Level-set MCMC Curve Sampling

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Outline

1. Overview
2. Curve evolution
3. Markov chain Monte Carlo
4. Curve sampling
5. Examples
6. Conditional simulation

Overview

- Curve evolution attempts to find a curve C (or curves C_i) that best segment an image (according to some model)
- Goal is to minimize an energy functional E(C) (view as a negative log likelihood)
- Find a local minimum using gradient descent

Sampling instead of optimization

- Draw multiple samples from a probability distribution **p** (e.g., uniform, Gaussian)
- Advantages:
  - Naturally handles multi-modal distributions
  - Can get out of local minima
  - Higher-order statistics (e.g., variances)
  - Conditional simulation

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Planar curves

- A curve is a function $\vec{C} : [0, 1] \to \mathbb{R}^2$
- We wish to minimize an energy functional with a data fidelity term and regularization term:

$$\mathsf{E}(\vec{C}) = \mathsf{D}(y|\vec{C}) + \mathcal{R}(\vec{C})$$

- This results in a gradient flow:

$$\frac{\mathrm{d}\vec{C}}{\mathrm{d}t}(p) = \vec{\mathsf{F}}(p)$$

- We can write any flow in terms of the normal:

$$\frac{\mathrm{d}\vec{C}}{\mathrm{d}t}(p) = \mathsf{f}(p)\vec{\mathcal{N}}(p)$$

Level-Set Methods

- A curve is a function (infinite dimensional)
- A natural implementation approach is to use marker points on the boundary (snakes)
  - Reinitialization issues
  - Difficulty handling topological change
- Level set methods instead evolve a surface (one dimension higher than our curve) whose zeroth level set is the curve (Sethian and Osher)

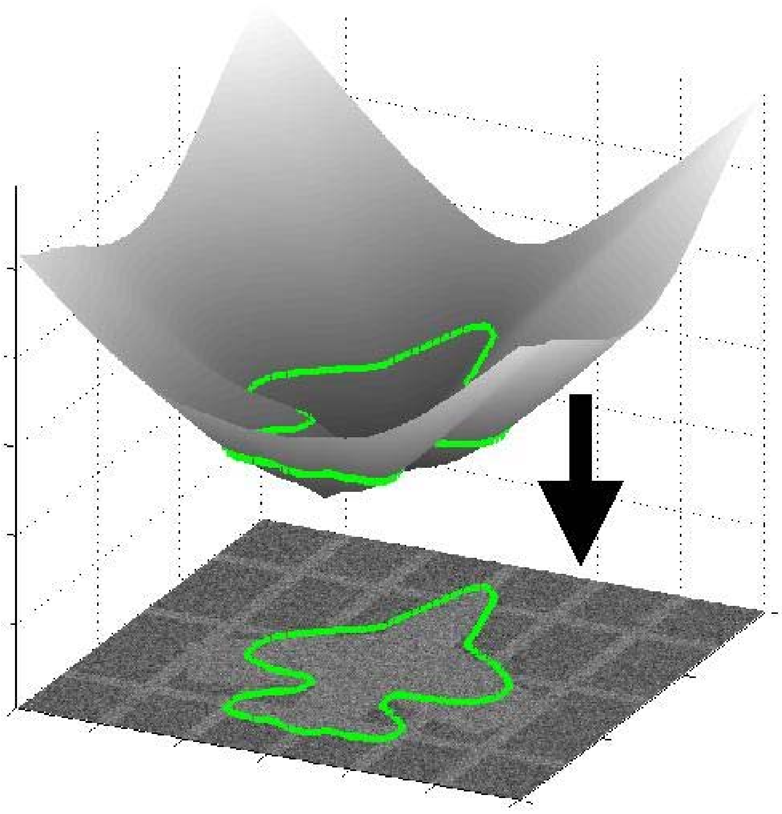
Embedding the curve

- Force level set Ψ to be zero on the curve

$$\Psi(\vec{C}(p)) = 0$$
$$\forall p \in [0, 1]$$

- Chain rule gives us

$$\begin{aligned}\frac{d\Psi}{dt} &= -\frac{d\vec{C}}{dt} \cdot \nabla\Psi \\ &= -f||\nabla\Psi||\end{aligned}$$



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General MAP Model

$$\mathrm{p}(x|y;S) \propto \mathrm{p}(y|x;S)\mathrm{p}(x;S)$$

- For segmentation:
  - *x* is a curve
  - *y* is the observed image (can be vector)
  - S is a shape model
  - Data model usually IID given the curve
- We wish to sample from p(*x*|*y*;S), but cannot do so directly

Markov Chain Monte Carlo

- Class of sampling methods that iteratively generate candidates based on a previous iterate (forming a Markov chain)
- Instead of sampling from $p(x|y;S)$, sample from a proposal distribution **q** and keep samples according to an acceptance rule **a**
- Examples include Gibbs sampling, Metropolis-Hastings

Metropolis-Hastings

- Metropolis-Hastings algorithm:
  1. Start with x^0
  2. At time t, generate candidate φ^t (given x^{t-1})
  3. Calculate Hastings ratio:

$$r^t = \frac{\mathsf{p}(\phi^t)}{\mathsf{p}(x^{t-1})} \cdot \frac{\mathsf{q}(x^{t-1}|\phi^t)}{\mathsf{q}(\phi^t|x^{t-1})}$$

  4. Set $x^t = \varphi^t$ with probability $\min(1, r^t)$, otherwise $x^t = x^{t-1}$
  5. Go back to 2

Asymptotic Convergence

- We want to form a Markov chain such that its stationary distribution is p(*x*):

$$\mathsf{p}(x) = \int \mathsf{p}(\phi)\mathsf{T}(x|\phi)\mathsf{d}\phi$$

- To guarantee asymptotic convergence, sufficient conditions are:

1) Ergodicity

2) Detailed balance

$$\mathsf{p}(x^{t-1})\mathsf{q}(\phi^t|x^{t-1})\mathsf{a}(\phi^t|x^{t-1}) = \mathsf{p}(\phi^t)\mathsf{q}(x^{t-1}|\phi^t)\mathsf{a}(x^{t-1}|\phi^t)$$

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MCMC Curve Sampling

- Generate perturbation on the curve:

$$\vec{C}'(s) = \vec{C}(s) + f(s)\vec{\mathcal{N}}(s)\mathrm{d}t$$

- Sample by adding smooth random fields:

$$f \sim \mathsf{N}(0, \Sigma)$$

- Σ controls the degree of smoothness in field
- Note for portions where *f* is negative, shocks may develop (so called prairie fire model)
- Implement using white noise and circular convolution

Smoothness issues

- While detailed balance assures asymptotic convergence, may need to wait a very long time
- In this case, smooth curves have non-zero probability under **q**, but are very unlikely to occur
- Can view accumulation of perturbations as
  $$F(s) = \sum_i f_i(s) = h \circledast \sum_i n_i$$
  (h is the smoothing kernel, n_i is white noise)
- Solution: make **q** more likely to move towards high-probability regions of **p**

Adding mean force

- We can add deterministic elements to f (i.e., a mean to $\mathbf{q}$):

$$f \sim \mathrm{N}(-\kappa + \gamma, \Sigma)$$

$$f(s) = \beta r(s) - \alpha\kappa(s) + \gamma$$

- The average behavior should then be to move towards higher-probability areas of $\mathbf{p}$
- In the limit, setting f to be the gradient flow of the energy functional results in always accepting the perturbation

Coverage/Detailed balance

- It is easy to show we can go from any curve C_1 to any other curve C_2 (shrink to a point)
- For detailed balance, we need to compute probability of generating C' from C (and vice versa)

$$\begin{aligned}
\vec{C}'(s) &= \vec{C}(s) + f(s)\vec{\mathcal{N}}(s)\mathrm{d}t \\
\vec{C}(s) &= \vec{C}'(s) + f'(s)\vec{\mathcal{N}}'(s)\mathrm{d}t
\end{aligned}$$

- Probability of going from C to C' is the probability of generating f (which is Gaussian) and the reverse is the probability of f' (also Gaussian)

Approximations to q

- Relationship between *f* and *f* ´ complicated due to the fact that the normal function changes
- *f* ´ does not always exists (given an *f*). Unknown what conditions on f are necessary to guarantee existence.
- Various levels of exactness
  - Assume $\vec{\mathcal{N}} = \vec{\mathcal{N}}'$ (then *f* ´= -*f*)
  - Infinitesimal approximation (ignore tangential)

$$f'(s) = < -f(s)\vec{\mathcal{N}}(s), \vec{\mathcal{N}}'(s) >$$

  - Trace along $\vec{\mathcal{N}}$ (technical issues)
- Unknown how approximations affect convergence

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SAD Target

- We define symmetric area difference (SAD) as:

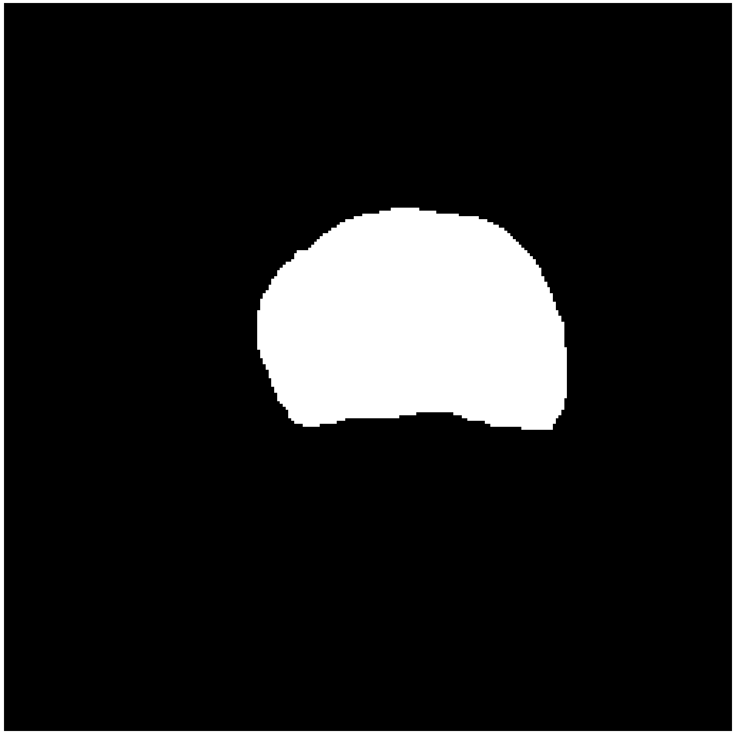
$$\mathrm{d}_{\mathrm{SAD}}(\Psi_1, \Psi_2) = \int_\Omega (\mathcal{H}(-\Psi_1(x)) - \mathcal{H}(-\Psi_2(x)))^2 \mathrm{d}x$$

