Open Problems from Dagstuhl Seminar 07281:

Structure Theory and FPT Algorithmics for Graphs, Digraphs and Hypergraphs

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The following is a list of the problems presented on Monday, July 9, 2007 at the open-problem session of the Seminar on Structure Theory and FPT Algorithmics for Graphs, Digraphs and Hypergraphs, held at Schloss Dagstuhl in Wadern, Germany.

Directed Maximum Leaf

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Is it fixed-parameter tractable to find an out-directed spanning tree of a given digraph with the maximum possible k of leaves?

More precisely, consider a digraph D. An *out-tree* of D is a subtree in which all vertices but one have in-degree exactly 1. The vertices of out-degree 0 are called *leaves*. Let $\ell(D)$ denote the maximum number of leaves of any out-tree in D. An *out-branching* of D is an out-tree that spans all vertices of D. Let $\ell_{\mathrm{s}}(D)$ denote the maximum number of leaves of any out-tree in D ("s" for "spanning").

A recent result presented at this workshop [AFGKS07] is that deciding $\ell(D) \geq k$ is fixed-parameter tractable. Is deciding $\ell_{\mathrm{s}}(D) \geq k$ also fixed-parameter tractable? This problem was originally posed by Michael Fellows in 2005.

References

[AFGKS07] Noga Alon, Fedor V. Fomin, Gregory Gutin, Michael Krivelevich, and Saket Saurabh. Parameterized algorithms for directed maximum leaf problems. In *Proceedings of 34th International Colloquium on Automata, Languages and Programming*, Wroclaw, Poland, July 2007, to appear.

Minimum Strong Spanning Subdigraph

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The Minimum Strong Spanning Subdigraph (MSSS) problem is as follows: given a strongly connected digraph D, find a strongly connected spanning subgraph of D with the fewest possible edges. Is the following parameterization fixed-parameter tractable? Given a strongly connected digraph D and a parameter k, determine whether D has a strongly connected spanning subgraph D' with at most $2n-2-k$ arcs.

Why this parameterization? For any digraph D on n vertices, the optimum solution to MSS uses at most $2n-2$ edges. This fact can be seen by picking a vertex v and taking the union of the arcs in an out-branching from v and an in-branching into v. (An out-branching rooted at v is a spanning subdigraph with no directed cycles so that every vertex except v has in-degree exactly 1, i.e., a spanning tree oriented away from v. An in-branching is the opposite.)

In fact, if D' is a minimum spanning strongly connected subdigraph of D, then for every vertex v, D' is exactly the union of an out-branching from v and an in-branching into v, and every such pair will use all arcs of D'. Hence, the question above can also be phrased as follows: given a strongly connected digraph D and a parameter k, is there a pair F_v^+, F_v^- of branchings such that F_v^+ is an out-branching from v and F_v^- an in-branching into v and F_v^+ and F_v^- share k arcs?

Anders Yeo points out another equivalent formulation of the problem: given a strongly connected digraph D, is it possible to "contract" k edges in D and keep the graph strongly connected? Here the contraction operation is as follows: when contracting the directed edge $x \to y$, also delete all arcs that had x as their tail and all arcs that had y as their head.

Minimum Vertex Multicut
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The parameterized vertex multicut problem is as follows: given a graph G, a parameter k, and ℓ pairs of vertices $(s_1,t_1),(s_2,t_2),\ldots,(s_\ell,t_\ell)$, is there a set S of k vertices such that $G-S$ contains no s_i–t_i path for any i? Is this problem fixed-parameter tractable (parameterized by k only)?

It is known that multicut is fixed-parameter tractable when parameterized by both k and ℓ [Marx06]. The problem is NP-hard even for $\ell = 3$ [Cunn91], so it is not fixed-parameter tractable when parameterized just by ℓ. The parameterization by k can also be asked when G is a digraph, or a directed acyclic graph, or of bounded treewidth; or when S is a set of edges instead of a set of vertices.

References

[Cunn91] William H. Cunningham. The optimal multiterminal cut problem. In *Reliability of computer and communication networks* (New Brunswick, NJ, 1989), DIMACS Ser. Discrete Math. Theoret. Comput. Sci. 5, AMS, 1991, pages 105–120.

