Continuous Flattening of Orthogonal Polyhedra

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We consider continuously folding polyhedral surfaces down to a multilayered flat folded state. It was asked by E. Demaine, M. Demaine and A. Lebiw, and proposed in [4] if there is a continuous motion for flattening any given polyhedron. For example, it was showed by J.-i. Itoh and C. Nara [5] that a box in Fig 1(a) is continuously flat-folded by pushing four side faces in without cutting and stretching the surface, where the shapes of those four faces are changed continuously by infinitely many creases showed by long dotted line segments in Fig 1(b) and the box reaches to the multilayered flat folded state in Fig 1(c).

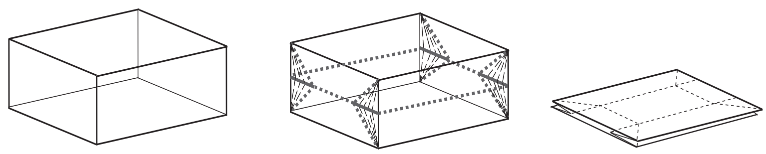

Figure 1: (a) A box; (b) mountain and valley creases are showed by bold grey line segments and bold grey dotted line segments respectively for the final flat folded state, together with long dotted line segments for moving creases; (c) the final flat folded state.

There are very important two theorems related to our topic. One of them is the Cauchy rigidity theorem [2, 4]: Any convex polyhedron is rigid if all faces are rigid. The other is the bellow theorem [3]: The volume of any (non-convex) polyhedron with rigid faces is invariant even if it is flexible. To flatten a polyhedron, the volume should be changed continuously, and so some faces are not rigid and their shapes should be changed by infinitely many (moving) creases.

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For any convex polyhedron P, by using the cut locus and Alexandrov gluing theorem, J.-i. Itoh, C. Nara and C. Vîlcu [6] gave a continuous flattening motion, and very recently, by using the straight skeleton gluing, the authors et al. [1] gave another continuous flattening motion. However, it remains as an open problem for *non-convex* polyhedra to find continuous flattening motions.

Our main result is the continuous flattening of orthogonal polyhedra which are not necessary convex or genus zero; see Fig. 2(a). Moreover we generalize the notion of orthogonal polyhedra, and show that there is a flat folding motion for any *semi-orthogonal polyhedron* which is defined as a polyhedron whose faces are orthogonal to the z-axis or the xy-plane in the Euclidean space with xyz-axes (see Fig. 2(b)), such that all faces parallel to the xy-axes have no crease during the motion.

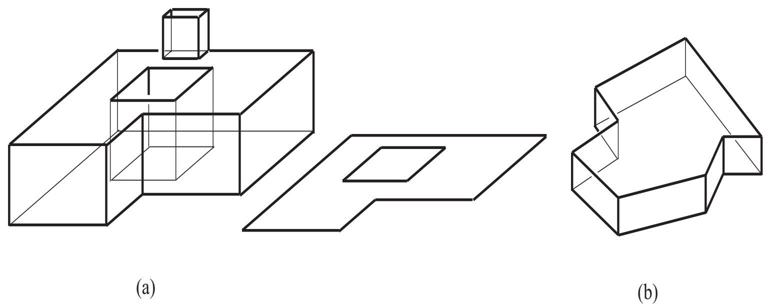

Figure 2: (a) An example $\mathcal{P}$ of an orthogonal polyhedron with the figure of the base floor of $\mathcal{P}$. (b) An example of “semi-orthogonal” polyhedra.

Theorem 1. *Every semi-orthogonal polyhedron $\mathcal{P}$ in $\mathbb{R}^3$ can be continuously flat folded such that all faces orthogonal to the z-axis are rigid and translated along the z-axis.*

References

[1] Z. Abel, E. Demaine, M. Demaine, J. Itoh, A. Lebiw, C. Nara, J. O’Rourke, Continuously flattening polyhedra using straight skeletons, Proc. 30th Annual Symposium on Computational Geometry (SoCG), 2014, pp. 396–405.

[2] A. Cauchy, Sur les polygons et les polyedres. J. Ecole Polytech. 9: 87, 1813.

[3] R. Connelly, I. Sabitov, A. Walz, The bellows conjecture, *Beiträge Algebra Geom.*, 38: 1–10, 1997.

[4] E. D. Demaine and J. O’Rourke, *Geometric Folding Algorithms; Linkages, Origami, Polyhedra*, Cambridge University Press, 2007.

[5] J.-i. Itoh and C. Nara, Continuous flattening of Platonic polyhedra In J. Akiyama, J. Bao, M. Kano and X. Tan (eds.) CGGA 2010. *Computational geometry, Graphs and Applications*, LNCS, vol. 7033, pp. 108–121. Springer, 2011.

[6] J.-i. Itoh, C. Nara, and C. Vîcu, Continuous flattening of convex polyhedra. In Computational Geometry, LNCS, vol. 7579, pp. 85–97. Springer, 2012.