- Use a Boltzmann distribution:

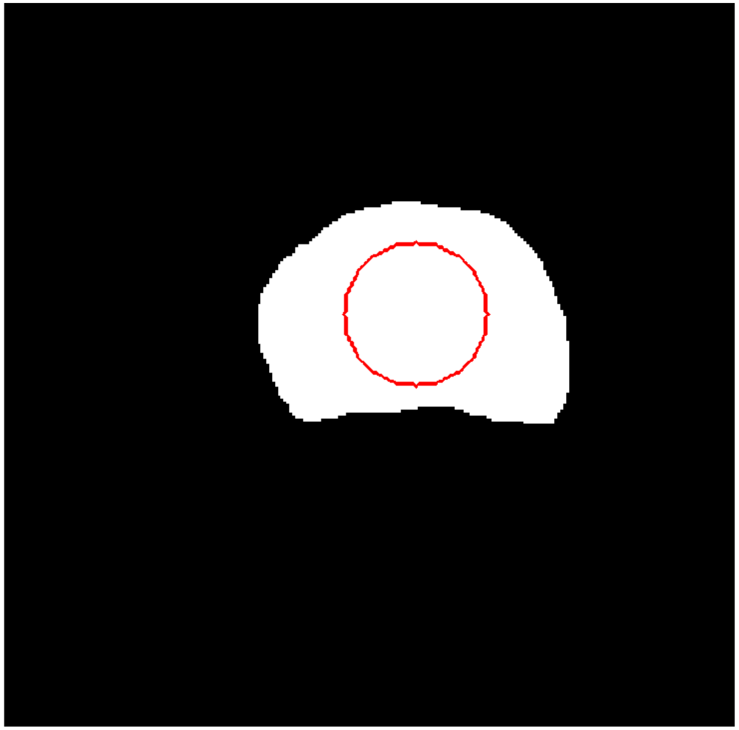
$$\mathrm{p}(\vec{C}|\vec{C}_0) = \frac{1}{Z}\exp(-\mathrm{d}_{\mathrm{SAD}}(\vec{C}, \vec{C}_0)/T)\mathrm{p}(\vec{C})$$

- T is a parameter we can use to control how likely we are to keep less likely samples
- We will keep a sample with T log(2) additional errors with probability ½
- Single mode distribution

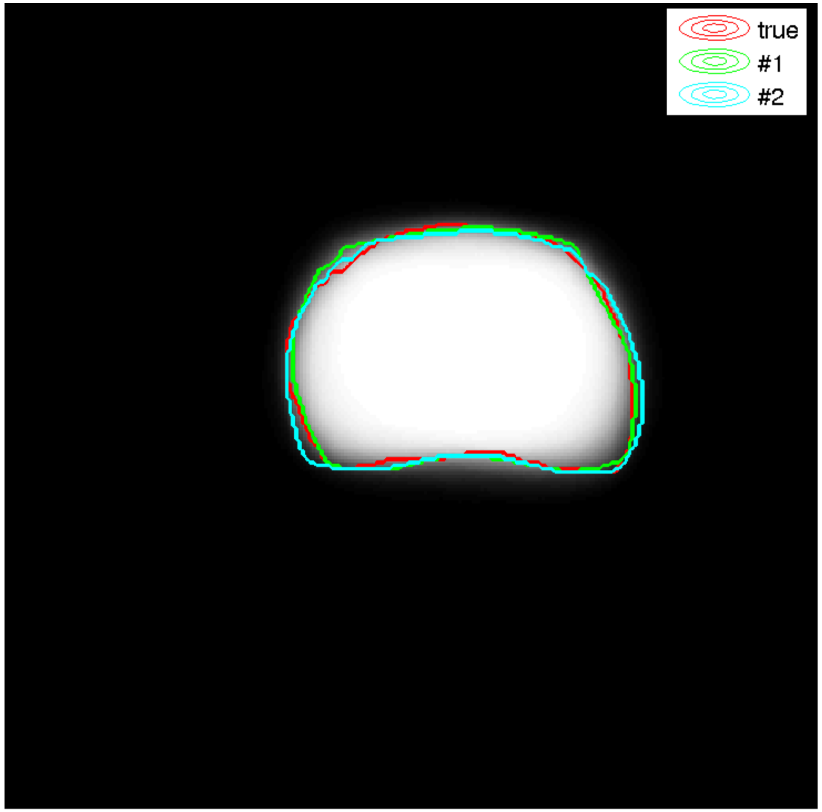
Target Shape



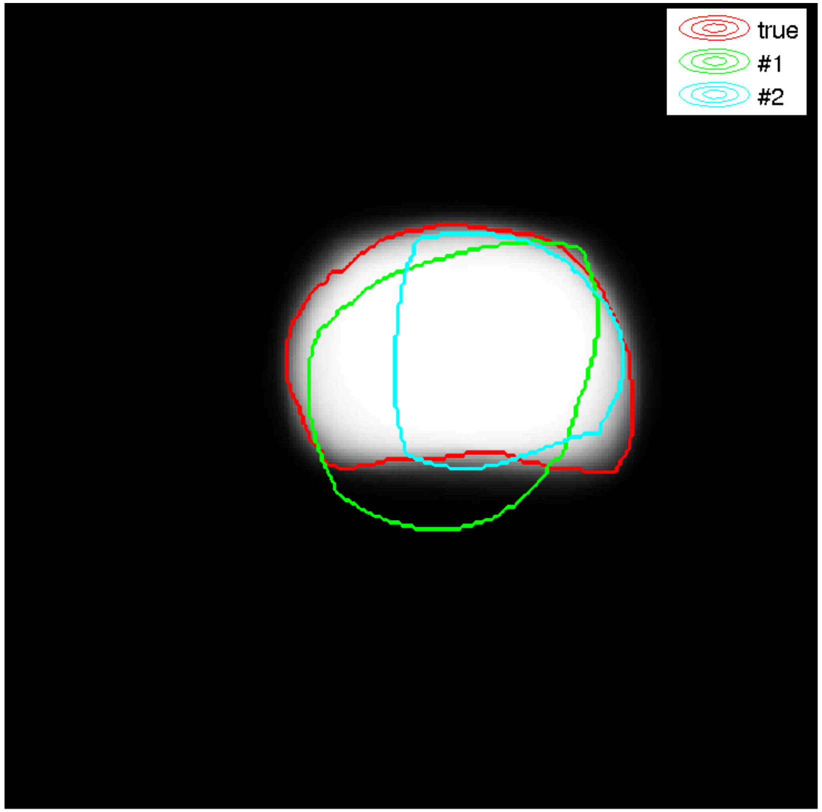
Initialization



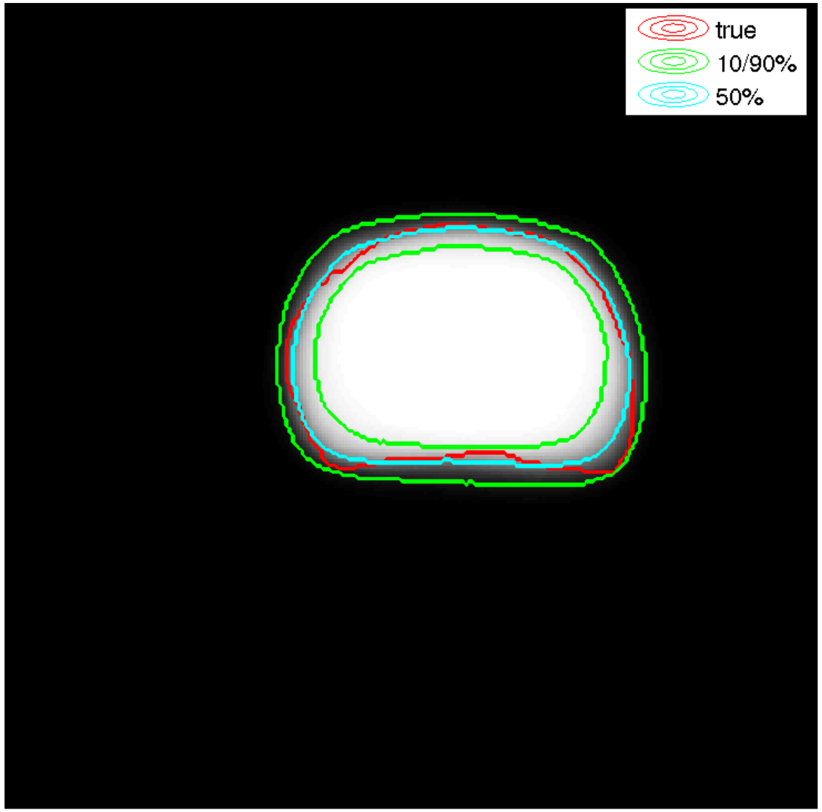
Most likely samples



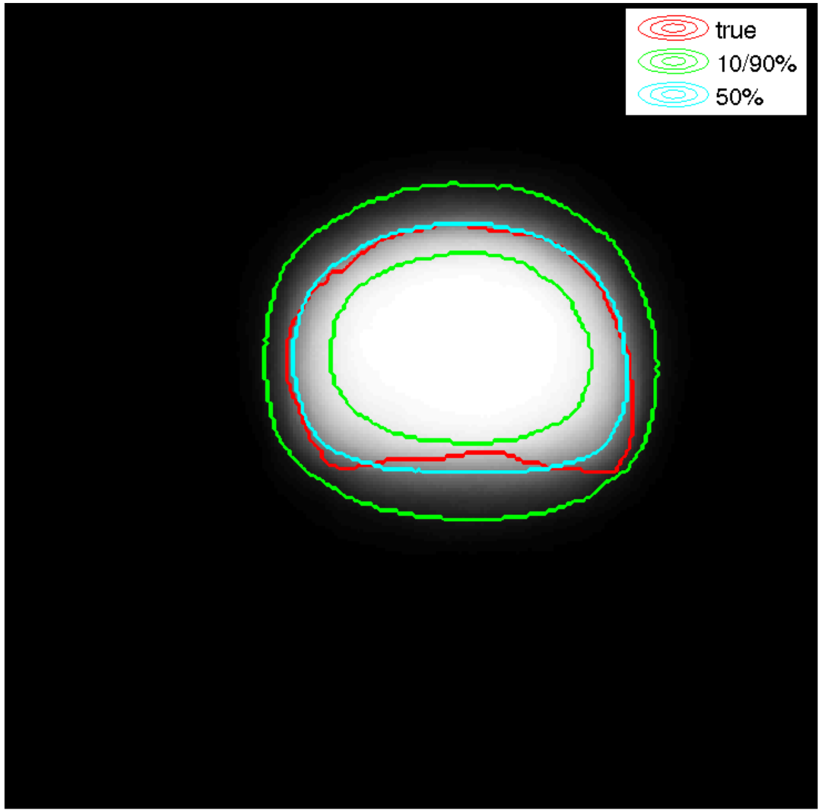
Least likely samples



"Confidence intervals"