[Marx06] Dániel Marx. Parameterized graph separation problems. *Theoretical Computer Science* 351(3):394–406, 2006. doi:10.1016/j.tcs.2005.10.007

P-sequences
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Given an alphabet Σ on n symbols, a *P-sequence* (phylogenetic sequence) is a string over Σ having no repeated symbols. Is the following problem fixed-parameter tractable? Given a set of P-sequences, determine whether there is a common supersequence of length at most $n+k$. This problem is at least as hard as directed feedback vertex set [FHS03], which was recently solved (and presented at this workshop) [CLL07, RS07].

References

[CLL07] Jianer Chen, Yang Liu, and Songjian Lu. Directed feedback vertex set problem is FPT. Preprint, 2007. http://kathrin.dagstuhl.de/files/Materials/07/07281/07281.ChenJianer.Paper.pdf

[FHS03] Michael Fellows, Michael Hallett, and Ulrike Stege. Analogs & duals of the MAST problem for sequences & trees. *Journal of Algorithms* 49:192–216, 2003. doi:10.1016/S0196-6774(03)00081-6

[RS07] Igor Razgon and Barry O'Sullivan. Directed feedback vertex set is fixed-parameter tractable. Preprint, arXiv:0707.0282 [cs.DS], 2007.

Almost 2-coloring

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Is the following problem fixed-parameter tractable? Given a graph G and a parameter k, determine whether G has a vertex 3-coloring such that one color class has at most k vertices. In other words, the goal is to remove an independent set of k vertices such that the remaining graph is bipartite.

Even Set / Minimum Distance in Linear Codes

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Is the following problem fixed-parameter tractable? Given a graph G and a parameter k, determine whether there is a set S of between 1 and k vertices such that, for every vertex v in G, $|N[v] \cap S|$ is even. (Here $N[v]$ denotes the set of neighbors of v in G.)

Almost 2-SAT

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Delete k Clauses 2-SAT is the following problem: given a 2-SAT formula, can you delete k clauses to make it satisfiable? This problem is equivalent in the fixed-parameter-tractability sense to the following problem: given a graph G having a perfect matching (so its minimum vertex cover has size at least $n/2$), does G have a vertex cover of size at most $n/2+k$? Are these problems fixed-parameter tractable?

Directed Biclique
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The Directed Biclique problem is the following: given a digraph D and a parameter k, is there a (k, k)-biclique, i.e., a set S of k vertices and another set T of k vertices in D such that (s, t) is an edge of D for all $s \in S$ and $t \in T$? (The goal does not care about edges within S or within T.) Is this problem fixed-parameter tractable?

Dániel Marx points out that this problem is interesting even in undirected bipartite graphs.

Combinatorics of Bicliques
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Let G be a graph on n vertices containing no $(k, k+1)$-biclique. What is the maximum number of (k, k)-bicliques in G? The trivial upper bound is n^{2k}. Is there an $f(k)n^{O(1)}$ upper bound? This combinatorial question seems to be at the heart of the fixed-parameter tractability of the Biclique problem.

Edge-Induced Vertex Cut
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Is the following problem fixed-parameter tractable? Given a graph G, two vertices $s, t \in V(G)$, and a parameter k, is there a set E' of at most k edges in G such that the set of vertices incident to the edges in E' forms an s–t vertex cut?

The nonparameterized version of this problem is NP-complete, and the parameterized version is W[2]-hard for hypergraphs [SS06]. There is also a trivial $2k$-approximation algorithm.

References

[SS06] Marko Samer and Stefan Szeider. Complexity and applications of edge-induced vertex-cuts. Preprint, arXiv:cs.DM/0607109, 2006.

Cliques in Line-Segment Intersection Graphs
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Is the following problem fixed-parameter tractable or $W[1]$-hard? Given a set of line segments in the plane, and given a parameter k, are there k line segments that pairwise intersect? In other words, this problem asks for a k-clique in the intersection graph of the line segments. This problem is not known to be in P or NP-complete (a question posed in [KN90]), but the fixed-parameter tractability might be easier to resolve first.

References

[KN90] Jan Kratochvíl and Jaroslav Nešetřil. INDEPENDENT SET and CLIQUE problems in intersection-defined classes of graphs. *Commentationes Mathematicae Universitatis Carolinae* 31(1):85–93, 1990.

