFHT04]. In this case, the notion of "grid-like graph" is precisely the regular $(r \times r)$-square grid. However, contraction-bidimensional problems (such as dominating set) have proved substantially harder. In particular, the largest class of graphs for which a treewidth-parameter bound can be obtained is apex-minor-free graphs instead of general H-minor-free graphs [DFHT04]. ("Apex-minor-free" means "H-minor-free" where H is a graph in which the removal of one vertex leaves a planar graph.) Such a treewidth-parameter bound has been obtained for all contraction-bidimensional problems in apex-minor-free graphs [DFHT04]. In this case, the notion of a "grid-like graph" is an $r \times r$ grid augmented with additional edges such that each vertex is incident to $O(1)$ edges to nonboundary vertices of the grid. (Here $O(1)$ depends on H.) Unfortunately, this treewidth-parameter bound is large: $f(k) = (\sqrt{k})^{O(\sqrt{k})}$. For a subexponential algorithm, we essentially need $f(k) = o(k)$. For apex-minor-free graphs, such a bound is known only for the special cases of dominating set and vertex cover [DH04b, DFHT].

The biggest graph classes for which we know a sublinear (indeed, $O(\sqrt{k})$) treewidth-parameter bound for many contraction-bidimensional problems are single-crossing-minor-free graphs and bounded-genus graphs. ("Single-crossing-minor-free" means "H-minor-free" where H is a minor of a graph that can be drawn in the plane with one crossing.) For single-crossing-minor-free graphs [DHT05, DHN+04] (in particular, planar graphs [DFHT05]), all contraction-bidimensional problems have a bound of $f(k) = O(\sqrt{k})$. In this case, the notion of "grid-like graph" is an $r \times r$ grid partially triangulated by additional edges that preserve planarity. For bounded-genus graphs [DFHT], a bound of $f(k) = O(\sqrt{k})$ has been shown, for the same notion of "grid-like graphs" but only for contraction-bidimensional problems with an additional property called α-*splittability*: upon splitting a vertex, the parameter should increase by at most $\alpha = O(1)$ (or decrease).

[1] A set S of vertices is an r-*dominating set* of graph G if any vertex of G has distance at most r from some vertex in S.

In this paper we extend the theory of bidimensionality for bounded-genus graphs by establishing a sublinear ($f(k) = O(\sqrt{k})$) treewidth-parameter bound for general contraction-bidimensional problems in bounded-genus graphs. Our notion of "grid-like graph" is somewhat broader: a partially triangulated $r \times r$ grid (as above) with up to g additional edges ("handles"), where g is the genus of the original graph. This form of contraction-bidimensionality is more general than α-splittability,[2] and thus we generalize the results for α-splittable contraction-bidimensional problems from [DFHT]. It is easy to construct a parameter that is contraction-bidimensional but not α-splittable, although these parameters are not "natural". So far all "natural" contraction-bidimensional parameters we have encountered are α-splittable, though we expect other interesting problems to arise that violate α-splittability.

Our results show that a partially triangulated grid with g additional edges is the prototype graph of genus g, as observed by Lovász [Lov03]. At a high level, this property means that, to solve an (algorithmic or combinatorial) problem on a general graph of genus g, the "hardest" instance on which we should focus is the prototype graph. This property generalizes the well-known result in graph theory that the grid is the prototype planar graph. This also extends our theory of constructing such prototypes for bidimensional problems.

Further algorithmic applications of our results follow from the graph-minor theory of Robertson and Seymour (e.g., [RS85]) and its extensions [DFHT, DH04b]. In particular, [RS03, DFHT] show how to reduce many problems on general H-minor-free graphs to subproblems on bounded-genus graphs. Essentially, the difference between bounded-genus graphs and H-minor-free graphs are "apices" and "vortices", which are usually not an algorithmic barrier. Applying our new theory for bounded-genus graphs, we generalize the algorithmic extensions of [DFHT, DH04b]. Indeed, we simplify the approaches of both [DFHT] and [DH04b], where it was necessary to "split" bounded-genus graphs into essentially planar graphs because of a lack of general understanding of bounded-genus graphs. Specifically, we remove the necessity of Lemmas 7.4–7.8 in [DH04b].

Last but not least are the combinatorial aspects of our results. In a series of 20 papers (so far), Robertson and Seymour (e.g., [RS85]) developed the seminal theory of graphs excluding a minor, which has had many algorithmic and combinatorial applications. Our understanding of contraction-bidimensional parameters can be viewed as a step toward generalizing the theory of graph minors to a theory of graph contractions. Specifically, we show that any graph of genus g can be contracted to its core of a partially triangulated grid with at most g additional edges; this result generalizes an analogous result from [RS03] when permitting arbitrary minor operations (contractions and edge deletions). Avoiding edge deletions in this sense is particularly important for algorithmic applications because many parameters are not closed under edge deletions, while many parameters are closed under contraction.

This paper is part of a series of papers on bidimensionality [DHT05, DHN$^+$04, DFHT05, DH04a, DFHT, DH04b, DFHT04, DH05b, DH05a]. The theory of bidimensionality has become a comprehensive body of algorithmic and combinatorial results, with consequences including tight parameter-treewidth bounds, direct separator theorems, linearity of local treewidth, subexponential fixed-parameter algorithms, and polynomial-time approximation schemes for a broad class of problems on graphs that

[2]This statement is the contrapositive of the following property: if the parameter is k for the partially triangulated grid with g additional edges, then by α-splitting the additional edges, the parameter is at most $k + \alpha g$ on the partially triangulated grid.

exclude a fixed minor. See [DH04c] for a survey of this work and the role of this paper. In particular, the results of this paper are used in the subsequent papers [DH05b, DH05a].