Doubling the temperature



Synthetic noisy image

- Piecewise-constant observation model:

$$y(x) = \mu(x) + n(x)$$

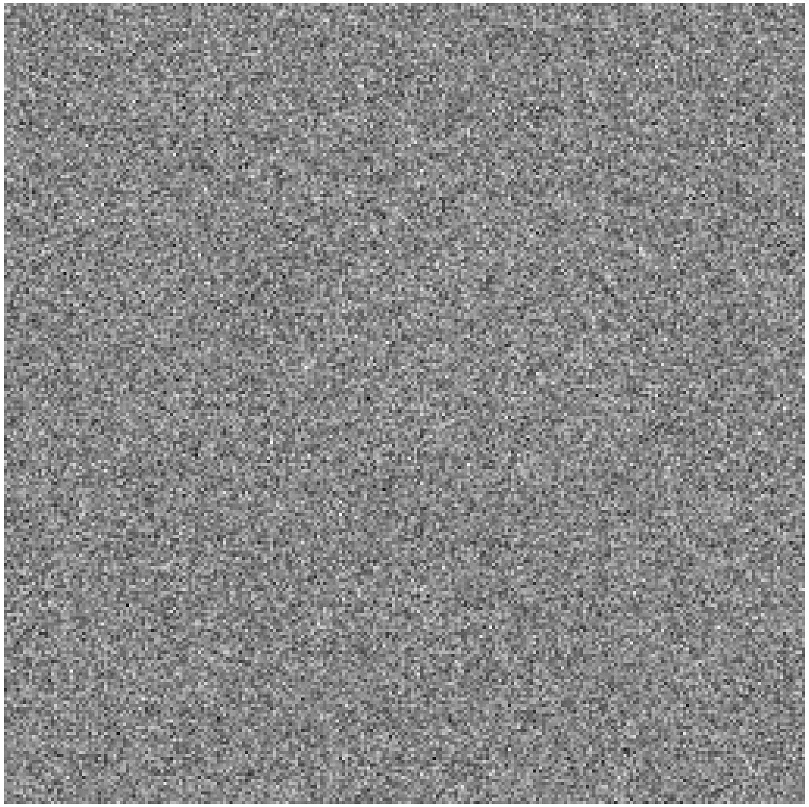
- Chan-Vese energy functional:

$$E(\vec{C}) = \iint_{R_0} (y-\mu_0)^2 dx + \iint_{R_1} (y-\mu_1)^2 dx + \alpha \oint_{\vec{C}} ds$$

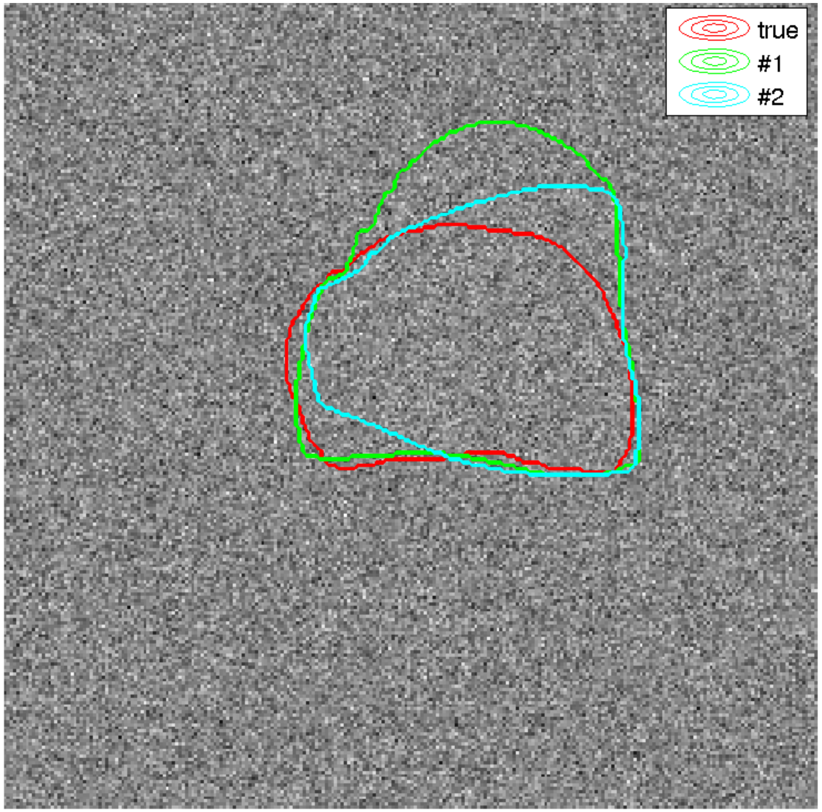
- Probability distribution ($T=2\sigma^2$):