Low-Diameter Dominating Set
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Is there an fixed-parameter approximation algorithm for k-dominating set in graphs of diameter 2? More precisely, is there an algorithm that, given a graph G and a parameter k, either determines that G has no dominating set of size k or finds a dominating set of size $g(k)$, in $f(k)n^{O(1)}$ time, for some functions f and g.

This problem is a natural target for fixed-parameter approximation because k-dominating set is still W[1]-hard for graphs of diameter 2 [MRF07]. Forcing diameter 2 makes it hard to find certificates that the dominating set must be large, because two vertices never have a distance of 3.

References

[MRF07] Catherine McCartin, Peter Rossmanith, and Michael Fellows. Frontiers of intractability for Dominating Set. Preprint, 2007.

Computing Excluded Minors of Bounded Local Treewidth
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The *local treewidth* of a graph G is a function measuring the treewidth of varying-size neighborhoods in G: $\mathrm{ltw}(G,r) = \max\{\mathrm{tw}(G[N_r(v)]) \mid v \in V(G)\}$, where $N_r(v)$ denotes the radius-r neighborhood of vertex v and $G[S]$ denotes the subgraph of G induced by vertex set S. Let $\mathcal{L}(\lambda)$ denote the family of graphs G whose minors H all have local treewidth $\mathrm{ltw}(H,r)$ bounded above by $\lambda\, r$.

Is there a computable function $f : \mathbb{N} \to \mathbb{N}$ such that every graph $G \notin \mathcal{L}(\lambda)$ has a minor $G' \notin \mathcal{L}(\lambda)$ of treewidth at most $f(\lambda)$? This combinatorial problem is posed in [AGK07] (presented at this workshop) in the context of computing excluded minors for the graphs of linear local treewidth.

At the open problem session, the following related problem was posed. The *$k \times k$ pyramid graph* is formed by taking the $k \times k$ grid graph and attaching one additional vertex to all other vertices. Define $K(\lambda)$ to be the class of graphs that do not have the $(\lambda-1)\times(\lambda-1)$ pyramid as a minor. Is there a computable function $f : \mathbb{N} \to \mathbb{N}$ such that any graph $G \notin K(\lambda)$ contains a minor $G' \notin K(\lambda)$ of treewidth at most $f(\lambda)$? While closely related to the local treewidth problem ($L(\lambda) \subseteq K(\lambda')$), they are not equivalent ($K(\lambda) \not\subseteq L(\lambda')$), and in fact this K problem has a relatively easy positive solution, found during the workshop.

References

[AGK07] Isolde Adler, Martin Grohe, and Stephan Kreutzer. Computing excluded minors. Preprint, 2007.

Correlation Clustering
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One form of "correlation clustering", introduced in [BBC04], is as follows. A *cluster graph* is a disjoint union of cliques. A *fuzzy graph* is a graph where every two vertices are connected by either an *edge*, a *nonedge*, or an *unknown*. An *edit* is the replacement of one type of an edge with another: editing an edge into a nonedge, or vice versa, costs 1; and editing an unknown into an edge or nonedge costs 0. Given a fuzzy graph, can we edit it into a cluster graph using cost at most k? Is this problem fixed-parameter tractable?

References

[BBC04] Nikhil Bansal, Avrim Blum, and Shuchi Chawla. Correlation clustering. *Machine Learning* 56(1–3):89–113, 2004.

Directed Odd-Cycle-Free Edge Deletion
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The Directed Odd-Cycle-Free Edge Deletion problem is the following: given a digraph D and a parameter k, can we delete at most k arcs from D such that the remaining graph has no directed cycle of odd length? Is this problem fixed-parameter tractable?

This problem is motivated from an application in systems biology. It is NP-complete by a reduction from the undirected case. The undirected version can be solved in $O(2^k \cdot m^2)$ time [GGHNW06] by iterative compression. However, that algorithm relies on the characterization of odd-cycle-free graphs by 2-colorings, which does not work for general digraphs. (It works only for strongly connected digraphs.)

[This problem was not posed during the open problem session, but was mentioned later.]

References

[GGHNW06] Jiong Guo, Jens Gramm, Falk Hüffner, Rolf Niedermeier, and Sebastian Wernicke. Compression-based fixed-parameter algorithms for feedback vertex set and edge bipartization. *Journal of Computer and System Sciences* 72(8):1386–1396, December 2006. doi:10.1016/j.jcss.2006.02.001