2. Preliminaries. All the graphs in this paper are undirected without loops or multiple edges. Given a graph G, we denote by $V(G)$ the set of its vertices and by $E(G)$ the set of its edges. For any vertex $v \in V(G)$ we denote by E_v the set of edges incident to v. Moreover, we use the notation $N_G(v)$ (or simply $N(v)$) for the set of neighbors of v in G (i.e., vertices adjacent to v).

Given an edge $e = \{x, y\}$ of a graph G, the graph obtained from G by contracting the edge e is the graph we get if we identify the vertices x and y and remove all loops and duplicate edges. A graph H obtained by a sequence of edge-contractions is said to be a *contraction* of G. A graph class $\mathcal{C}$ is a *contraction-closed* class if any contraction of any graph in $\mathcal{C}$ is also a member of $\mathcal{C}$. A contraction-closed graph class $\mathcal{C}$ is H-*contraction-free* if $H \notin \mathcal{C}$. Given any graph class $\mathcal{H}$, we say that a contraction-closed graph class $\mathcal{C}$ is $\mathcal{H}$-*contraction-free* if $\mathcal{C}$ is H-contraction-free for *any* $H \in \mathcal{H}$.

2.1. Treewidth and Branchwidth. The notion of treewidth was introduced by Robertson and Seymour [RS86] and plays an important role in their fundamental work on graph minors. To define this notion, first we consider the representation of a graph as a tree, which is the basis of our algorithms in this paper. A *tree decomposition* of a graph G, denoted by $TD(G)$, is a pair (χ, T) in which T is a tree and $\chi = \{\chi_i \mid i \in V(T)\}$ is a family of subsets of $V(G)$ such that (1) $\bigcup_{i \in V(T)} \chi_i = V(G)$; (2) for each edge $e = \{u, v\} \in E(G)$ there exists an $i \in V(G)$ such that both u and v belong to χ_i; and (3) for all $v \in V(G)$, the set of nodes $\{i \in V(T) \mid v \in \chi_i\}$ forms a connected subtree of T. To distinguish between vertices of the original graph G and vertices of T in $TD(G)$, we call vertices of T *nodes* and their corresponding χ_i's *bags*. The maximum size of a bag in $TD(G)$ minus one is called the *width* of the tree decomposition. The *treewidth* of a graph G ($\mathbf{tw}(G)$) is the minimum width over all possible tree decompositions of G.

A *branch decomposition* of a graph (or a hypergraph) G is a pair (T, τ), where T is a tree with vertices of degree 1 or 3 and τ is a bijection from the set of leaves of T to $E(G)$. The *order* of an edge e in T is the number of vertices $v \in V(G)$ such that there are leaves t_1, t_2 in T in different components of $T(V(T), E(T) - e)$ with $\tau(t_1)$ and $\tau(t_2)$ both containing v as an endpoint.

The *width* of (T, τ) is the maximum order over all edges of T, and the *branchwidth* of G, $\mathbf{bw}(G)$, is the minimum width over all branch decompositions of G. (In the case where $|E(G)| \leq 1$, we define the branchwidth to be 0; if $|E(G)| = 0$, then G has no branch decomposition; if $|E(G)| = 1$, then G has a branch decomposition consisting of a tree with one vertex—the width of this branch decomposition is considered to be 0).

It is easy to see that, if H is a minor of G, then $\mathbf{bw}(H) \leq \mathbf{bw}(G)$. The following result is due to Robertson and Seymour [RS91, Theorem 5.1].

LEMMA 2.1 ([RS91]). *For any connected graph G where $|E(G)| \geq 3$, $\mathbf{bw}(G) \leq \mathbf{tw}(G) + 1 \leq \frac{3}{2}\mathbf{bw}(G)$.*

The main combinatorial result of this paper is Theorem 4.8 (see the end of Section 4.2), which determines, for any k and g, a family of graphs $\mathcal{H}_{k,g}$ such that any $\mathcal{H}_{k,g}$-contraction-free graph G with genus g will have branchwidth $O(gk)$. To describe such a family, we will need some definitions on graph embeddings.

2.2. Graph Embeddings. Most of the notions defined in this subsection can be found in [MT01].

A *surface* Σ is a compact 2-manifold without boundary. We will always consider connected surfaces. We denote by $\mathbb{S}_0$ the sphere $\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$. A *line* in Σ is a subset homeomorphic to $[0, 1]$. An *O-arc* is a subset of Σ homeomorphic to a circle. A subset of Σ is an *open disk* if it is homeomorphic to $\{(x, y) \mid x^2 + y^2 < 1\}$, and it is a *closed disk* if it is homeomorphic to $\{(x, y) \mid x^2 + y^2 \leq 1\}$.

A 2-*cell embedding* of a graph G in a surface Σ is a drawing of the vertices as points in Σ and the edges as lines in Σ such that every face (connected component of $\Sigma - E(G) - V(G)$) is an open disk. To simplify notations we do not distinguish between a vertex of G and the point of Σ used in the drawing to represent the vertex or between an edge and the line representing it. We also consider G as the union of the points corresponding to its vertices and edges. That way, a subgraph H of G can be seen as a graph H where $H \subseteq G$. We use the notation $V(G)$, $E(G)$, and $F(G)$ for the set of the vertices, edges, and faces of the embedded graph G. For $\Delta \subseteq \Sigma$, $\overline{\Delta}$ is the *closure* of Δ. The boundary of Δ is $\mathbf{bd}(\Delta) = \overline{\Delta} \cap \overline{\Sigma - \Delta}$, and the interior is $\mathbf{int}(\Delta) = \overline{\Delta} - \mathbf{bd}(\Delta)$.

A subset of Σ meeting the drawing only in vertices of G is called *G-normal*. If an O-arc is G-normal, then we call it a *noose*. The length of a noose is the number of vertices it meets.

Representativity [RS88] is the measure of the "density" of the embedding of a graph in a surface. The *representativity* (or *facewidth*) $\mathbf{rep}(G)$ of a graph G embedded in surface $\Sigma \neq \mathbb{S}_0$ is the smallest length of a noncontractible noose in Σ. In other words, $\mathbf{rep}(G)$ is the smallest number k such that Σ contains a noncontractible (non-null-homotopic in Σ) closed curve that intersects G in k points.