$$\mathrm{p}(\vec{C}) = \frac{1}{Z}\exp(-\mathrm{E}(\vec{C})/T)$$

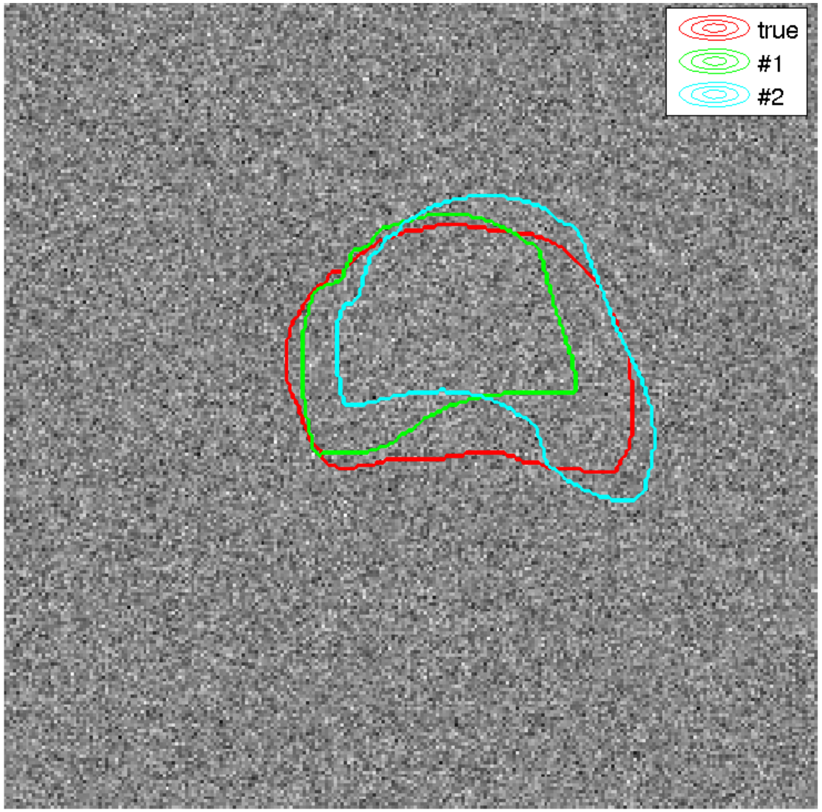
Prostate in a Haystack



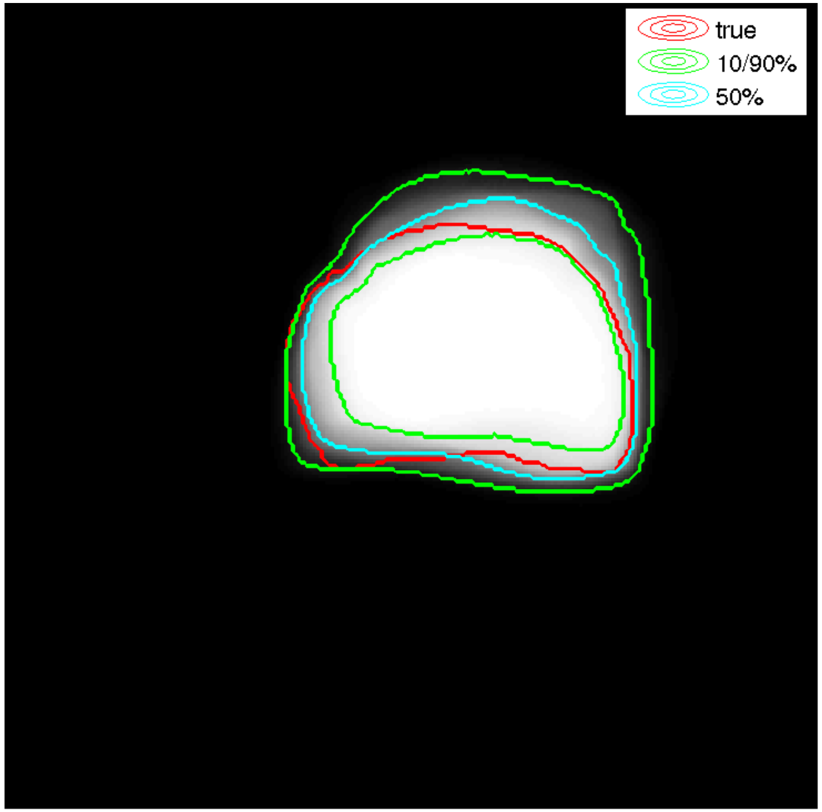
Most likely samples



Least likely samples



Confidence intervals

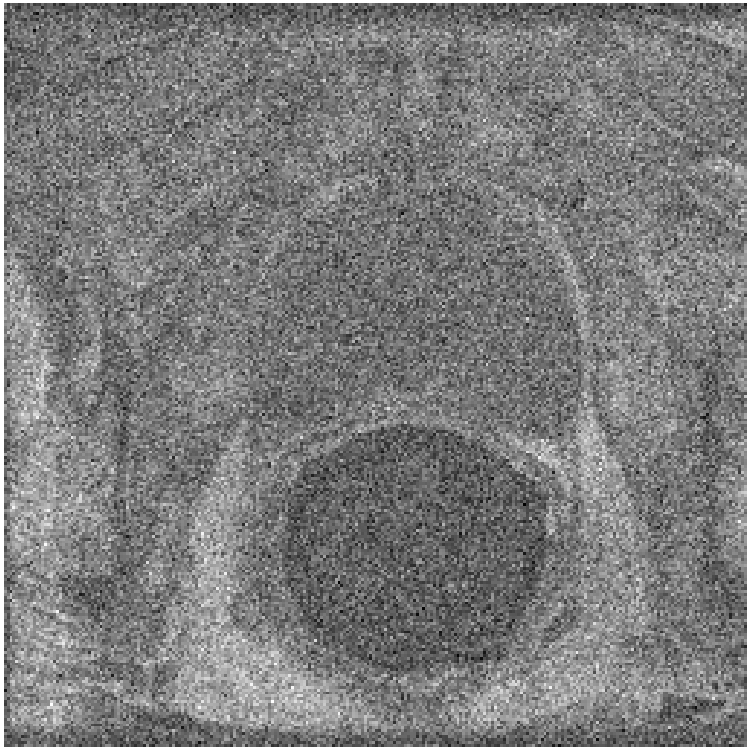


When “best” is not best

- In this example, the most likely samples under the model are not the most accurate according to the underlying truth
- 10%/90% “confidence” bands do a good job of enclosing the true answer
- Histogram image tells us more uncertainty in upper-right corner
- “Median” curve is quite close to the true curve
- Optimization would result in subpar results

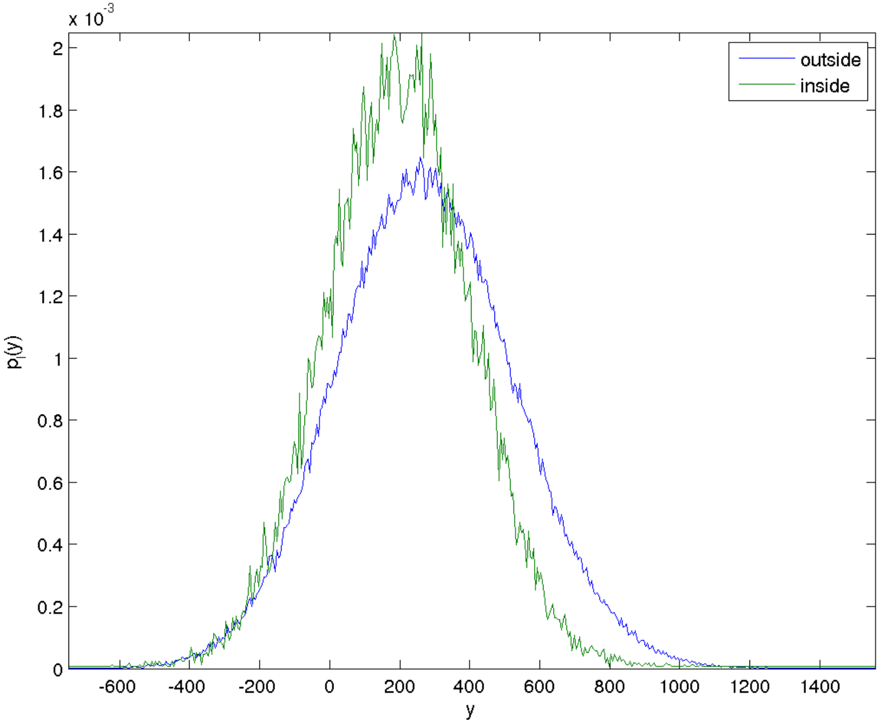
Bias-corrected prostate

- “Expert” segmentation, add noise (simulate body coil image)



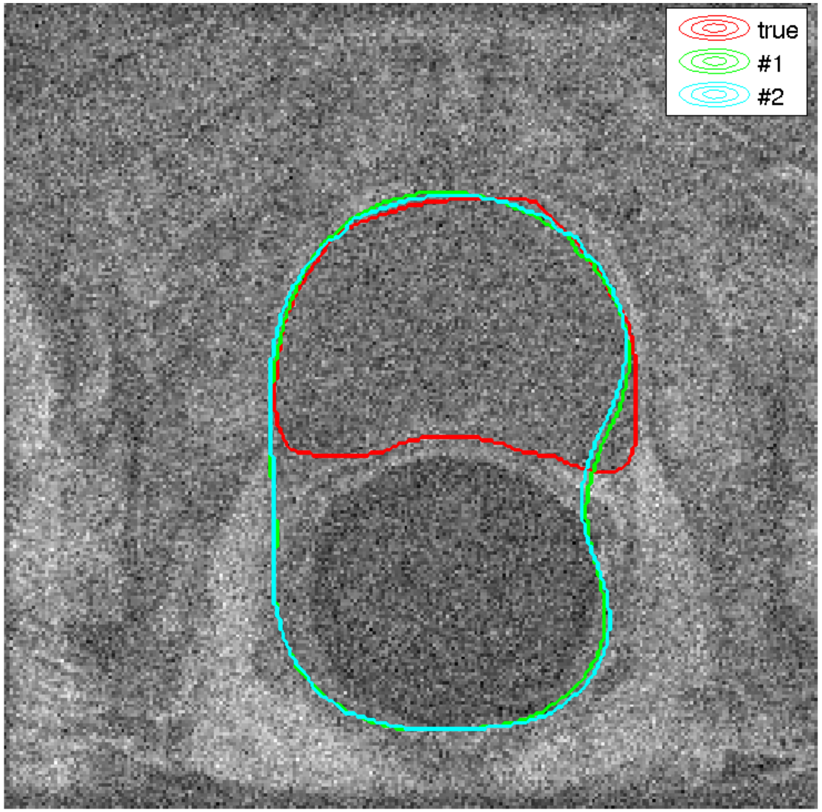
Learn probability densities

- Use histograms
- Learn pdf inside p(y|1) and pdf outside p(y|0) and assume iid given curve:

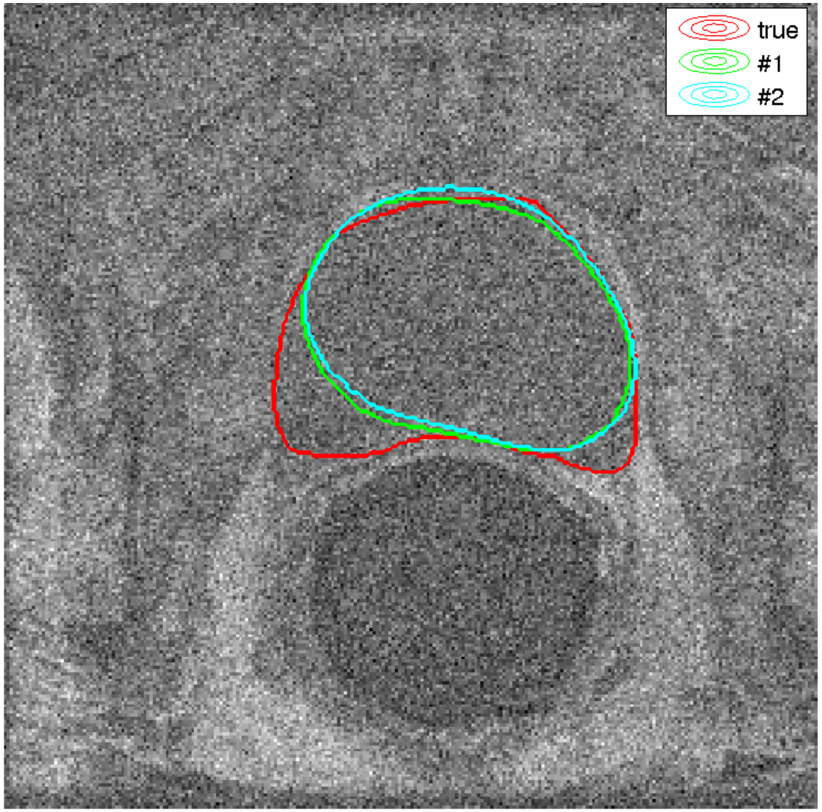


$$E(\vec{C}) = -\int_{\Omega} \log \mathrm{p}(y(x)|\mathcal{H}(-\Psi(x)))\mathrm{d}x + \alpha \oint_{\vec{C}} ds$$

