Minimal Unfoldable Polyhedron

Hugo A. Akitaya[1], Erik D. Demaine[2], David Eppstein[3], Tomohiro Tachi[4], and Ryuhei Uehara[5]

[1] Department of Computer Science, Tufts University, USA.
[2] CSAIL, MIT, USA.
[3] Department of Computer Science, University of California, Irvine, USA.
[4] Department of General Systems Studies, The University of Tokyo, Japan.
[5] School of Information Science, JAIST, Japan.

1 Introduction

In computational origami, one of the most remarkable problems asks if every convex polyhedron has an edge-unfolding to a simple and nonoverlapping polygon [3, Open Problem 21.1]. Here, an *edge-unfolding* is a set of edges of the polyhedron such that cutting along these edges unfolds the polyhedron.

There are several evidences for/against the conjecture (see [3, Sections 22.2 and 22.3]); however, it is far from to be settled. As an evidence for this conjecture, it is known that any convex polyhedron P is edge-unfoldable if P is a prismoid or a dome; however, they form a quite limited set of convex polyhedra. On the other hand, as an evidence against this conjecture, there are several edge-ununfoldable nonconvex polyhedra. In this context, a key property of a polyhedron is *topological convexity*; a polyhedron P is *topologically convex* iff there is a convex polyhedron P' such that the graph induced by its vertices and edges is isomorphic to the one induced by P. In fact, if we do not restrict to topologically convex polyhedra, it is easy to make an edge-ununfoldable polyhedron with 7 faces: take a tetrahedron and add a tetrahedral bump in the middle of one of its faces.

Research on small edge-ununfoldable nonconvex polyhedra gives us insight of the open problem. In [5], Grünbaum gives a polyhedron with 13 vertices and 13 faces that is not edge-unfoldable. We sharpen such edge-unfoldability of nonconvex polyhedra:

Theorem 1.1 *(1) There exists a topologically convex polyhedron with 7 vertices and 6 faces that is not edge-unfoldable. (2) There exists a topologically convex polyhedron with 6 vertices and 7 faces that is not edge-unfoldable. (3) Any polyhedron with less than 6 vertices or 6 faces is edge-unfoldable.*

That is, we give much smaller edge-ununfoldable polyhedra, and we also prove that they are optimal with respect to the number of faces and vertices, respectively. Note that they are not only topologically convex, but also all their faces are simple polygons (or simply connected and no holes).

2 Edge-ununfoldable polyhedra

In order to prove Theorem 1.1(1), we show a polyhedron P with 7 vertices and 6 faces s.t. any edge-unfolding of P causes an overlapping. It is as shown in Figure 1: P has an *apex* G, and the *base* of P consists of one triangle ABF and one concave pentagon $BCDEF$. The *side* of P consists of 2 triangles CDG and EDG and 2 concave quadrilaterals $ABCG$ and $AFEG$. We make P symmetric to its mirror image across the plane AGD. The key properties of the polyhedron P are the following; (a) total angle around $D > 360°$, (b) total angle around $G > 360°$, (c) $\angle CBF + \angle CBA > 360°$, and (d) $\angle ABF + \angle CBA > 300°$ and $BF < BC$.

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be the graph induced by P. Then any edge-unfolding of P induces a spanning tree $\mathcal{T}$ of $\mathcal{G}$. Let $\mathcal{T}$ be any spanning tree of $\mathcal{G}$. We show that any edge-unfolding given by $\mathcal{T}$ produces overlapping. Since $\mathcal{T}$ is a tree, it has at least 2 leaves. However, from the properties (a)-(c), none of B, F, D, G can be a leaf since overlapping occurs at leaves. On the other hand, vertices A, C, E cannot be three leaves since we cannot have any spanning tree $\mathcal{T}$ that contains the edge BF with 3 leaves A, C, E. Thus $\mathcal{T}$ has only 2 leaves, i.e., $\mathcal{T}$ is a Hamilton path of $\mathcal{G}$. In most cases, $ABCG$ overlaps with $CBFED$ at vertex B by (c). The other case is shown in Figure 2. Thus the polyhedron P in Figure 1 has no edge-unfolding without overlapping. In order to prove Theorem 1.1(2), we show a polyhedron P with 6 ver-

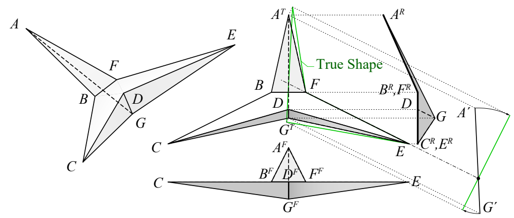

Figure 1: An edge-ununfoldable polyhedron with 7 vertices and 6 faces.