It is more convenient to work with Euler genus. The *Euler genus* $\mathbf{eg}(\Sigma)$ of a surface Σ is equal to the nonorientable genus $\tilde{g}(\Sigma)$ (or the crosscap number) if Σ is a nonorientable surface. If Σ is an orientable surface, $\mathbf{eg}(\Sigma)$ is $2g(\Sigma)$, where $g(\Sigma)$ is the orientable genus of Σ. Given a graph G, its Euler genus $\mathbf{eg}(G)$ is the minimum $\mathbf{eg}(\Sigma)$ where Σ is a surface in which G can be embedded.

2.3. Splitting Graphs and Surfaces. In this section we describe precisely how to cut along a noncontractible noose in order to decrease the genus of the graph until we obtain a planar graph.

Let G be a graph and let $v \in V(G)$. Also suppose we have a partition $\mathcal{P}_v = (N_1, N_2)$ of the set of the neighbors of v. Define the *splitting* of G with respect to v and P_v to be the graph obtained from G by (i) removing v and its incident edges; (ii) introducing two new vertices v^1, v^2; and (iii) connecting v^i with the vertices in $N_i, i = 1, 2$. If H is the result of the consecutive application of the above operation on some graph G, then we say that H is a *splitting* of G. If additionally in such a splitting process we do not split vertices that are results of previous splittings, then we say that H is a *fair splitting* of G.

The following lemma defines how to find a fair splitting for a given noncontractible noose. It will serve as a link between Lemmas 4.4 and 4.7 in the proof of the main result of this paper. Its proof is straightforward, following lines similar to those of [DFHT].

Lemma 2.2. *Let G be a connected graph 2-cell embedded in a nonplanar surface Σ, and let N be a noncontractible noose of Σ. Then there is a fair splitting G' of G affecting the set $S = (v_1, \ldots, v_\rho)$ of the vertices of G met by N, such that (i) G' has at most two connected components; (ii) each connected component of G'*

can be 2-cell embedded in a surface with Euler genus strictly smaller than the Euler genus of Σ; and (iii) there are two faces f_1 and f_2, each in the 2-cell embedding of a connected component of G' (and the connected components are different for the two faces if G' is disconnected), such that the boundary of f_i, for $i \in \{1,2\}$, contains $S_i = (v_1^i, \ldots, v_\rho^i)$, where v_j^1 and v_j^2 are the vertices created after the splitting of the vertex v_j, for $j = 1, \ldots, \rho$.

3. Incomplete Embeddings and Their Properties. In this section we give a series of definitions and results that support the proof of the main theorem of the next section. In particular, we will need special embeddings of graphs that are incomplete, i.e., only *some* of the edges and vertices of the graph are embedded in a surface. Moreover, we will extend the definition of a contraction so that it will also consider contractions of faces for the part of the graph that is embedded.

Let Σ be a surface (orientable or not). Given a graph G, a vertex set $V \subseteq V(G)$, and an edge set $E \subseteq E(G)$ such that $\cup_{v \in V} E_v \subseteq E$, we denote by G^- the graph obtained by G by removing all vertices in V and all edges in E, i.e., the graph $G^- = (V(G) - V, E(G) - E)$.[3] We also say that G is (V,E)-*embeddable in* Σ if G^- has a 2-cell embedding in Σ. We call the graph G^- the *ground* of G and we call the edges and vertices of G^- *landed*. On the other hand, we call the vertices in V and E *flying*. Notice that the flying edges are partitioned into three categories: those that have both endpoints in $V(G) - V$ (we call them *bridges*), those with one endpoint in $V(G) - V$ and one endpoint in V (we call them *pillars*), and those with both endpoints in V (we call them *clouds*). From now on, whenever we refer to a graph (V,E)-embeddable in Σ we will accompany it with the corresponding 2-cell embedding of G^- in Σ.

The set of *atoms* of G with respect to some (V,E)-embedding of G in Σ is the set $A(G) = V(G) \cup E(G) \cup F(G)$, where $F(G)$ is the set of faces of the 2-cell embedding of G^- in Σ. Notice that a flying atom can only be a vertex or an edge. In this paper, we will consider the faces as open sets whose boundaries are cyclic sequences of edges and vertices.

3.1. Contraction Mappings. A strengthening of a graph being a contraction of another graph is for there to be a "contraction mapping" which preserves some aspects of the embedding in a surface during the contractions. See Fig. 3.1 for an example. Given two graphs G and H that are $(V^{(G)}, E^{(G)})$- and $(V^{(H)}, E^{(H)})$-embeddable in Σ and Σ', respectively, we say that $\phi : A(G) \to A(H)$ is a *contraction mapping* from G to H with respect to their corresponding embeddings if the following conditions are satisfied:

1. For any $v \in V(G)$, $\phi(v) \in V(H)$.
2. For any $e \in E(G)$, $\phi(e) \in E(H) \cup V(H)$.
3. For any $f \in F(G)$, $\phi(f) \in F(H) \cup E(H) \cup V(H)$.
4. For any $v \in V(H)$, $G[\phi^{-1}(v)]$ is a connected subgraph of G.
5. $\{\phi^{-1}(v) \mid v \in V(H)\}$ is a partition of $V(G)$.
6. If $\phi(\{x,y\}) = v \in V(H)$, then $\phi(x) = \phi(y) = v$.
7. If $\phi(\{x,y\}) = e \in E(H)$, then $\{\phi(x), \phi(y)\} \in E(H)$.
8. If $f \in F(G)$ and $\phi(f) = v \in V(G)$ and $f = (x_0, \ldots, x_{r-1})$, then $\phi(\{x_i, x_{i+1}\}) = \phi(x_i) = v$ for any $i = 0, \ldots, r-1$ (where indices are taken modulo r).

