Edge-Unfolding Prismatoids: Tall or Rectangular Base

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Abstract

We show how to edge-unfold a new class of convex polyhedra, specifically a new class of prismatoids (the convex hull of two parallel convex polygons, called the top and base), by constructing a nonoverlapping "petal unfolding" in two new cases: (1) when the top and base are sufficiently far from each other; and (2) when the base is a rectangle and all other faces are nonobtuse triangles. The latter result extends a previous result by O'Rourke that the petal unfolding of a prismatoid avoids overlap when the base is a triangle (possibly obtuse) and all other faces are nonobtuse triangles. We also illustrate the difficulty of extending this result to a general quadrilateral base by giving a counterexample to our technique.

1 Introduction

A famous open problem known as Dürer's problem [2, Open Problem 21.11, p. 298] asks whether every convex polyhedron has an ***edge unfolding***, that is, a set of edges to cut such that the remaining surface unfolds into the plane without overlap. Despite the simple statement of the problem, a solution remains elusive. One approach to making partial progress on this problem is to prove that special classes of convex polyhedra have edge unfoldings.

One of the simplest yet still-open cases is ***prismatoids***, defined as the convex hull of two parallel convex polygons, called the ***top*** and ***base*** (bottom). Aloupis [1] showed that, if we omit the top and base, the resulting "band" of side faces has an edge unfolding. The challenge is thus to place the top and base without overlap; indeed, O'Rourke [3] showed that it is impossible to simply attach these polygons to an unfolded band without overlap (a "band unfolding").

A simpler goal is to unfold a prismatoid with the top removed, resulting in a polyhedron homeomorphic to a disk called a ***topless prismatoid***. At CCCG 2013, O'Rourke [4] constructed an edge unfolding for any topless prismatoid whose faces other than the base are triangles. Specifically, the edge unfolding has a strong property called ***petal unfolding***, meaning that it does not cut any of edges incident the base.

A topless prismatoid can be viewed as the local neighborhood of a single face on an arbitrary convex polyhedron; indeed, this work extends past petal unfoldings of a single face and its edge-adjacent faces ("edge-neighborhood patch") on a convex polyhedron [5] and of "domes" where all faces except a base share a single vertex [2, Section 22.5.2, p. 319] (which introduced petal unfoldings as "volcano unfoldings"). On the negative side, O'Rourke [4] showed that the larger neighborhood of faces sharing a vertex with a single face on a convex polyhedron ("vertex-neighborhood patch") does not always have a nonoverlapping petal unfolding. On the positive side, O'Rourke [4] showed that such a neighborhood has a nonoverlapping petal unfolding if the base is a triangle (possibly obtuse) and all other incident faces are nonobtuse triangles.

The latter result also leads to an edge unfolding of prismatoids with both the top and base, provided the base B is a triangle (possibly obtuse) and all other faces (including the top A) are nonobtuse triangles. In this setting, the definition of ***petal unfolding*** extends to mean that it does not cut any of edges incident to the base B, and cuts all but one of the edges incident to the top A. (Thus, in all cases, the side faces unfold by simple rotation around one edge of the base B.) O'Rourke [4] in fact showed that *all* petal unfoldings of such prismatoids avoid overlap.

1.1 Our Results

We expand O'Rourke's methods to encompass a broader family of prismatoids, showing that petal unfoldings never overlap in two new situations. Our first result is a step toward O'Rourke's conjecture that the base can be any convex polygon, provided the other faces are nonobtuse triangles:

Theorem 1.1 *For any prismatoid where the base B is a rectangle and all other faces are nonobtuse triangles, every petal unfolding avoids overlaps.*

Our second result takes a different approach, showing that "tall" prismatoids always petal unfold, and thus thin prismatoids form the remaining hard case:

Theorem 1.2 *For any prismatoid whose top A and base B are sufficiently far apart, every petal unfolding*

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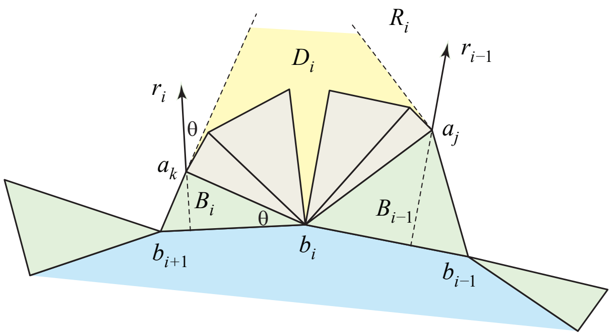