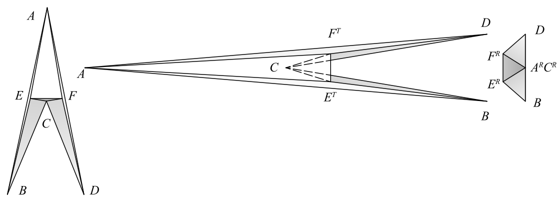

Figure 3: An edge-ununfoldable polyhedron with 6 vertices and 7 faces.

tices and 7 faces s.t. any edge-unfolding of P causes an overlapping. This polyhedron is composed of a quadrilateral $ABCD$ concave at C, connected to 2 apices E and F as in Figure 3. This polyhedron is designed to have the following properties: (a) $\angle BCE > \angle BCD$, and (b) $\angle AEB + \angle BEC + \angle CEF > 2\pi$. Property (a) forces that edges BC and CD must be cut, otherwise triangle BCE would overlap quadrilateral $ABCD$. Because A,B,D are the only vertices with positive curvature, two leaves of the spanning tree exist either on A, B, or D. If B and D are both leaves, then, the only route to A is through path $CFEA$. However, corner FEA would create an overlapping vertex by condition (b). If A and D are the only two leaves, the tree is a path $AEFDCB$. Now this again contains corner AEF, thus the unfolding overlaps by (b).

3 Edge-unfoldable polyhedra with less than 6 vertices or 6 faces

In order to prove Theorem 1.1(3),we first recall that DiBiase established that all convex polyhedra with 4, 5, or 6 vertices can be edge-unfolded [4]. Based on this result, there are only two cases to be considered; P is (a) a nonconvex square pyramid (with 5 faces and 5 vertices), or (b) a nonconvex polyhedron that consists of 6 triangles and 5 vertices with degree sequence 3,3,4,4,4. The proof that these two polyhedra are edge-unfoldable is omitted here.

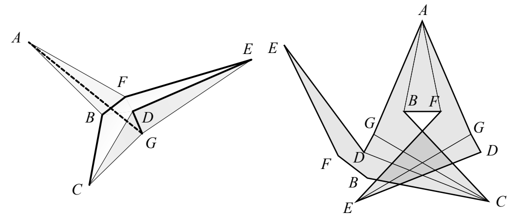

Figure 2: An edge-unfolding that causes overlapping.

4 Concluding remarks

We investigated minimal edge-ununfoldable polyhedra. We show two edge-ununfoldable polyhedra; one has 6 faces and 7 vertices, and the other has 7 faces and 6 vertices. They are optimal with respect to the number of faces and vertices for topologically convex polyhedra. Grünbaum shows an edge-ununfoldable convex faced polyhedron with 13 faces and 13 vertices [5], and Bern et al. give an edge-ununfoldable triangular faced polyhedron with 36 faces and 20 vertices [1]. Improving these upper bounds and/or showing lower bounds are future work.

Acknowledgements

David Eppstein was supported in part by NSF grants CCF-1618301 and CCF-1616248. Tomohiro Tachi was supported in part by KAKENHI 16H06106. Ryuhei Uehara was supported in part by MEXT/JSPS KAKENHI Grant Number 17H06287 and 18H04091. This work was discussed at the 34th Bellairs Winter Workshop on Computational Geometry, co-organized by Erik D. Demaine and Godfried Toussaint, held on March 22–29, 2019 in Holetown, Barbados. We thank the other participants of that workshop for providing a stimulating research environment.

References

[1] M. Bern, E. D. Demaine, D. Eppstein, E. Kuo, A. Mantler, and J. Snoeyink. Ununfoldable polyhedra with convex faces. *Computational Geometry: Theory and Applications*, 24(2):51–62, 2003.

[2] T. Biedl, E. D. Demaine, M. L. Demaine, A. Lubiw, M. Overmars, J. O'Rourke, S. Robbins, and S. Whitesides. Unfolding Some Classes of Orthogonal Polyhedra. *CCCG 98*, 1998.

[3] E. D. Demaine and J. O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, 2007.

[4] J. DiBiase. *Polytope Unfolding.* Undergraduate thesis, Smith College, 1990.

[5] B. Grünbaum. No-Net Polyhedra. *Geombinatorics*, 11:111–114, 2002.