[3] In this paper, the vertices and edges of a graph G are referred to as $V(G)$ and $E(G)$, respectively, while V and E are subsets.

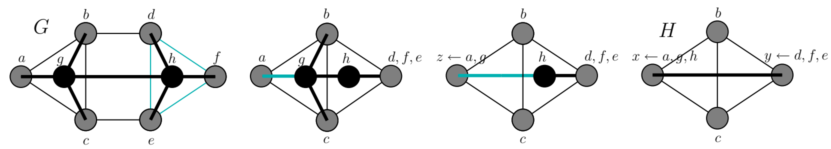

FIG. 3.1. *An example of a contraction of a graph (V,E)-embeddable in $\mathbb{S}_0$ where $V = \{g,h\}$ and $E = \{\{g,a\},\{g,b\},\{g,c\},\{h,g\},\{h,d\},\{h,f\},\{h,e\}\}$. The contraction is shown in a three-step sequence: contracting the edges of the face $\{d,e,f\}$, then the edge $\{a,g\}$, and then edge $\{z,h\}$. A contraction mapping from G to H is defined as follows: $\phi(a)=\phi(g)=\phi(h)=\phi(\{a,g\})=\phi(\{g,h\})=x$, $\phi(b)=b$, $\phi(c)=c$, $\phi(d)=\phi(f)=\phi(e)=\phi(\{f,d\})=\phi(\{d,e\})=\phi(\{e,f\})=\phi(\{d,e,f\})=y$, $\phi(\{a,b\})=\phi(\{g,b\})=\{x,b\}$, $\phi(\{a,c\})=\phi(\{g,c\})=\{x,c\}$, $\phi(\{b,c\})=\{b,c\}$, $\phi(\{b,d\})=\{b,y\}$, $\phi(c,e)=\{c,y\}$, $\phi(\{a,b,c\})=\{x,b,c\}$, $\phi(\{b,d,e,c\})=\{b,c,y\}$, $\phi(\{h,d\})=\phi(\{h,e\})=\phi(\{h,f\})=\{x,y\}$, $\phi(\{a,b,d,f,e,c\})=\{x,b,y,c\}$.*

9. If $f \in F(G)$ and if $\phi(f) = e$ (an edge of H), then there are two edges of f contained in $\phi^{-1}(e)$.
10. If $f \in F(G)$ and if $\phi(f) = g$ (a face of H), then each edge of g is landed and is the image of some edge in f.

Notice that, from Conditions 1, 2, and 3, the preimages of the faces of H are faces of G.

The following lemma is easy.

LEMMA 3.1. *If there exists some contraction mapping from a graph G to a graph H with respect to some embedding of G and H, then H is a contraction of G.*

Proof. Observe that H can be obtained from G if we contract all the edges of $\bigcup_{v\in V(H)} G[\phi^{-1}(v)]$. $\square$

3.2. Properties of Contraction Mappings. It is important that the two notions (contraction and existence of a contraction mapping) are identical in the case where G and H have no flying atoms, i.e., $V^{(G)} = V^{(H)} = E^{(G)} = E^{(H)} = \emptyset$. We choose to work with contraction mappings instead of simple contractions because they include stronger information that is sufficient to build the induction argument of Lemma 4.7.

LEMMA 3.2. *If G and H are graphs and H is a contraction of G then for any $(\emptyset,\emptyset)$-embedding of G and H on the same surface Σ there exists a contraction mapping from G to H with respect to their corresponding embeddings.*

Proof. We partition the contracted edges of H into connected subsets such that no edges belonging in different subsets are connected by a path of contracted edges. We map all edges of each such subset to the vertex of H that remains after their contraction. We also observe that an edge that does not belong to such a subset survives after the contraction and we map it to its copy in H. Notice that no edges incident to the same vertex belong to different subsets. Using this fact, we map each vertex of G to a vertex of H as follows: if v is incident to some contracted edge, then we map it to the same vertex to which this contracted edge is mapped. If not, then this vertex survives after the contraction procedure and therefore it is mapped to its copy in H. It remains now to map any face f of G to atoms of H. Notice that, if some face of G is incident to noncontractible edges, then these edges should be at least two. Using this fact, we distinguish three cases: in the first, all the edges in $\mathbf{bd}(f)$ belong to the same subset of the partition. Then we map f to the vertex occurring by the construction of the edges in this subset. In the second case, there are exactly two noncontractible edges of G in $\mathbf{bd}(f)$. Then these two edges should be mapped to the

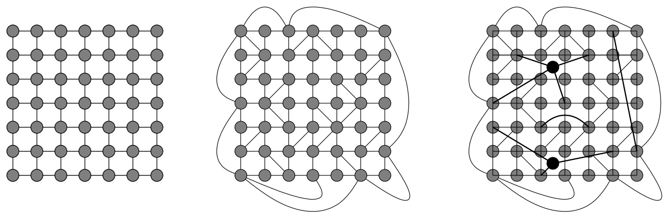

FIG. 3.2. *A (7×7)-grid, a partially triangulated (7×7)-grid, and a $(7, 9)$-gridoid (the flying edges and vertices are the distinguished ones).*

same edge e of H, and we map f to e. In the third case, we have that $\mathbf{bd}(f)$ contains more than 2 noncontractible edges of G. We observe that the noncontractible edges in $\mathbf{bd}(f)$ define a face g in H and we map f to g. It is now easy to verify that the mapping we just defined satisfies Conditions 1–10. □

The following lemma is a useful generalization of Lemma 3.2.