Figure 1: The diamond region D_i and the A-triangles it contains. [Based on Figure 12(a) of [4], used with permission.]

avoids overlaps. More precisely, petal unfolding avoids overlap if

$$z \geq \frac{3\pi P_A + 4d_{AB}}{2\Delta_B},$$

where

- *z is the distance between the planes containing the two bases A, B of the prismatoid;*
- *P_A is the perimeter of the top A;*
- *$\Delta_B = \pi - \max_i \angle_B(b_i)$ is the smallest turn angle in the base B (in radians);*
- *A' is the projection of A onto the plane of B; and*
- *d_{AB} is the diameter of the region $A' \cup B$.*

We prove these theorems in Sections 3 and 4 respectively, after covering the relevant background from [4] in Section 2. In Section 3.1, we give counterexamples for extending our technique of Theorem 1.1 to general quadrilaterals.

2 Background

We follow the notation given in O'Rourke's paper [4]. Let A and B be the top and base of the prismatoid, respectively. Let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the vertices of A and B respectively. Let B_i be the triangle with one vertex on A and two vertices at b_i and b_{i+1}, where indices are treated modulo n. Call these triangles ***B*-triangles**, and define ***A*-triangles** similarly.

Consider two consecutive B-triangles $B_{i-1} = b_{i-1}b_i a_j$ and $B_i = b_i b_{i+1} a_k$ in the unfolding, as in Figure 1. Define a diamond region D_i bounded by line segments $b_i a_j$ and $b_i a_k$, and by the rays through a_j and a_k perpendicular to $b_i a_j$ and $b_i a_k$ respectively. Because all the A-triangles are nonobtuse, all the A-triangles attached to edges $b_i a_j$ or $b_i a_k$ stay within the region D_i.

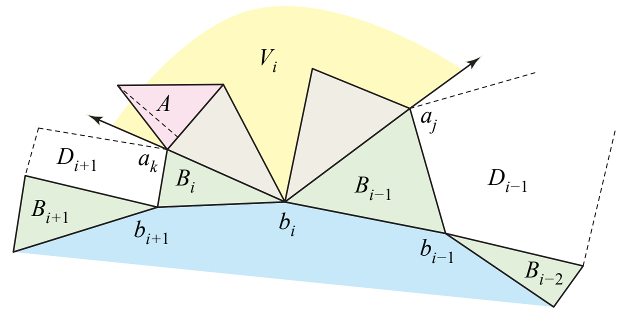

Figure 2: The region V_i containing A-triangles and the top A. [Based on Figure 13 of [4], used with permission.]

Define a larger wedge region V_i bounded by rays $\overrightarrow{b_i a_j}$ and $\overrightarrow{b_i a_k}$ (and disjoint from B, B_{i-1}, and B_i), as shown in Figure 2. Wedge V_i contains all the A-triangles attached to $b_i a_j$ or $b_i a_k$, as well as the top A, should it be attached to one of these A-triangles.

3 Unfolding Rectangular-Base Prismatoids (Proof of Theorem 1.1)

O'Rourke [4] showed that petal unfoldings never overlap for prismatoids with a convex base B and all other faces nonobtuse triangles *provided* that the region V_i does not intersect any B-triangles or any diamonds D_j for $j \neq i$ (which contain all other A-triangles). He showed that this property holds when the base B is a triangle (possibly obtuse). We extend this result to include the case where B is a rectangle, as in Figure 3.

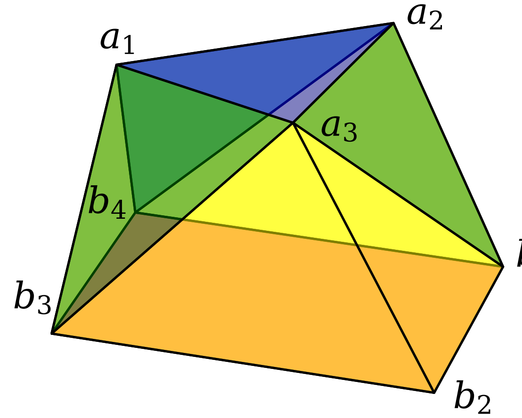

Figure 3: An acutely triangular prismatoid with a rectangular base.