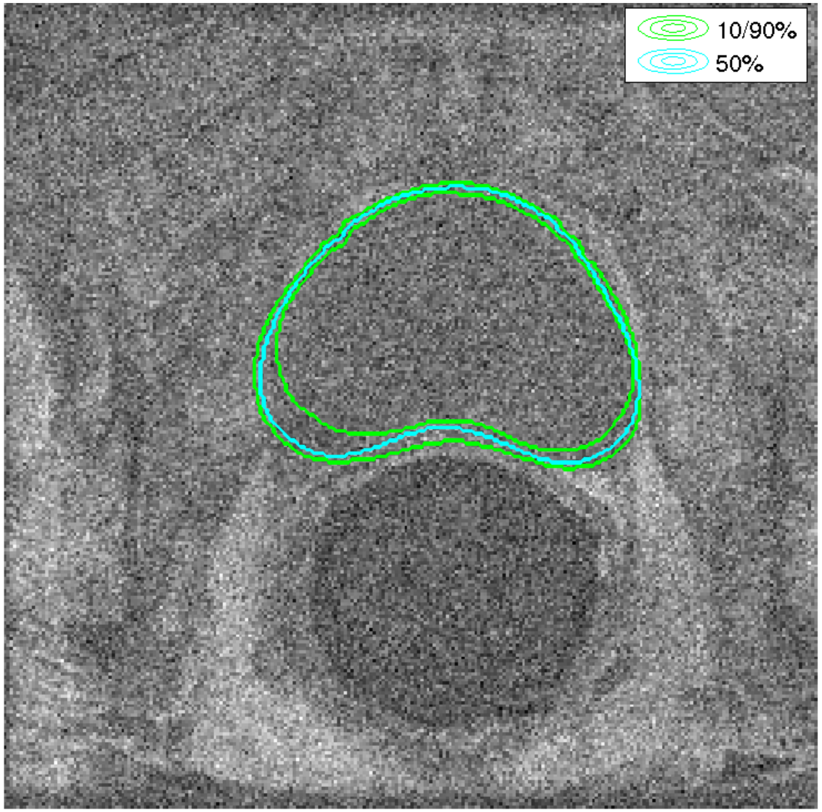
Most likely samples



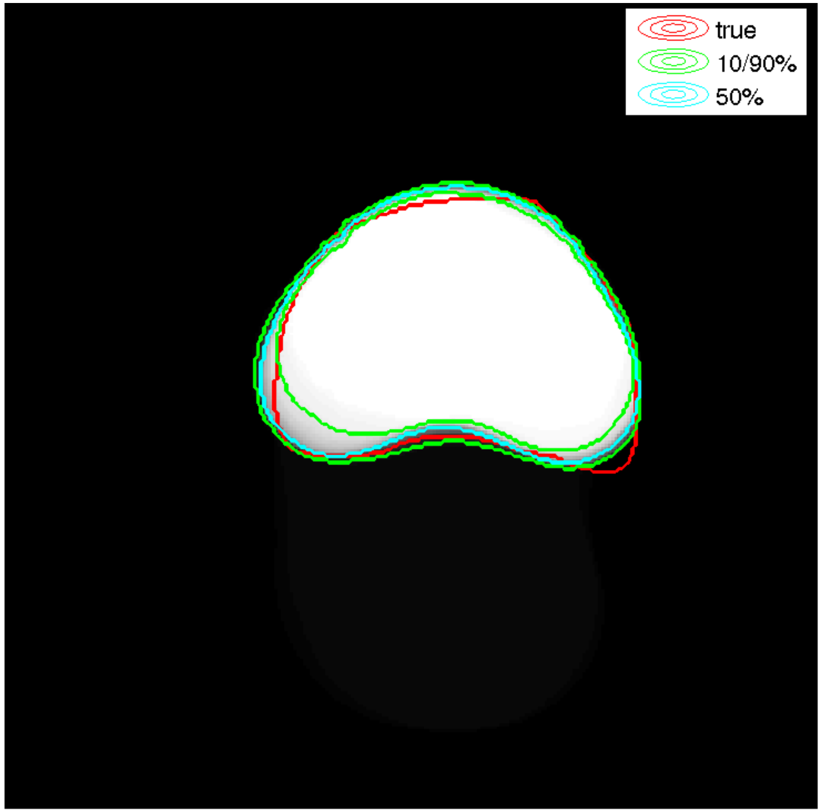
Least likely samples



Confidence intervals



Confidence intervals (with histogram)



Multimodality and convergence

- Natural multimodal distribution
- Burn-in time is not long enough (otherwise more samples would have clustered near the more-likely mode)
- When starting near one mode, need a lot of time to traverse valley between modes
- Clustering could help with presenting results

Gravity inversion

- Supplement to standard seismic data to segment bottom salt using an array of surface gravimeters (~10^{-15} N accuracy)
- Subtract base effects (geoid, centrifugal force, etc.) to leave salt effects:

$$\vec{g}(i) = G \int_{\Omega} \frac{\rho(x;\vec{C})\widehat{r}_i(x)}{||\vec{r}_i(x)||^2} \mathrm{d}x$$

- Assume constant density inside and outside:

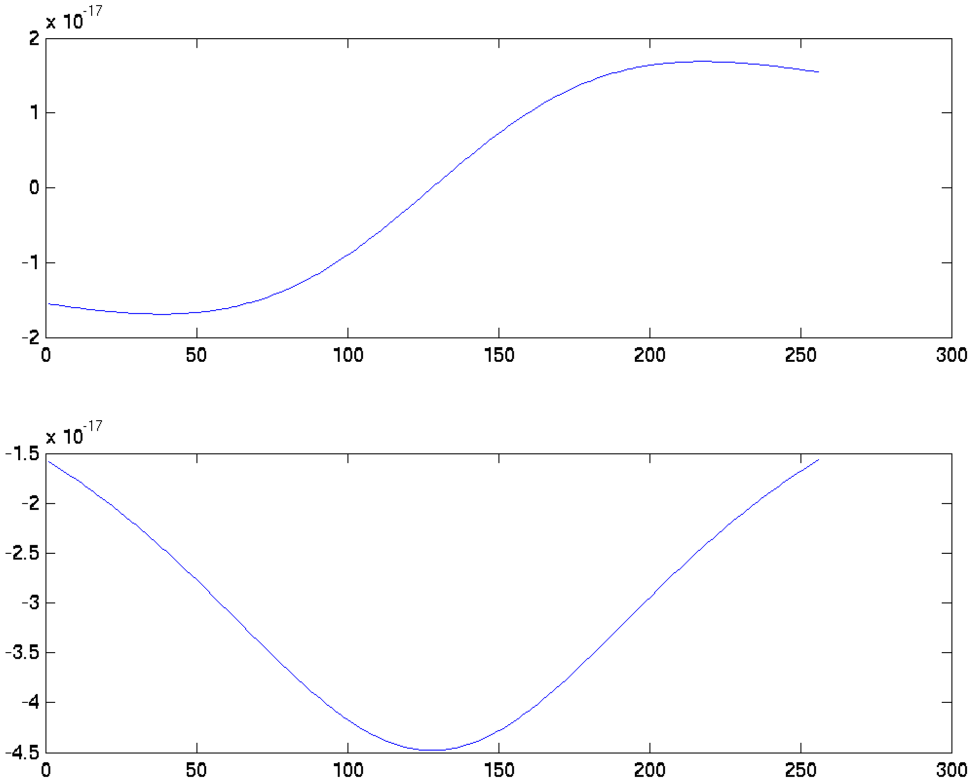
$$\rho(x;\vec{C}) = \triangle\rho\mathcal{H}(-\Psi(x))$$

- Model energy as L2 estimation error (probability as Boltzmann distribution):

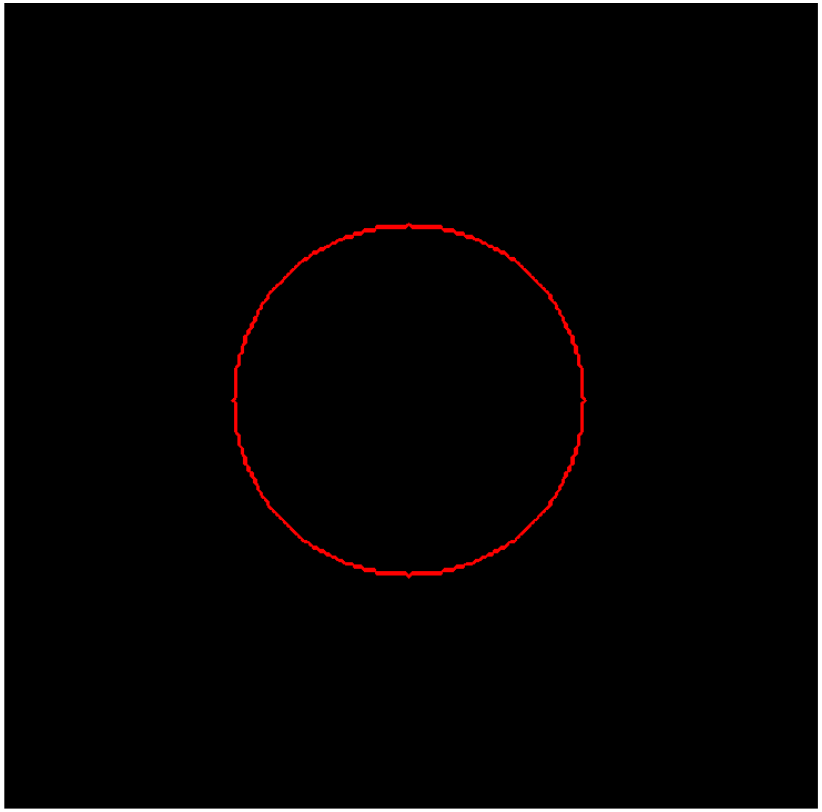
$$\mathrm{E}(\vec{C}) = \sum_{i=1}^{N_{\mathrm{array}}} ||\vec{g}_{\mathrm{obs}}(i) - \vec{g}(i;\vec{C})||^2 + \alpha \oint_{\vec{C}} \mathrm{d}s$$

A strange segmentation problem

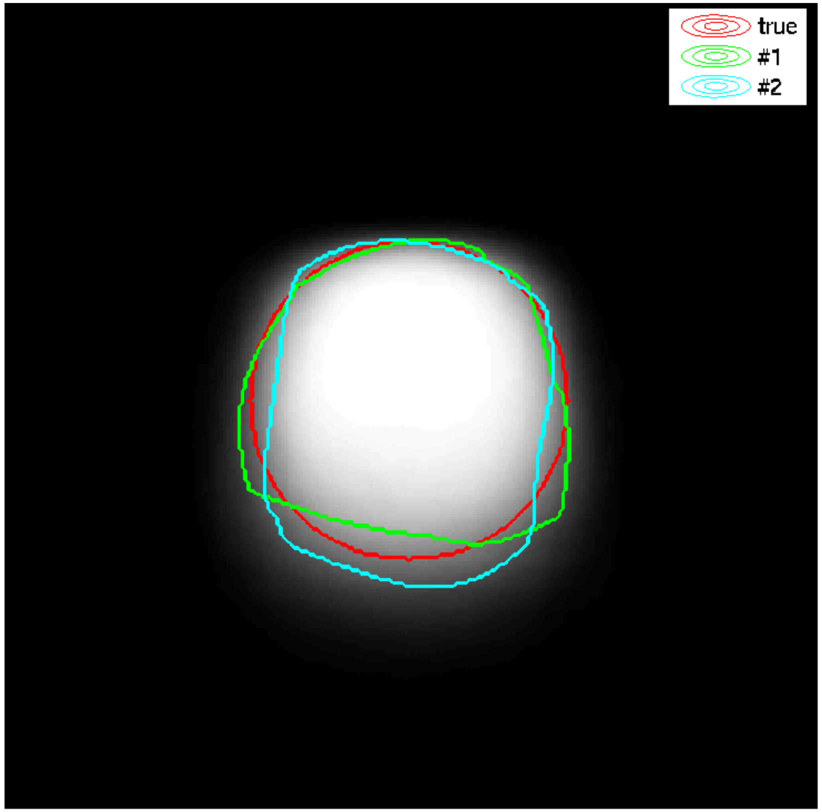
X- and z-components of gravity profile for synthetic salt body



