Tetris is NP-hard even with $O(1)$ columns

Sualeh Asif* Erik D. Demaine* Jayson Lynch* Mihir Singhal*

Abstract

We prove that the classic falling-block video game *Tetris* remains NP-complete even when restricted to 8 columns, settling an open problem posed over 15 years ago [BDH$^+$04]. Our reduction is from 3-Partition, similar to the previous reduction for unrestricted board sizes [BDH$^+$04], but with a better packing of buckets.

In the (perfect-information) Tetris problem [BDH$^+$04], we are given an initial board state of filled squares and a sequence of pieces that will arrive, and the goal is to place the pieces in sequence to either survive (not go above the top row) or clear the entire board. This problem was proved NP-hard for arbitrary board sizes in 2002 [BDH$^+$04], and more recently for other polyomino pieces [DDE$^+$17]. The variant we consider here is the *c-column Tetris problem* (abbreviated cC-Tetris), which is the Tetris problem restricted to boards with exactly c columns. The original Tetris paper [BDH$^+$04] asked specifically about the complexity of cC-Tetris for $c = O(1)$, motivated by real-world Tetris where $c = 10$. Our main result is the following:

Theorem 1. *It is NP-complete to survive or clear the board in cC-Tetris for any $c \geq 8$.*

Membership in NP follows from the same result for general Tetris [BDH$^+$04, Lemma 2.1]. Like [BDH$^+$04], we reduce from the strongly NP-hard 3-Partition problem: given a multiset of nonnegative integers $\{a_1, \ldots, a_{3s}\}$ and a nonnegative integer T satisfying the constraints $\sum_{i=1}^{3s} a_i = sT$ and $\frac{T}{2} < a_i < \frac{T}{4}$ for all $1 \leq i \leq 3s$, determine whether $\{a_1, \ldots, a_{3s}\}$ can be partitioned into s (disjoint) triples, each of which sum to exactly T. For the reduction, we exhibit a mapping from 3-Partition instances to 8C-Tetris instances so that the following is satisfied:

Lemma 2 (Tetris $\iff$ 3-Partition). *For a "yes" instance of* 3-Partition*, there is a way to drop the pieces that clears the entire board without triggering a loss. Conversely, if the board can be cleared, then the* 3-Partition *instance has a solution.*

Proof sketch. The initial board, illustrated in Figure 1(a) (where filled cells are grey and the rest of the cells are unfilled), will have 8 columns and $12sT + 48s + 26$ rows. The reduction is polynomial size.

The piece sequence is as follows. First, for each a_i, we send the following a_i *sequence* (see Figures 1(i–m)): $\langle$ , $\langle$ , ,  $\rangle^{a_i}$, ,  $\rangle$. After all these pieces, we send the following *clearing sequence* (see Figures 1(n) and (b–h)): $\langle\langle$ , ,  $\rangle^s$, , $\langle$  $\rangle^{6sT+24s+6}$, , $\langle$  $\rangle^{3sT+12s+4}\rangle$.

Figures 1(b–n) illustrate that a solution to 3-Partition will clear the board. To show the other direction, we progressively constrain any 8C-Tetris solution to a form that directly encodes a 3-Partition solution. Because the area of the pieces sent is exactly equal to $8(12sT + 48s + 26)$, no square can be left empty. We enumerate all possible cases to show that this goal is impossible to meet (some square must be left empty) if there is no 3-Partition solution. Figures 1(o–w) show some of the cases. □

References

[BDH$^+$04] Ron Breukelaar, Erik D. Demaine, Susan Hohenberger, Hendrik Jan Hoogeboom, Walter A. Kosters, and David Liben-Nowell. Tetris is hard, even to approximate. *International Journal of Computational Geometry and Applications*, 14(1–2):41–68, 2004.

[DDE$^+$17] Erik D. Demaine, Martin L. Demaine, Sarah Eisenstat, Adam Hesterberg, Andrea Lincoln, Jayson Lynch, and Y. William Yu. Total Tetris: Tetris with monominoes, dominoes, trominoes, pentominoes, …. *Journal of Information Processing*, 25:515–527, 2017.