Theorem 1.1 *For any prismatoid where the base B is a rectangle and all other faces are nonobtuse triangles, every petal unfolding avoids overlaps.*

Proof. O'Rourke [4] showed that it suffices to prove that V_i does not intersect any B-triangles or any diamonds D_j for $j \neq i$. He already showed that V_i does not intersect B_j, for $i-2 \leq j \leq i+1$. For a rectangle, this covers all four B-triangles. Because B_i and B_{i+1} are acute, the rays bounding D_{i+1} and D_{i-1} lie strictly outside V_i, so V_i cannot intersect those diamonds. Thus, all that remains is to show that V_i does not intersect D_{i+2}.

By symmetry, it suffices to show that V_1 does not intersect D_3, as shown in Figure 4. In fact, we claim that D_3 is contained within the region S bounded by rays $\overrightarrow{b_1b_2}$ and $\overrightarrow{b_1b_4}$ containing b_3. We will show that the line segments and rays bounding D_3 never leave S.

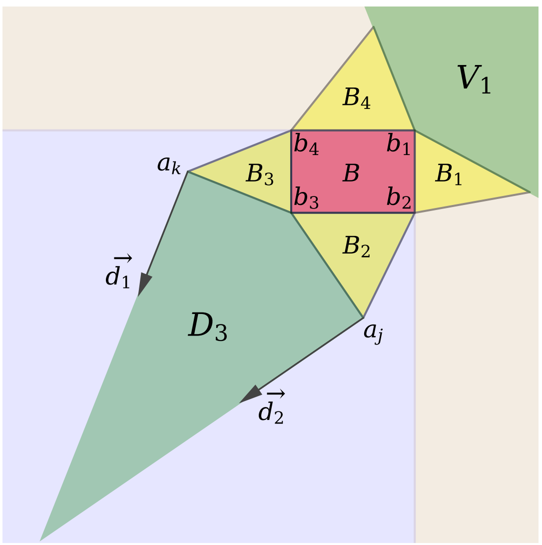

Figure 4: The regions V_1 and D_3 for the rectangular prismatoid in Figure 3. Note that D_3 always lies in the lower left quarter-plane, and V_1 always lies in the remaining three quarters of the plane.

Let a_j and a_k be the apices of triangles B_2 and B_3, so D_3 is bounded by the line segments b_3a_k and b_3a_j, and by the rays $\vec{d_1}$ and $\vec{d_2}$ perpendicular to b_3a_k and b_3a_j at a_k and a_j respectively.

First, if b_3a_k intersected line b_1b_4, then $\angle b_3b_4a_k$ of B_3 would be obtuse. Also, b_3a_k cannot intersect ray b_1b_2, as it is on the wrong side of line b_3b_4. Thus, b_3a_k is contained in S. Similarly, b_3a_j is contained in S.

Now consider ray $\vec{d_1}$. Suppose it intersected $\overrightarrow{b_1b_2}$ at some point x. Then, in quadrilateral $b_2b_3a_kx$, we have $\angle b_3a_kx = \angle xb_2b_3 = 90°$, meaning $\angle b_2b_3a_k = 180° - \angle a_kxb_2 < 180°$. However, this would make $\angle b_4b_3a_k = 360° - \angle b_2b_3b_4 - \angle b_2b_3a_k > 90°$, contradicting B_3 being nonobtuse.

Similarly, suppose that $\vec{d_1}$ intersects $\overrightarrow{b_1b_4}$ at some point y. Then, in triangle ya_kb_4, we have $\angle ya_kb_4 < 180°$, so $\angle b_4a_kb_3 = 360° - \angle ya_kb_4 - \angle b_3a_ky > 360° - 180° - 90° = 90°$, contradicting the assumption that B_3 is nonobtuse. Hence, $\vec{d_1}$ never intersects $\overrightarrow{b_1b_2}$ or $\overrightarrow{b_1b_4}$, and is thus contained in S. A similar argument shows that $\vec{d_2}$ is contained in S.

Finally, we show that V_1 intersects S only at point b_1. This claim holds because the two rays bounding V_1 only ever intersect $\overrightarrow{b_1b_2}$ and $\overrightarrow{b_1b_4}$ at b_1. Therefore, all petal unfoldings do not overlap. □

3.1 Difficulty of Quadrilateral Bases

It is natural to hope that Theorem 1.1 can be extended to all quadrilateral bases, or any convex base. However, our technique above relies on the fact that each angle of B is nonobtuse. Specifically, showing that V_i and D_{i+2} do not intersect requires the assumption that $\angle b_1b_2b_3 \leq 90°$, and $\angle b_1b_4b_3 \leq 90°$. Every angle of polygon B is nonobtuse only when B is a rectangle or a nonobtuse triangle, so other quadrilaterals will require a more careful treatment.