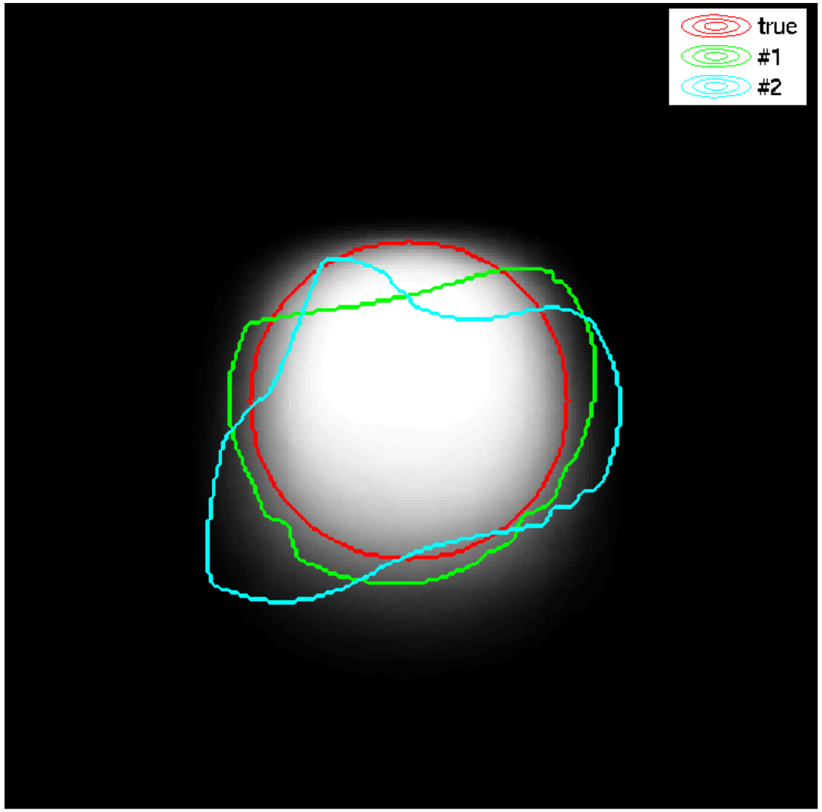
Circle salt



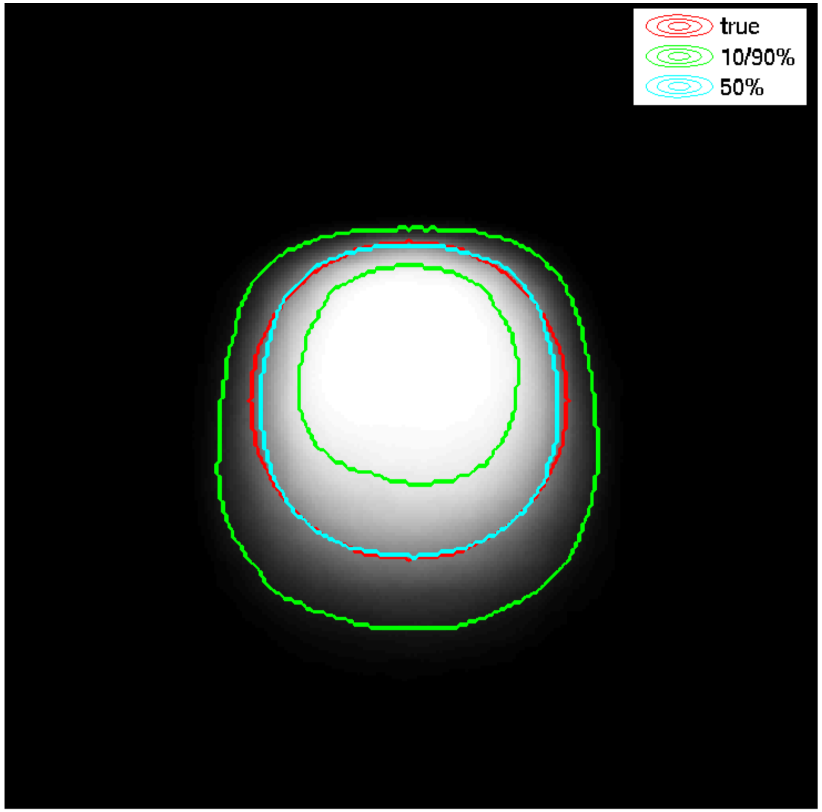
Most likely samples



Least likely samples



Confidence intervals

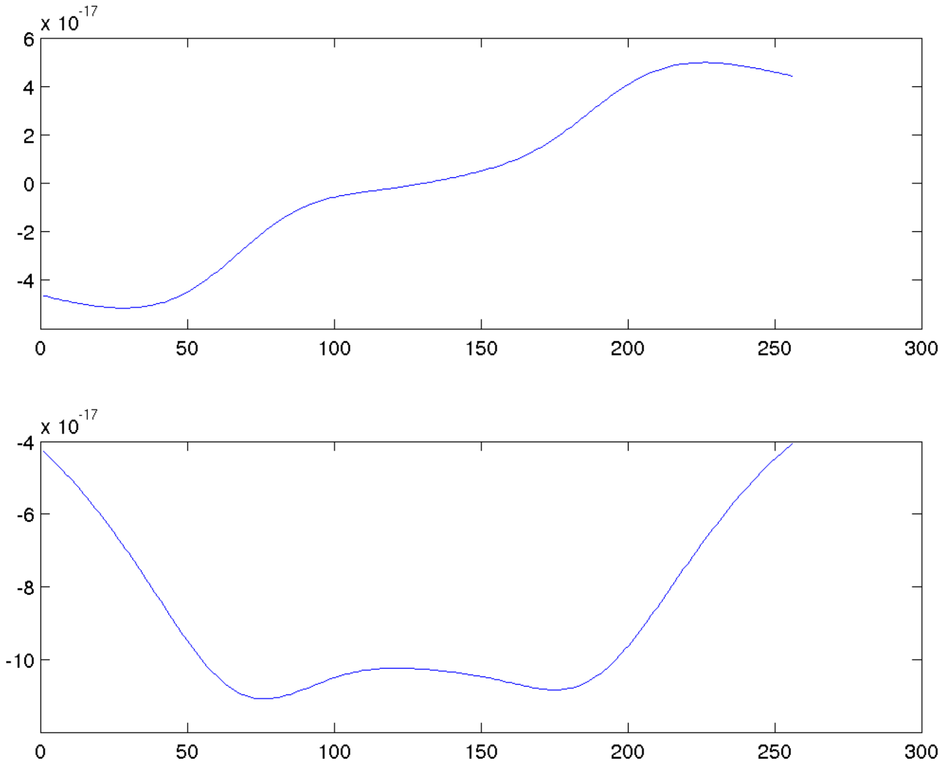


Notable features

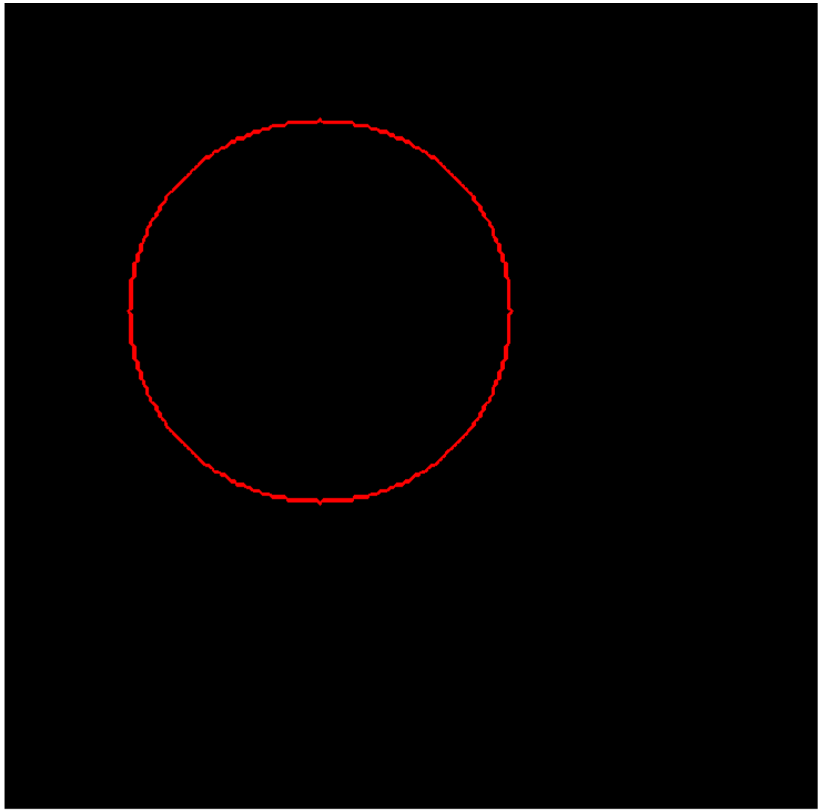
- Measurement points << image pixels, but we can do a reasonable job
- Much higher uncertainty at the bottom than the top (weaker measurements)
- Less uncertainty in middle than on sides
- Median of histogram not necessarily related to median of distribution

Mystery example

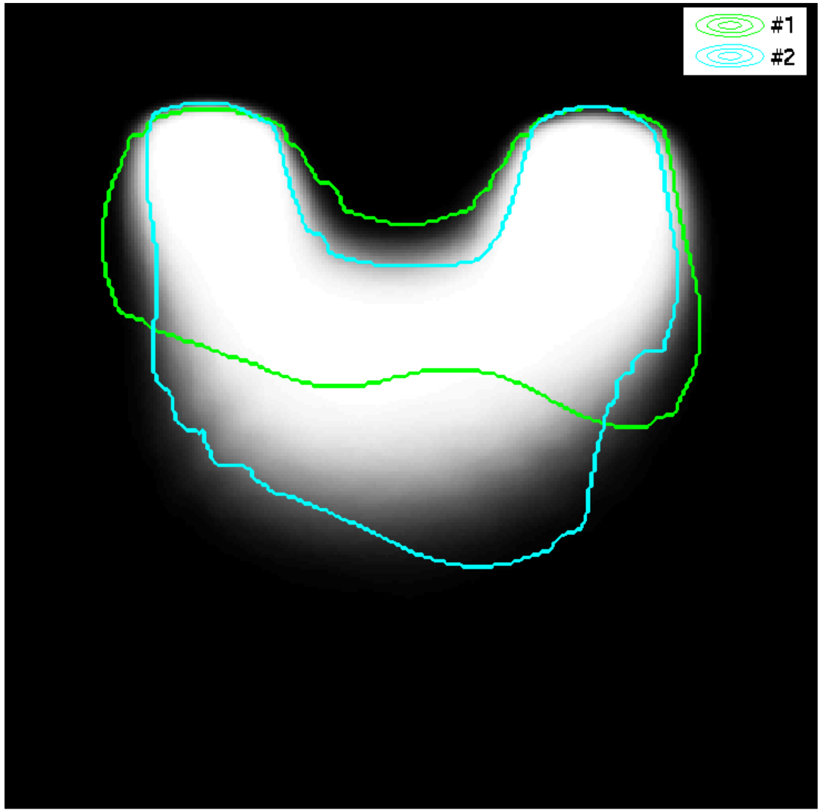
- Same x- and z-components of gravity
- Synthetic image with unknown truth