LEMMA 3.3. *Let G be a graph (V, E)-embeddable on some surface Σ and let H be the graph occurring from G after contracting edges in $E(G^-)$. Then $G[V] = H[V]$, H is also (V, E)-embeddable in Σ, and there exists a contraction mapping ϕ from G to H with respect to their corresponding embeddings.*

Proof. Let H^- be the result of the application of the same contractions on G^- embeddable on the surface Σ. From Lemma 3.2, there exists a mapping ϕ' from G^- to H^-. Add in H^- all the flying vertices and all the clouds of G. This implies that $G[V] = H[V]$. Then, for any flying vertex v, add in H^- all pillar edges connecting it to the vertices in $\{\phi^{-1}(u) \mid u \in V(G^-)$ and $u \in N_G(v)\}$. Finally, for any bridge $\{v, u\}$ of G where $\phi'(v) \neq \phi'(u)$, we add in H^- the bridge $\{\phi'(v), \phi'(u)\}$. Notice that after the aforementioned edge additions transform H^- to H that is embeddable in Σ.

We now construct the required map ϕ as follows: for any $a \in A(G^-), \phi(a) = \phi'(a)$; for any $v \in V, \phi(v) = v$. Finally, for any $\{x, y\} \in E$, we define $\phi(\{x, y\})$ as follows. If $\phi(x) \neq \phi(y)$, we set $\phi(\{x, y\}) = \{\phi(x), \phi(y)\}$; and if $\phi(x) = \phi(y)$, we set $\phi(\{x, y\}) = \phi(x)$. □

3.3. Gridoids. A *partially triangulated* $(r \times r)$-*grid* is any graph that contains an $(r \times r)$-grid as a subgraph and is a subgraph of some triangulation of the same $(r \times r)$-grid.

We call a graph G an (r, k)-*gridoid* if it is (V, E)-embeddable in $\mathbb{S}_0$ for some pair V, E, where $|E| \leq k$, $E(G[V]) = \emptyset$ (i.e., G does not have clouds), and G^- is a partially triangulated $(r' \times r')$-grid embedded in $\mathbb{S}_0$ for some $r' \geq r$. For an example of a $(7, 9)$-gridoid and its construction, see Fig. 3.2.

4. Main Result. In this section we will prove that, if a graph G has branchwidth more than $4k(\mathbf{eg}(G)+1)$, then G contains as a contraction some $(k-12\mathbf{eg}(G), \mathbf{eg}(G))$-gridoid, where $k \geq 12\mathbf{eg}(G)$.

4.1. Transformations of Gridoids. LEMMA 4.1. *Let G be an (r, k)-gridoid $(\emptyset, E)$-embeddable in $\mathbb{S}_0$ and let $v \in V(G^-)$. Then there exists some contraction mapping ϕ from G to some $(r-4, k+1)$-gridoid $(\{v\}, E \cup \{\{v, y\}\})$-embeddable in $\mathbb{S}_0$ such that $\phi(v) = v$.*

Proof. Let G^* be the grid from which G is constructed. Let (x, y) denote the coordinates of the vertex v in G^*. We define the required map ϕ by distinguishing two cases.

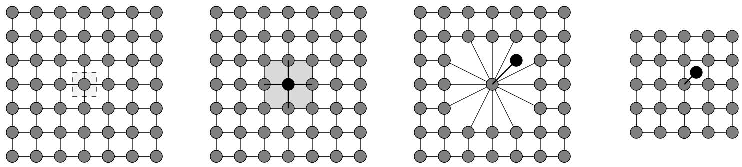

FIG. 4.1. *An example of the first case in the proof of Lemma 4.1.*

Case 1: (x, y) is a vertex of degree 4 in G^*, i.e., $x, y \notin \{1, r\}$. Refer to Figure 4.1 for an example. Let $f_1, \ldots, f_\rho$ be the faces of G^- containing v, cyclically ordered in the way they appear in the embedding of G^- in Σ, and set $f = \cup_{i=1,\ldots,\rho}\overline{f}_i$. We first consider a modified embedding of G where now v is a flying vertex (we add it in V) and the remaining ground graph has the same embedding as before with the difference that now $f - \mathbf{bd}(f)$ is a face replacing the faces $f_1, \ldots, f_r$ that disappear. We construct a graph J that is $(\{v\}, E \cup \{\{v, y\}\})$-embeddable in Σ by contracting all the edges in $\mathbf{bd}(f)$ to a single vertex y. This makes the face $f - \mathbf{bd}(f)$ "disappear" toward creating y and the pillars adjacent to v shrink to a single edge connecting v with y. We construct a mapping $\phi' : A(G) \to A(J)$ as follows. Notice that any atom a of G that is not contained in f is also an atom of J. If a is such an atom, then set $\phi'(a) = a$. If $a \in \mathbf{bd}(f)$, then $\phi'(a) = y$. If $a \in f - \mathbf{bd}(f) - \{v\}$, then set $\phi'(a) = \{y, v\}$ and, finally, set $\phi'(v) = v$. It is easy to verify that ϕ' is a contraction mapping G to J such that $\phi'(v) = v$.

We now further contract in J^- all the edges in $\{\{(x-1, i), (x, i)\}, \{(x, i), (x+1, i)\} \mid i = 1, \ldots, y-2, y+2, \ldots, r\}$ and in $\{\{(i, y-1), (i, y)\}, \{(i, y), (i, y+1)\} \mid i = 1, \ldots, x-2, x+2, \ldots, r\}$, and we call H the resulting graph (these contractions are well defined because these edges are not contracted during the previous transformation of G to J). Observe that H is an $(r-2, k+1)$-gridoid and that applying Lemma 3.3 we construct a contraction mapping ϕ'' from J to H with respect to their $(\{v\}, E \cup \{\{v, y\}\})$-embeddings in $\mathbb{S}_0$, where $\phi''(v) = v$. It remains to observe that $\phi = \phi' \circ \phi''$ is the required map and $\phi(v) = v$.