Furthermore, the prismatoid $\mathcal{P}_c$, shown in Figure 5 and coordinatized in Table 1, is counterexample to the conjecture that the regions do not overlap when B is a general quadrilateral. Figure 6 shows the overlap.

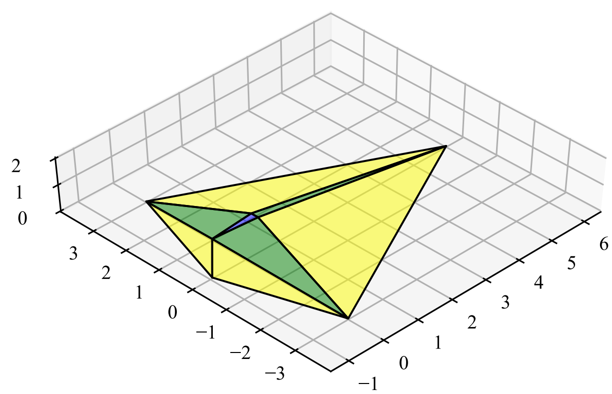

Figure 5: Prismatoid $\mathcal{P}_c$ has a quadrilateral base and all other faces nonobtuse triangles. The largest angle among the triangular faces is 89.7°.

The points of this prismatoid can be moved so that the base B is cyclic (vertices lie on a common circle), forming a new prismatoid $\mathcal{P}_{cyc}$ with coordinates given by Table 2. To find the coordinates of $\mathcal{P}_{cyc}$, we used a gradient descent method to minimize $|\angle b_1b_2b_3 - 90°|$ while maintaining that all triangles are nonobtuse. The overlap of the regions in $\mathcal{P}_{cyc}$ is much more difficult to see (refer to Figure 6): the angle formed at the intersection point is less than 0.003°.

These examples mean that extending the proof of Theorem 1.1, even to just cyclic quadrilaterals, requires a more precise treatment than considering the regions V_i and D_i. On the other hand, all petal unfoldings of $\mathcal{P}_c$ and $\mathcal{P}_{cyc}$ have no overlap, so O'Rourke's conjecture about petal unfoldings with an arbitrary convex base remains plausible.

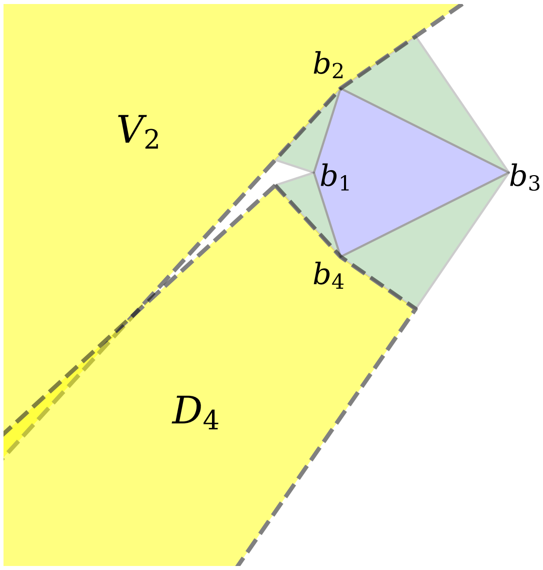

Figure 6: The regions V_2 and D_4 intersect.

Point(s)	Coordinates		
b_1	$(-0.95,$	$0.00,$	$0.00)$
b_2, b_4	$(0.00,$	$\pm 3.00,$	$0.00)$
b_3	$(6.00,$	$0.00,$	$0.00)$
a_1	$(-0.90,$	$0.00,$	$1.45)$
a_2, a_3	$(0.30,$	$\pm 0.10,$	$1.45)$

Table 1: The coordinates of the vertices of $\mathcal{P}_c$.