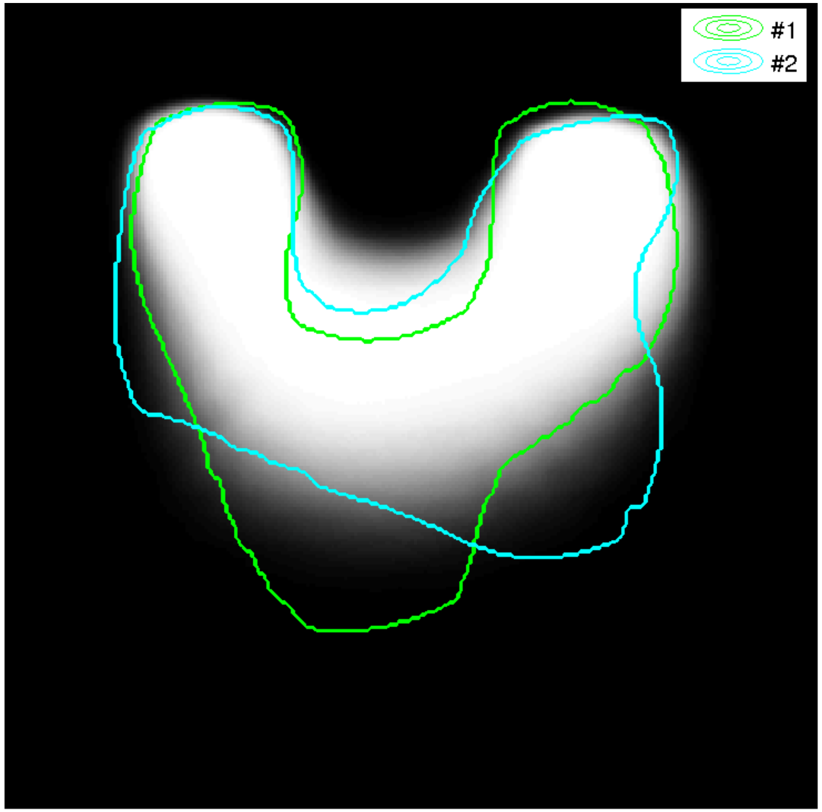
Initial state



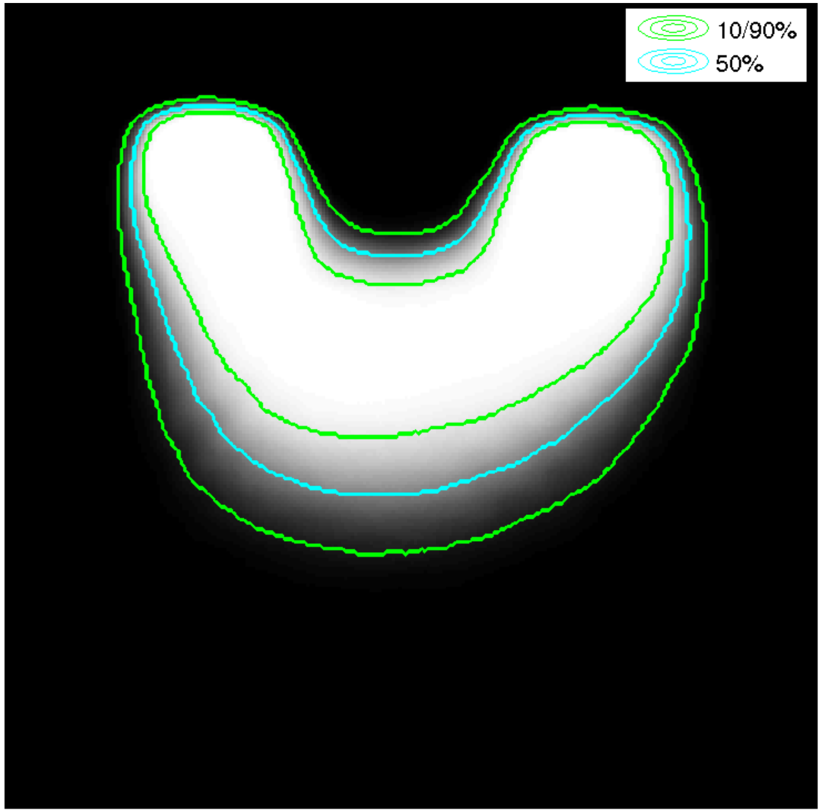
Most likely samples



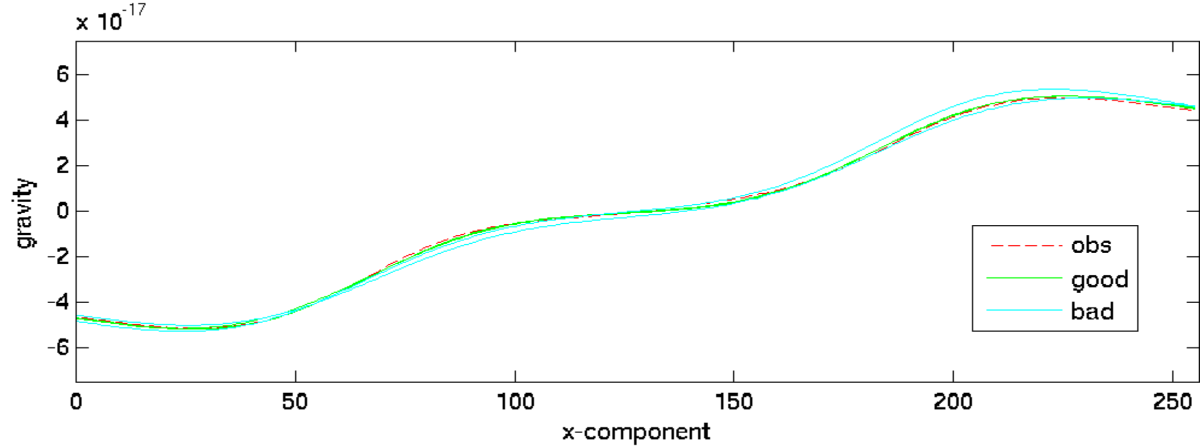
Least likely samples

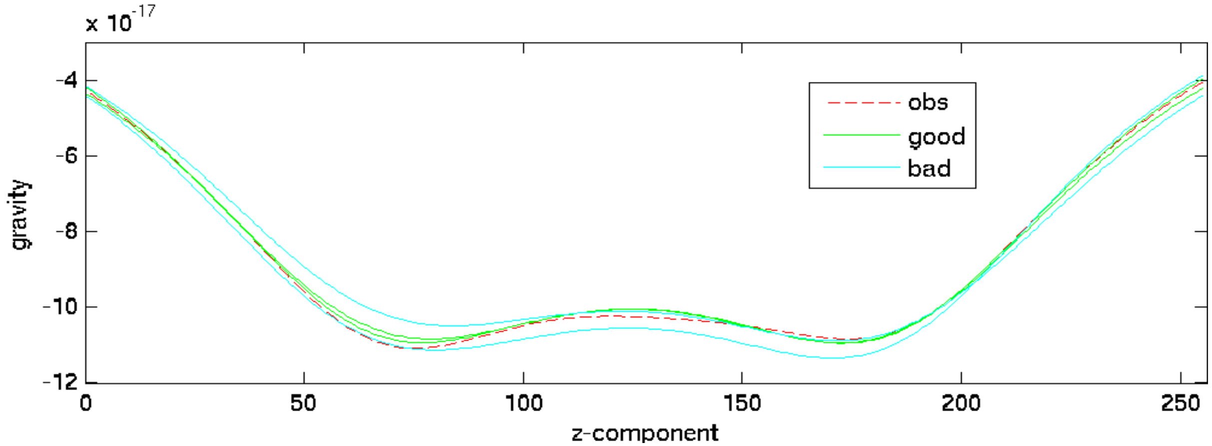


Confidence intervals



Gravity error





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User Information

- In many problems, the model admits many reasonable solutions
- Currently user input largely limited to initialization
- We can use user information to reduce the number of reasonable solutions
  - Regions of inclusion or exclusion
  - Partial segmentations
- Can help with both convergence speed and accuracy
- Interactive segmentation

Conditional simulation

- With conditional simulation, we are given the values on a subset of the variables
- We then wish to generate sample paths that fill in the remainder of the variables (e.g., simulating Brownian motion)

Simulating curves

- Say we are given C_s, a subset of C (with some uncertainty associated with it)
- We wish to sample the unknown part of the curve C_u
- One way to view is as sampling from:

$$\mathsf{p}(y|\vec{C})\mathsf{p}(\vec{C}) = \mathsf{p}(y|\vec{C})\mathsf{p}(\vec{C}_u|\vec{C}_s)\mathsf{p}(\vec{C}_s)$$

- Difficulty is being able to evaluate the middle term as theoretically need to integrate p(C)

Simplifying Cases

Under special cases, evaluation of $\mathsf{p}(\vec{C}_u|\vec{C}_s)$ is tractable:

1. When C is low-dimensional (can do analytical integration or Monte-Carlo integration)
2. When C_s is assumed to be exact
3. When p(C) has special form (e.g., independent)
4. When willing to approximate

Chan-Vese in 3D

- Energy functional with surface area regularization:

$$E(\vec{C}) = \frac{1}{2\sigma_1^2}\iiint_{R_1}(y-\mu_1)^2 d\mathbf{x} + \frac{1}{2\sigma_2^2}\iiint_{R_2}(y-\mu_2)^2 d\mathbf{x}$$
$$+\alpha \oiint_{\vec{C}} dA$$