*Massachusetts Institute of Technology, Cambridge, MA, USA

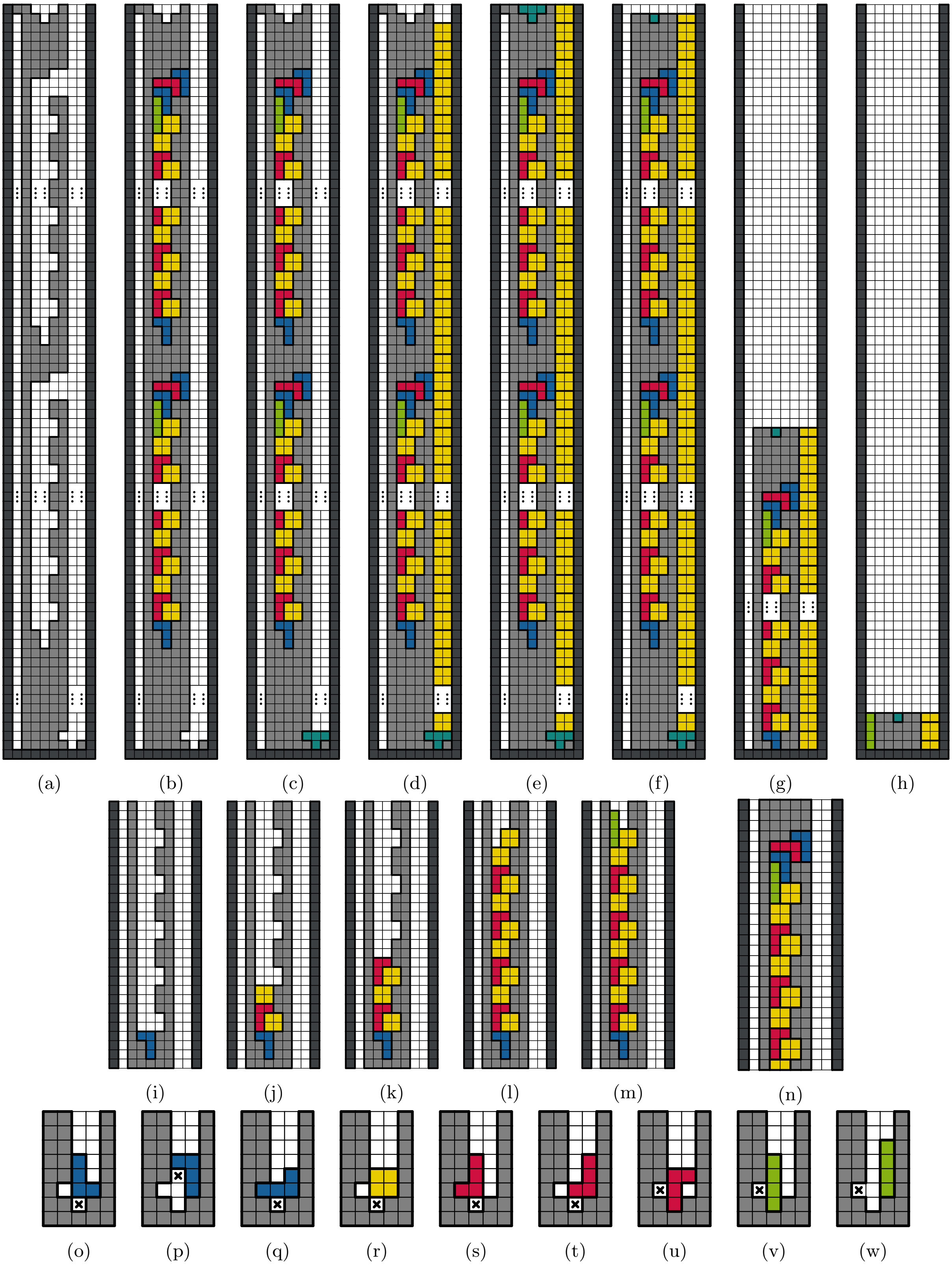

Figure 1: (a) shows the initial board. (b–d) demonstrate filling and clearing the board in the final clearing sequence. (i–m) show a valid sequence of moves for $a_i = 5$. (n) shows our bucket terminator. (o–w) show invalid possibilities for various pieces in the bucket.