Point(s)	Coordinates		
b_1	$(-1.5633,$	$0.0000,$	$0.0000)$
b_2, b_4	$(0.0000,$	$\pm 3.7169,$	$0.0000)$
b_3	$(8.8372,$	$0.0000,$	$0.0000)$
a_1	$(-1.5581,$	$0.0000,$	$1.6435)$
a_2, a_3	$(0.2225,$	$\pm 0.0299,$	$1.6435)$

Table 2: The coordinates of the vertices of $\mathcal{P}_{cyc}$.

4 Unfolding Tall Prismatoids (Proof of Theorem 1.2)

For a given prismatoid, let z denote the distance between the planes of the top and base. We show that, for prismatoids with large enough z, all petal unfoldings avoid overlap.

Theorem 1.2 *For any prismatoid whose top A and base B are sufficiently far apart, every petal unfolding avoids overlaps. More precisely, petal unfolding avoids overlap if*

$$z \geq \frac{3\pi P_A + 4d_{AB}}{2\Delta_B},$$

where

- *z is the distance between the planes containing the two bases A, B of the prismatoid;*
- *P_A is the perimeter of the top A;*
- *$\Delta_B = \pi - \max_i \angle_B(b_i)$ is the smallest turn angle in the base B (in radians);*
- *A' is the projection of A onto the plane of B; and*
- *d_{AB} is the diameter of the region $A' \cup B$.*

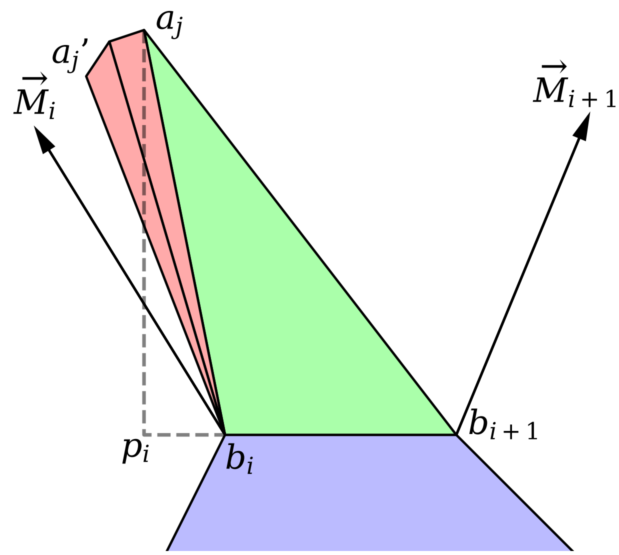

Figure 7: One of the B-triangles, along with some A-triangles attached to its left. In this configuration, p_i is on the opposite side of b_i as b_{i+1}, so $\angle b_{i+1}b_ia_j$ is obtuse.

Proof. We show that, in any petal unfolding, every face that gets attached to a B-face B_i will stay in a region S_i bounded by the edge b_ib_{i+1} and the rays $\overrightarrow{M_i}$ and $\overrightarrow{M_{i+1}}$ bisecting the exterior angles of B at b_i and b_{i+1} respectively, as shown in Figure 7. Note that the angle between edge b_ib_{i+1} and $\overrightarrow{M_i}$ is at least $\frac{\pi}{2} + \frac{\Delta_B}{2}$, and every edge of the form b_ia_j has length at least z.

Let $0 < \ell < 1$ be a constant. Consider a B-face B_i with vertices $b_ib_{i+1}a_j$. First we claim that the angle $\angle b_{i+1}b_ia_j$ will be at most $\frac{\pi}{2} + \frac{\Delta_B}{2} \cdot \ell$, as long as $z \geq \frac{2d_{AB}}{\Delta_B \ell}$.

Consider the projection p_i of a_j onto b_ib_{i+1}. If it lies on the same side of b_i as b_{i+1}, then $\angle b_{i+1}b_ia_j$ is acute, and we are done. Otherwise, the angle is obtuse, but we can use the fact that the length of b_ip_i is at most d_{AB}.

In this case, we know $\angle b_{i+1}b_ia_j = \frac{\pi}{2} + \arctan \frac{b_ip_i}{p_ia_j}$. Also $p_ia_j \geq z$, so

$$\angle b_{i+1}b_ia_j \leq \frac{\pi}{2} + \arctan \frac{d_{AB}}{z} \leq \frac{\pi}{2} + \frac{d_{AB}}{z}.$$

Substituting in our assumption that $z \geq \frac{2d_{AB}}{\Delta_B \ell}$, we get that $\angle b_{i+1}b_ia_j \leq \frac{\pi}{2} + \frac{\Delta_B}{2} \cdot \ell$, as desired.

