Simple Folding is Really Hard

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One of the most researched subsets of computational origami studies *flat foldings*—folded states of a polygonal paper that lie in a plane. If we unfold such a folding, we obtain a *flat-foldable crease pattern* which is the planar straight-line graph formed by the creases. Each crease originates from one of two types of fold: mountain (the paper folds backwards) or valley (the paper folds forwards). A crease pattern is called *assigned*, if each of its creases are labeled either mountain or valley, or *unassigned*, if no crease is labeled. The FLAT-FOLDABILITY problem asks whether a given crease pattern comes from some flat folding. This decision problem is known to be NP-complete for both assigned and unassigned crease patterns [2].

A *simple fold* is an operation that transforms a flat folding into another by a rigid 180° rotation of a subset of the paper around an axis ℓ. During the motion, the paper is not allowed to tear or self-cross. This restriction is motivated by practical sheet-metal bending, where a single robotic tool can fold the sheet material at once. The SIMPLE-FOLDABILITY problem asks whether a given crease pattern can be folded by a sequence of simple folds (unfolding is not allowed). Arkin et al. [1] introduced many models of simple folds with respect to the number of layers folded: they are *one-layer* (Fig. 1 (1), (5)), *all-layers* (Fig. 1 (1), (2), (3)), and *some-layers* (which imposes no restriction). They prove that SIMPLE-FOLDABILITY is weakly NP-complete for: one-layer (assigned), some-layers (assigned/unassigned) and all-layers (assigned/unassigned) if the paper is an orthogonal polygon and the creases are paper-aligned orthogonal (abbreviated); and for some-layers (assigned) and all-layers (assigned) if the paper is square and the creases are paper aligned at multiple of 45° (abbreviated). They also provide a polynomial-time algorithm for SIMPLE-FOLDABILITY with rectangular paper with paper-aligned orthogonal creases (abbreviated). Arkin et al. pose as an open problem whether there exist a pseudo-polynomial time algorithm for the models proven weakly NP-hard.

We settle this long standing open problem by proving strong NP-completeness for all models with crease patterns (assigned/unassigned), and for some-layers and all-layers models with crease patterns (assigned/unassigned). We reduce from 3-PARTITION, which is NP-complete [3]: can a set of integers $A = \{a_1, \dots, a_n\}$ be partitioned into $n/3$ triples each with sum $\sum A/(n/3) = t$? Given an instance of 3-PARTITION, we construct the crease pattern shown in Fig. 2, where $\infty = 10nt$. The vertical creases force the long vertical uncreased strip of paper on the left of the construction to pass through the right part. Collision is only avoided by folding through horizontal creases that encode the integers a_i if and only if the 3-PARTITION instance has a solution. To prove the results for we create a crease pattern that forces the square paper to be folded into a long rectangular strip and then, using turn gadgets, into the orthogonal polygon shown in Fig. 2. We also point out an error in the NP-hardness proof in [1](Theorem 7.1) when the crease pattern is unassigned.

If it is hard to decide simple-foldability, a natural question arises: how close can we estimate the number of possible simple folds that can be performed? We define MAXFOLD, the natural optimization version of the decision problem asking for the maximum number of simple folds that can be folded given a crease pattern. We show that given a crease pattern admitting a maximum sequence of m simple folds, approximating MAXFOLD to within a factor of $m^{1-\varepsilon}$ for any constant $\varepsilon > 0$ is NP-complete in the some-layers and all-layers models. To achieve such result, we transform the reduction in Figure 2 into a gap-producing reduction by adding $O(n^{1/\varepsilon})$ horizontal creases splitting the existing vertical creases. The new creases can only be

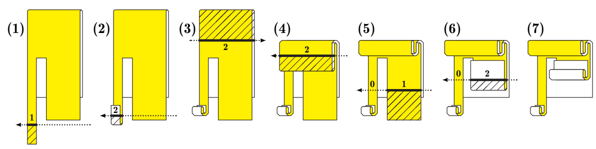

Figure 1: Example folding steps demonstrating the differences between simple folding models. The axis ℓ is a directed dotted line and the simple fold rotates the textured subset of the paper.

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folded if all the preexisting creases of the construction are already folded.