Case 2: We now examine the case where $v = (x, y)$ is a vertex of G^* with degree 2 or 3. Let q be the union of all the squares of G^* that have common edges with the unique face of G^* that is not a square (we call the cycle defined by the boundary of this face the *exterior cycle*). We construct a minor of G^- by contracting all the edges in $\mathbf{bd}(\bar{q})$. $\mathbf{bd}(\bar{q})$ contains two connected components that are disjoint cycles, and one of them is the exterior cycle of G^*. These components are shrunk to two distinct adjacent vertices v and u, and we can assume that v is the one of degree 1. We further contract some edge incident to u that is different from $\{v, u\}$. The remaining graph is a partially triangulated $(r-4, r-4)$-grid with some additional pending edge adjacent to its exterior cycle. Let G' be the graph occurring from G after applying to G the same sequence of contractions as in G^-. From Lemma 3.3 we have that G' is also $(\emptyset, E)$-embeddable in Σ and there exists a contraction mapping ϕ' from G to G' with respect to their corresponding embeddings. Moreover, as v is an endpoint of the edges contracted toward forming the vertex v of G', we have $\phi'(v) = v$. Now we update the embedding of G' so that v becomes a flying vertex (we move it in V) and the remaining ground graph has the same embedding as before with the difference that now $\{v, u\}$ is not drawn anymore in the surface (it becomes a pillar). We will use the notation H in order to denote G' $(\{v\}, E \cup \{\{v, u\}\})$-embeddable in Σ in the updated way. We also define a mapping ϕ'' from G' to H with respect to their corresponding

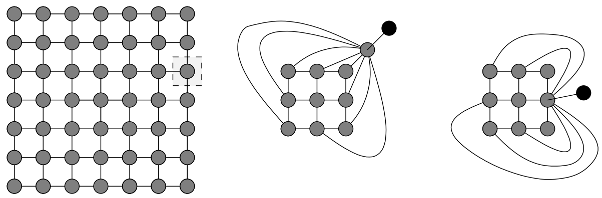

FIG. 4.2. *An example of the second case in the proof of Lemma 4.1.*

embeddings so that $\phi''(a) = a$ for any $a \in A(G')$ that is not the face of G' containing the edge $\{v, u\}$. For this face f we set $\phi''(f) = f'$, where f' is the face created in H^- after the removal of $\{v, u\}$ from the interior of f in G'^-.

Observe that H is an $(r-4, k+1)$-gridoid and that $\phi = \phi' \circ \phi''$ is a contraction mapping from G to H with respect to the $(\emptyset, E)$-embedding of G and the $(\{v\}, E \cup \{\{v, y\}\})$-embedding of H in $\mathbb{S}_0$ where $\phi(v) = v$. This completes the proof as $\phi(v) = v$. ∎

LEMMA 4.2. *Let G be an (r, k)-gridoid $(\emptyset, E)$-embeddable in $\mathbb{S}_0$, and let e be some of its flying edges. Then there exists some $(r-4, k)$-gridoid H $(\emptyset, E')$-embeddable in $\mathbb{S}_0$ for some E' and a contraction mapping ϕ of G to H such that $\phi(e) \in V(H)$.*

Proof. Let $e = \{v, u\}$. Refer to Figure 4.3 for an example. According to Lemma 4.1, there exists some contraction mapping ϕ from G to some $(r-4, k+1)$-gridoid G' $(\{v\}, E \cup \{\{v, y\}\})$-embeddable in $\mathbb{S}_0$ such that $\phi(v) = v$. We construct a new graph H $(\emptyset, E - \{v, u\} \cup \{v, y\})$-embeddable in $\mathbb{S}_0$ by simply contracting the edge $\{v, u\}$ to the vertex v. We define a contraction mapping ϕ' from G' to H as follows: if $a \in A(G') - \{v, u, \{v, u\}\}$, then $\phi'(a) = a$; otherwise $\phi'(a) = v$. Finally, we observe that $\phi \circ \phi'$ is a contraction mapping ϕ from G to H such that $\phi(e) \in V(H)$. ∎

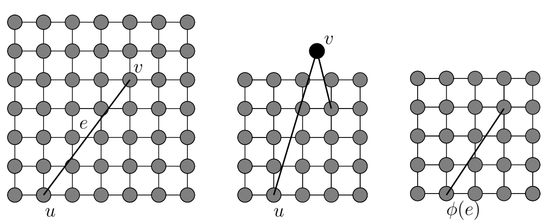

FIG. 4.3. *An example of the proof of Lemma 4.2.*

LEMMA 4.3. *Let G be an (r, k)-gridoid $(\emptyset, E)$-embeddable in $\mathbb{S}_0$, and let a be some of its atoms. Then there exists some $(r-4, k)$-gridoid $(\emptyset, E)$-embeddable in $\mathbb{S}_0$ and a contraction mapping ϕ from G to H with respect to their corresponding embeddings such that $\phi(a) \in V(H)$.*

Proof. We will denote as G^* the $(r \times r)$-grid that should be triangulated in order to construct G^-.

The lemma follows directly from Lemma 4.2 in the case where a is a flying edge of G. If this is not the case, then a is an atom of G^- that is either a vertex or an edge or a face. If a is a face, then either it is a square or triangular face included in some square $C = ((x, y), (x, y+1), (x+1, y), (x+1, y+1))$ of G^* or it is a face with all vertices in the exterior face of G^*.

We will first examine all the aforementioned cases except for the last one. We take the $(r-1, k)$-gridoid H^- that is constructed if we contract in G^- all the edges in $\{\{(x,i),(x+1,i)\} \mid i=1,\dots,r\}$ and in $\{\{(i,y),(i,y+1)\} \mid i=1,\dots,r\}$.

We will now examine the case where a is a face with all vertices in the exterior face of G^*. Then we take the $(r-2,k)$-gridoid H^- that is constructed if we contract in G^- all the edges included in the exterior face of G^* to a single vertex q and then contract some edge incident to q.