Second, we show that, as long as $z \geq \frac{3\pi}{2\Delta_B} P_A \cdot \frac{1}{1-\ell}$, the angle $\angle a_jb_ia_j'$ subtended by the A-triangles attached to edge a_jb_i is at most $\frac{\Delta_B}{3}(1-\ell)$. We start by bounding the measure of $\angle a_jb_ia_{j+1}$ for any edge a_ja_{j+1} of A. By the Law of Sines, $\frac{\sin\angle a_jb_ia_{j+1}}{a_ja_{j+1}} = \frac{\sin\angle a_ja_{j+1}b_i}{a_jb_i}$, so

$$\begin{aligned}\angle a_jb_ia_{j+1} &= \arcsin\frac{a_ja_{j+1}\sin\angle a_ja_{j+1}b_i}{a_jb_i}\\ &\le \arcsin\frac{a_ja_{j+1}}{z}.\end{aligned}$$

Because $\arcsin x \le \frac{\pi}{2}x$ for $x \ge 0$, we obtain $\angle a_jb_ia_{j+1} \le \frac{\pi}{2}\cdot\frac{a_ja_{j+1}}{z}$.

The sum of these lengths a_ja_{j+1} over all A-triangles is P_A, so the sum of the angles over all A-triangles is at most $\frac{\pi P_A}{2z}$. Because the angle $\angle a_jb_ia'_j$ is the sum of $\angle a_jb_ia_{j+1}$ over some subset of the edges a_ja_{j+1} of A, we can substitute $z \ge \frac{3\pi}{2\Delta_B}P_A\cdot\frac{1}{1-\ell}$ to get that $\angle a_jb_ia'_j \le \frac{\Delta_B}{3}(1-\ell)$.

Third, we show that, if $z \ge \min\left(\frac{2d}{\Delta_B\ell}, \frac{3\pi}{2\Delta_B}P_A\cdot\frac{1}{1-\ell}\right)$, then no matter where the top face A is attached in the unfolding, it will not exit the region S_i. We accomplish this by proving that the shortest distance $d_{\min}$ between the point a'_j and the ray $\overrightarrow{M_i}$ is at least $\frac{P_A}{2}$. By the triangle inequality, this means that A cannot intersect $\overrightarrow{M_i}$. Note that this shortest distance is $d_{\min} = b_ia'_j\sin\left(\frac{\pi}{2}+\frac{\Delta_B}{2}-\angle b_{i+1}b_ia_j-\angle a_jb_ia'_j\right)$.

We know that $b_ia'_j \ge z$, and from our previous results, we know that

$$\begin{aligned}&\frac{\pi}{2}+\frac{\Delta_B}{2}-\angle b_{i+1}b_ia_j-\angle a_jb_ia'_j\\ \ge{}&\frac{\pi}{2}+\frac{\Delta_B}{2}-\frac{\pi}{2}-\frac{\Delta_B}{2}\cdot\ell-\frac{\Delta_B}{3}(1-\ell)\\ ={}&\frac{\Delta_B}{6}(1-\ell).\end{aligned}$$

Using the fact that $\sin x \ge \frac{2x}{\pi}$ for $0 \le x \le \frac{\pi}{2}$, we obtain

$$d_{\min} \ge \frac{3\pi}{2\Delta_B}P_A\cdot\frac{1}{1-\ell}\cdot\frac{2}{\pi}\cdot\frac{\Delta_B}{6}(1-\ell) = \frac{P_A}{2},$$

as desired.

Repeating this argument for every side b_ia_j of every B-triangle, we obtain that, if

$$z \ge \min\left(\frac{2d}{\Delta_B\ell}, \frac{3\pi}{2\Delta_B}P_A\cdot\frac{1}{1-\ell}\right),$$

then no petal unfolding of $\mathcal{P}$ can overlap. This lower bound is minimized when the two inputs to the min are equal. This occurs when $\ell = \frac{4d}{4\pi P_A+4d}$, which when substituted yields the desired $z \ge \frac{3\pi P_A+4d_{AB}}{2\Delta B}$. □

The most room for improvement in this proof is the second step's bound $z \ge \frac{3\pi}{2\Delta_B}P_A\cdot\frac{1}{1-\ell}$, as it is impossible for all the A-triangles to be attached to a single point on A.

Acknowledgments

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