- With a slice-based model, we can write the regularization term as:

$$\oiint_{\vec{C}} dA = \sum_{i=1}^{n-1} \iint_{\vec{c}_i \oplus \vec{c}_{i+1}} dA$$

where $\vec{c}_i \oplus \vec{c}_{i+1}$ is the surface between c_i and c_{i+1}

Zero-order hold approximation

- Approximate volume as piecewise-constant "cylinders":

$$\vec{C}(s,z) = \vec{c}_i(s), \;\; \forall \; |z - i\triangle z| < \frac{\triangle z}{2}$$

- Then we see that the surface areas are:

$$\iint_{\vec{c}_i \oplus \vec{c}_{i+1}} dA = \frac{\triangle z}{2} \oint_{\vec{c}_i} ds + \frac{\triangle z}{2} \oint_{\vec{c}_{i+1}} ds + \iint_{R^{\text{diff}}_{i,i+1}} d\mathbf{x}$$

- We see terms related to the curve length and the difference between neighboring slices

- Upper bound to correct surface area

Overall regularization term

- Adding everything together results in:

$$\oiint_{\vec{C}} dA = \triangle z \underbrace{\sum_{i=1}^{n} \oint_{\vec{c}_i} ds}_{\text{self potentials}} + \underbrace{\sum_{i=1}^{n-1} \int\int_{R^{\text{diff}}_{i,i+1}} dx}_{\text{edge potentials}}$$

2.5D Approach

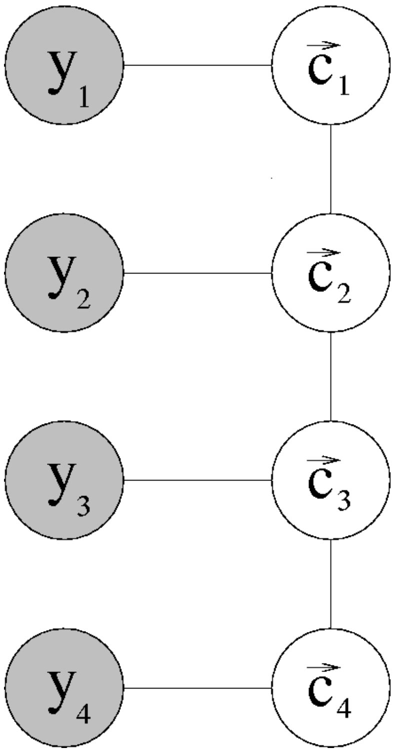
- In 3D world, natural (or built-in) partition of volumes into slices
- Assume Markov relationship among slices
- Then have local potentials (e.g., PCA) and edge potentials (coupling between slices)
- Naturally lends itself to local Metropolis-Hastings approach (iterating over the slices)

2.5D Model

- We can model this as a simple chain structure with pairwise interactions
- This admits the following factorization:

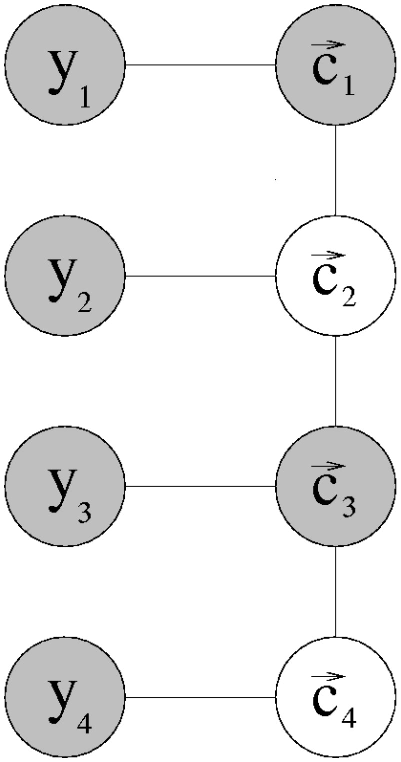
$$\mathsf{p}(Y|\vec{C}) = \prod_{i=1}^{n} \mathsf{p}(y_i|\vec{c}_i)$$

$$\mathsf{p}(\vec{C}) = \prod_{i=1}^{n} \psi_i(\vec{c}_i) \prod_{i=1}^{n-1} \psi_{i,i+1}(\vec{c}_i, \vec{c}_{i+1})$$



Partial segmentations

- Assume that we are given segmentations of every other slice
- We now want to sample surfaces conditioned on the fact that certain slices are fixed
- Markovianity tells us that c_2 and c_4 are independent conditioned on c_3

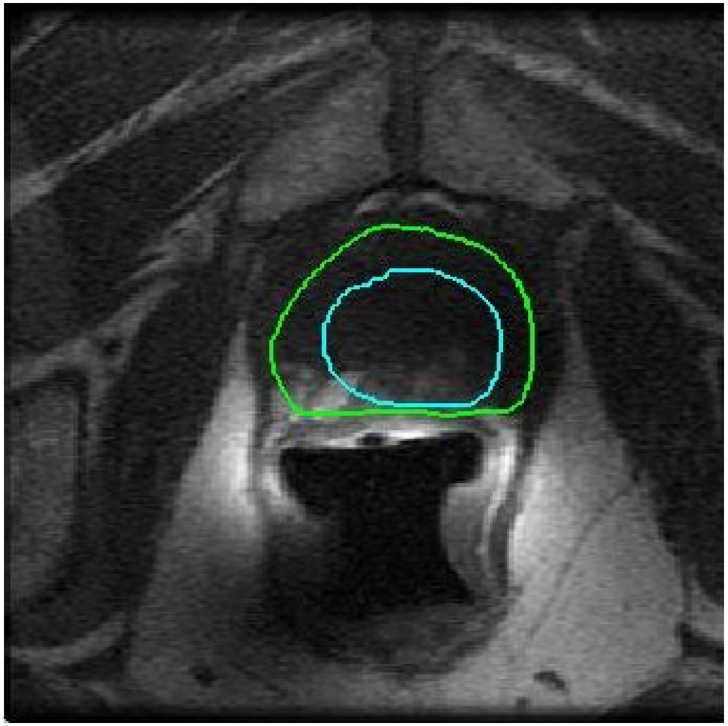


Log probability for c_2

- We can then construct the probability for c_2 conditioned on its neighbors using the potential functions defined previously:

$$\begin{aligned}\ell(\vec{c}_2|y_2,\vec{c}_1,\vec{c}_3) &= \ell(y_2|\vec{c}_2)+\\ &\alpha[\triangle z\oint_{\vec{c}_2} ds + \mathsf{d}_{\mathsf{SAD}}(\vec{c}_2,\vec{c}_1)+\mathsf{d}_{\mathsf{SAD}}(\vec{c}_2,\vec{c}_3)]\end{aligned}$$

Results



Neighbor segmentations

Our result (cyan) with expert (green)

Larger spacing

- Apply local Metropolis-Hastings algorithm where we sample on a slice-by-slice basis
- Theory shows that asymptotic convergence is unchanged
- Unfortunately larger spacing similar to less regularization
- Currently have issues with poor data models that need to be resolved

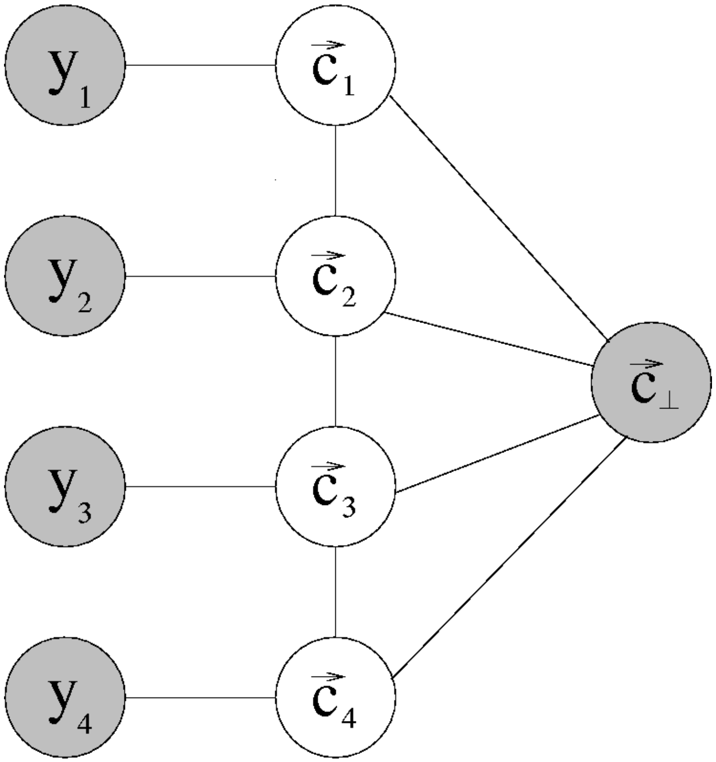
Full 3D

- In medical imaging, have multiple slice orientations (axial, sagittal, coronal)
- In seismic, vertical and horizontal shape structure expected
- With a 2.5D approach, this introduces complexity to the graph structure

Incorporating perpendicular slices

- $c_\perp$ is now coupled to all of the horizontal slices
- $c_\perp$ only gives information on a subset of each slice
- Specify edge potentials as, e.g.:

$$\Psi_{i,\perp}(\vec{c}_i, \vec{c}_\perp) = \exp\left(- \int_0^{L_y} (\Gamma(x_0, y, i\triangle z) - \Gamma_\perp(x_0, y, i\triangle z))^2 dy \right)$$



Other extensions

- Additional features can be added to the base model:
  - Uncertainty on the expert segmentations
  - Shape models (semi-local)
  - Exclusion/inclusion regions
  - Topological change (through level sets)

Conclusion

- Computationally feasible algorithm to sample from space of curves
- Approximate detailed balance
- Demonstrated utility for robustness to noise, multimodal distributions, displaying uncertainty
- Can generate arbitrary shapes with relatively complex geometry
- Conditional simulation provides a natural framework to incorporate partial user segmentations on a slice-by-slice level

Further research

- Multiple curves, known/unknown topology
- We would like q to naturally sample from space of smooth curves (different perturbation structure)
- Speed always an issue for MCMC approaches
  - Multiresolution perturbations
  - Parameterized perturbations (efficient basis)
  - Hierarchical models
- How to properly use samples?