Because in both cases H^- is a contraction of H, we can use Lemma 3.3 to construct a contraction $(\emptyset,E)$-mapping ϕ from G to H with respect to their $(\emptyset,E)$-embeddings in Σ. Notice also that $\phi(a) \in V(H^-)$ because, in both cases, all the edges of the cycle $(x,y),(x,y+1),(x+1,y),(x+1,y+1)$ are contracted (and therefore mapped) to a single vertex of H^-. □

4.2. Excluding Gridoids as Contractions. Lemma 4.4. *Let G be a graph $(\emptyset,\emptyset)$-embeddable on some surface Σ. Let H be an (r,k)-gridoid $(\emptyset,E)$-embeddable on the sphere, and assume that ϕ is a contraction mapping from G to H with respect to their corresponding embeddings.*

Let $\{v_1^i,\dots,v_\rho^i\}$, $i=1,2$, be subsets of the vertices of two faces f_i, $i=1,2$, of the embedding of G where $f_1 \cap f_2 = \emptyset$ (we assume that the orderings of the indices in each subset respect the cyclic orderings of the vertices in $f_i, i=1,2$). Let G' be the graph obtained if we identify in G the vertex v_i^1 with the vertex v_i^2. Then, the following hold:

- *(a) G' has some 2-cell embedding on a surface of bigger Euler genus.*
- *(b) There exists some $(r-12,k+1)$-gridoid H, $(\emptyset,E\cup\{\{e\}\})$-embeddable on the sphere such that there exists some contraction mapping from G' to H with respect to their corresponding embeddings.*

Proof. (a) Let Σ be the surface where G is embedded. We define a surface Σ^- from Σ by removing the two "patches" defined by the (internal) points of the faces f_1 and f_2. Notice that G is still embeddable on Σ^- and that Σ^- is a surface with boundary whose connected components are the boundaries B_1,B_2 of the faces f_1 and f_2. We now construct a new surface from Σ^- by identifying the boundaries B_1 and B_2 in a way that v_i^1 is identified with v_i^2. Notice that the embedding that follows is a 2-cell embedding and that the new surface has bigger Euler genus.

(b) From Conditions 1, 2, and 3 in Section 3.1, $\phi(f_1)$ is either a vertex, an edge, or a face of H. We apply Lemma 4.3 to construct a contraction mapping σ_1 from H to some $(r-4,k)$-gridoid H_1, where $\sigma_1(\phi(f_1)) \in V(H_1)$. Notice again that $\sigma_1(\phi(f_2))$ is either a vertex, an edge, or a face of H_1. We again use Lemma 4.3 to construct a contraction mapping σ_2 from H_1 to some $(r-8,k)$-gridoid H_2, where $\sigma_2(\sigma_1(\phi(f_i))) = v_i \in V(H_2), i=1,2$. We now apply Lemma 4.1 for v_1 and construct some contraction mapping σ_3 from H_2 to some $(r-12,k+1)$-gridoid H_3, $(\{v_1\},E\cup\{\{v_1,y\}\})$-embeddable in $\mathbb{S}_0$ such that $\sigma_3(v_1)=v_1$. Summing up, we have that $\phi' = \phi\circ\sigma_1\circ\sigma_2\circ\sigma_3$ is a map from G to H_3 with respect to the $(\emptyset,\emptyset)$-embedding of G on Σ and the $(\{v_1\},E\cup\{\{v_1,y\}\})$-embedding of H_3 in $\mathbb{S}_0$. Moreover, we have that $\phi'(f_1)=v_1$ and $\phi'(f_2)=v_2 \in V(H_3)$ (to facilitate the notation we assume that $\sigma_3(v_2)=v_2$).

Notice now that if v is the result of the identification in H_3 of the vertex v_1 with the vertex v_2, we take a new graph H $(\emptyset,E\cup\{\{v,y\}\})$-embeddable in $\mathbb{S}_0$. Let A' be all the atoms of G that are not included in the faces f_1 and f_2. Notice that these atoms are not harmed while constructing G' from G, and we set $\mu(a)=\phi'(a)$ for each $a\in A'$. Finally, for each atom $a \in A(G')-A$, we set $\mu(a)=v$. It now is easy to check that μ is a contraction mapping from G' to H with respect to their corresponding embeddings. Because H is an $(r-12,k+1)$-gridoid, we are done. □

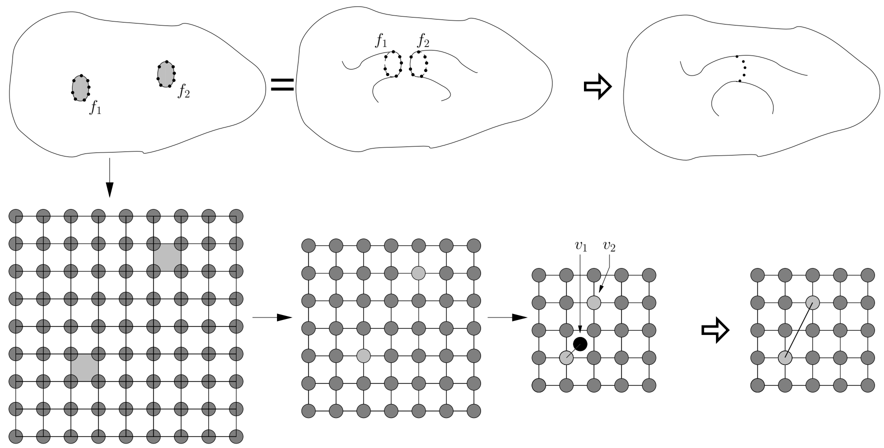

FIG. 4.4. *An example of the transformations in the proof of Lemma 4.7.*

The following is one of the main results in [DFHT].

THEOREM 4.5. *Let G be a graph 2-cell embedded in a nonplanar surface Σ of representativity at least θ. Then one can contract edges in G to obtain a partially triangulated $(\theta/4 \times \theta/4)$-grid.*

We also need the following easy lemma.

LEMMA 4.6. [DFHT] *Let G be a graph and let H be the graph occurring from G after splitting some vertex $v \in V(G)$. Then $\mathbf{bw}(G) \leq \mathbf{bw}(H) + 1$.*

We are now ready to prove the central result of this section.