Additionally, we propose three new simple folds models, namely *infinite-one-layer*, *infinite-some-layers* and *infinite all-layers* that require that exactly one, at least one, or all layers are folded at the intersection of ℓ and the flat folding, respectively. Examples are shown in Fig. 1 (1), Fig. 1 (1), (3), (4), and Fig. 1 (1), (3), respectively. We prove strong NP-hardness for the infinite-one-layer and infinite-some-layers models by a reduction from 3-PARTITION. Given an instance of 3-PARTITION, we construct the ⊞ crease pattern shown in Fig. 3, where $\delta = \frac{3}{2n}$. The uncreased portion of the construction forces the creases labeled $c_i, i \in \{1, \dots, 2\frac{n}{3}\}$ to be folded only when they are aligned with a crease labeled s_i after h_1 and h_2 are folded, or else the axis ℓ would intersect an uncreased region of the paper. Alignment is only possible by folding correct creases $v_j, j \in \{1, \dots, n\}$, whose positions encode integers in A, if and only if the 3-PARTITION instance has a solution.

Finally, we show a polynomial-time algorithm for ⊞ crease patterns in all infinite models based on the algorithm in [1]. These results motivate why rectangular maps have orthogonal, not diagonal creases.

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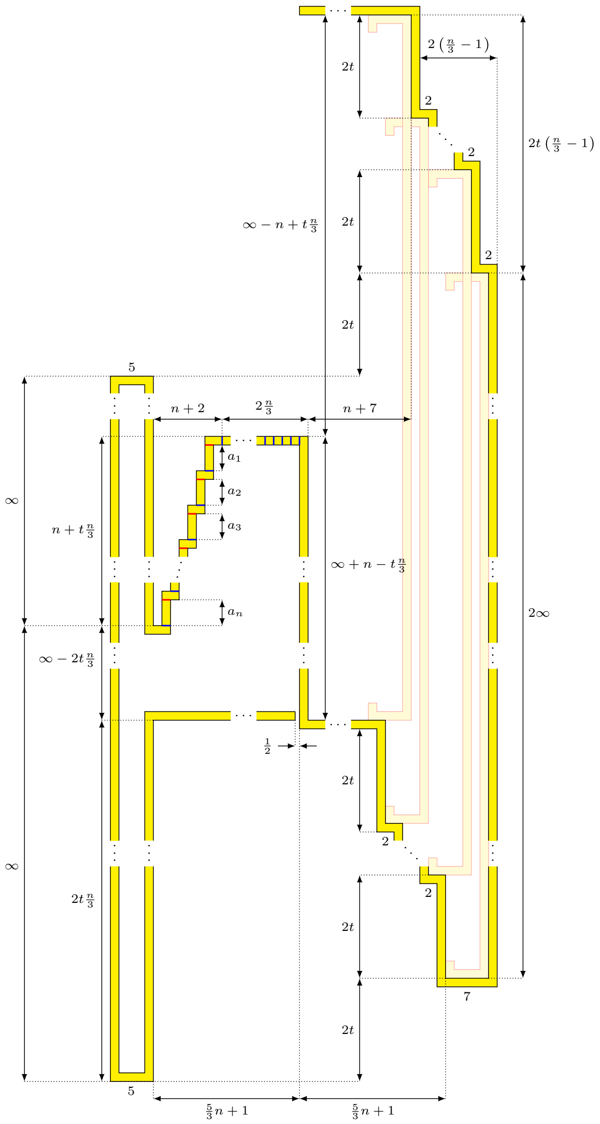

Figure 2: Reduction from 3-PARTITION to SIMPLE-FOLDABILITY for one-layer, some-layers, and all-layers. Mountains and valleys are drawn in red and blue respectively.

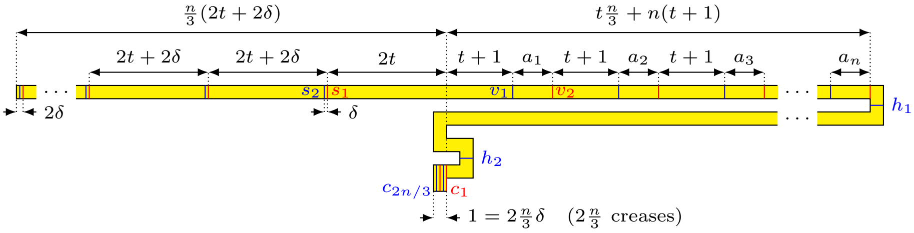

Figure 3: Reduction from 3-PARTITION to SIMPLE-FOLDABILITY in the infinite-one-layer and infinite-some-layers models. Crease assignment is drawn in red and blue for mountain and valley respectively.