LEMMA 4.7. *Let G be a graph $(\emptyset, \emptyset)$-embeddable on a surface Σ of Euler genus g and assume that $\mathbf{bw}(G) \geq 4(r - 12g)(g + 1)$. Then there exists some $(r - 12g, g)$-gridoid H, $(\emptyset, E)$-embeddable in $\mathbb{S}_0$ such that there exists some contraction mapping from G to H with respect to their corresponding embeddings.*

Proof. First, if the graph G is disconnected, we discard all but one connected component C such that $\mathbf{bw}(C) = \mathbf{bw}(G)$.

We use induction on g. Clearly, if $g = 0$, G is a planar graph and after applying Lemma 3.1, the result follows from the planar exclusion theorem of [RST94]. (The induction base relies heavily on the fact that for conventional embeddings the contraction relation is identical to our mapping.)

Suppose now that $g \geq 1$ and the theorem holds for any graph embeddable in a surface with Euler genus less than g. Refer to Fig. 4.4. If the representativity of G is at least $4(r - 12g)$, then by Theorem 4.5 we can contract edges in G to obtain a partially triangulated $((r - 12g) \times (r - 12g))$-grid (with no additional edges) and we are done. Otherwise, the representativity of G is less than $4(r - 12g)$. In this case, the smallest noncontractible noose has vertex set S of size less than $4(r - 12g)$. Let G' be a splitting of G with respect to S as in Lemma 2.2. Recall that G' is now $(\emptyset, \emptyset)$-embeddable on a surface of Euler genus $g' \leq g - 1$.

By Lemma 4.6, the branchwidth of G' is at least the branchwidth of G minus $|S|$. Because $|S| \leq 4(r - 12g)$, we have that $\mathbf{bw}(G') \geq 4(r - 12g)(g + 1) - 4(r - 12g) = 4(r - 12g)g \geq 4(r - 12g)(g' + 1)$. By the induction hypothesis there exists some $(r - 12g', g')$-gridoid H', $(\emptyset, E)$-embeddable in $\mathbb{S}_0$ such that there exists some contraction mapping from G' to H with respect to their corresponding embeddings. From Lemma 4.4, there exists some $(r - 12g' - 12, g' + 1)$-gridoid H, $(\emptyset, E \cup \{\{e\}\})$-

embeddable on the sphere such that there exists some contraction mapping from G to H with respect to their corresponding embeddings. Because $r - 12g' - 12 \geq r - 12g$ and $g' + 1 \leq g$, we are done. □

Now we have the conclusion of this section.

THEOREM 4.8. *If a graph G excludes all $(k - 12\mathbf{eg}(G), \mathbf{eg}(G))$-gridoids as contractions for some $k \geq 12\mathbf{eg}(G)$, then G has branchwidth at most $4k(\mathbf{eg}(G) + 1)$.*

By Lemma 2.1 we can obtain a treewidth-parameter bound as desired.

5. Algorithmic Consequences. Define the *parameter* corresponding to an optimization problem to be the function mapping graphs to the solution value of the optimization problem. In particular, *deciding* a parameter corresponds to computing whether the solution value is at most a specified value k. A parameter is *contraction-bidimensional* if (1) its value does not increase when taking contractions and (2) its value on an $(r, O(1))$-gridoid is $\Omega(r^2)$.[4]

THEOREM 5.1. *Consider a contraction-bidimensional parameter P such that, given a tree decomposition of width at most w for a graph G, the parameter can be decided in $h(w) \cdot n^{O(1)}$ time. Then we can decide parameter P on a bounded-genus graph G in $h(O(\sqrt{k})) \cdot n^{O(1)} + 2^{O(\sqrt{k})} n^{3+\varepsilon}$ time.*

Proof. The algorithm proceeds as follows. First we approximately compute the treewidth and a corresponding tree decomposition of the graph G. Specifically, given a number ω, Amir's algorithm [Ami01] either reports that the treewidth of G is at least ω or produces a tree decomposition of width at most $(3 + \frac{2}{3})\omega$ in time $O(2^{3.698\omega} n^3 \omega^3 \log^4 n)$. We use this algorithm to check whether $\mathbf{tw}(G) = O(\sqrt{k})$ for a sufficiently large constant in the O notation (similar algorithmic results on treewidth that also work for our purposes can be found in [Lag96, Ree92, RS95]). If not, Theorem 4.8 tells us that the graph G has an $(O(\sqrt{k}), O(1))$-gridoid as a contraction. Property 2 of contraction bidimensionality tells us then that the parameter value is $\Omega(k)$. By choosing the constant in the O notation (in $\mathbf{tw}(G) = O(\sqrt{k})$) large enough, we can make the constant in the Ω notation greater than 1. Then we conclude that the parameter value is strictly greater than k (assuming k is at least some constant), so we can answer the decision problem negatively. On the other hand, if $\mathbf{tw}(G) = O(\sqrt{k})$, we apply the $h(\mathbf{tw}(G)) \cdot n^{O(1)}$ algorithm to the tree decomposition produced by Amir's algorithm. The overall running time is $h(O(\sqrt{k})) \cdot n^{O(1)} + 2^{O(\sqrt{k})} n^{3+\varepsilon}$. □

COROLLARY 5.2. *Vertex cover, minimum maximal matching, dominating set, edge dominating set, r-dominating set (for fixed r), and clique-transversal set can be solved on bounded-genus graphs in $2^{O(\sqrt{k})} n^{3+\varepsilon}$ time, where k is the size of the optimal solution. Feedback vertex set and connected dominating set can be solved on bounded-genus graphs in $2^{O(\sqrt{k} \log k)} n^{3+\varepsilon}$ time.*

Acknowledgments. Thanks go to Fedor Fomin for early collaboration on this project, and to the anonymous referees for their helpful feedback on the paper.

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