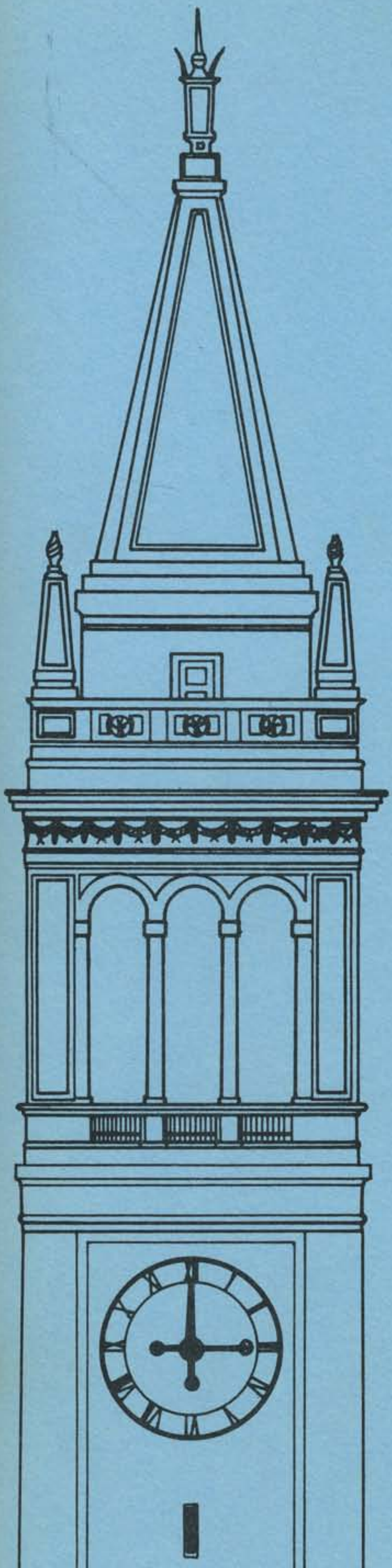

Second-Order Cardinal Characterizability

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Second-Order Cardinal Characterizability

Abstract

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The closure under isomorphism of classes of structures explicitly definable in a pure second-order relational language $\mathcal{L}$ with identity leads to a consideration of several specialized notions of definability. For any $\mathcal{L}$-sentence φ without parameters, the *spectrum* of φ is the class of all cardinals κ such that φ is true in universes of power κ, and φ *characterizes* a cardinal κ iff $\{\kappa\}$ is the spectrum of φ. Similarly for any $\mathcal{L}$-sentence φ containing a single binary relation parameter, φ *characterizes* an order type α iff φ implicitly defines an order structure of type α up to isomorphism. For any positive integer n, let $\bigvee^1_n$ $(\bigwedge^1_n)$ be the collection of all objects definable by prenex $\mathcal{L}$-sentences all of whose relation quantifiers precede any individual quantifier and which have n homogeneous blocks of relation quantifiers, the first quantifier being existential (universal); let $\Diamond^1_n = \bigvee^1_n \cap \bigwedge^1_n$.

The study of second-order spectra and characterizable cardinals reduces to the study of $\Diamond^1_2$ spectra and $\Diamond^1_2$ characterizable cardinals in the following sense: *Theorem*. For any second-order spectrum S and any sufficiently large integer n, $\{ (2^{\kappa^n})^+ : \kappa \in S \}$ is a $\Diamond^1_2$ spectrum; in particular, for any second-order characterizable cardinal κ, $(2^\kappa)^+$ is $\Diamond^1_2$ characterizable. (Cf. Zykov [*Izv. Akad. Nauk SSSR, Ser. Mat.* 17 (1953), 63-76] for an earlier reduction to

$\vee_2^1$ spectra.) Löwenheim - Skolem and compactness arguments show that this is the optimal reduction in terms of prefix classes, as the only cardinals characterizable by propositional combinations of $\vee_1^1$ sentences are the finite cardinals.

In order to study the $\Diamond_2^1$ - characterizable cardinals we first derive some results about the $\mathcal{L}$-characterizability of (order types of) ordinals. For any positive integer n, let δ_n be the least ordinal which is not the order type of a Δ_n^1 well ordering of integers, and let $\delta_{(\omega)} = \sup\{\delta_n : 0<n<\omega\}$.
Theorem. A countable ordinal α is $\Diamond_2^1$ characterizable iff α is $\vee_2^1$ characterizable iff $\alpha < \delta_2$; furthermore, if the axiom of constructibility holds, then a countable ordinal α is second-order characterizable iff $\alpha < \delta_{(\omega)}$. Theorem. The countable $\wedge_1^1$ - characterizable ordinals form a cofinal proper subset of δ_2 withorder type δ_2. Theorem. A cardinal is $\Diamond_2^1$ characterizable iff its order type is $\Diamond_2^1$ characterizable.

Incidental to the above results are several theorems about definability in an applied second-order language for number theory. Theorem. The class of Δ_2^1 sets of natural numbers is a Σ_2^1 but not a Π_2^1 class. Theorem. There is a Σ_2^1 but no Π_2^1 well ordering of natural numbers of type δ_2.

The above results are applied to the study of $\Diamond_2^1$ - characterizable cardinals, as follows. Theorem. Let F enumerate a $\Diamond_2^1$ spectrum in increasing order and let α be an ordinal in the domain of F; if α is $\Diamond_2^1$ characterizable (in particular, if $\alpha < \delta_2$), then F_α is $\Diamond_2^1$ characterizable; furthermore, if $\alpha < \delta_{(\omega)}$ and $2^{\aleph_0} < F_\alpha$, then F_α

is $\Diamond_2^1$ characterizable. In order to apply this theorem, various means of generating $\Diamond_2^1$ spectra and various operators on spectra which preserve $\Diamond_2^1$ spectra are studied. For example, it is shown that the class of fixed points in the enumeration of any $\Diamond_2^1$ spectrum is itself a $\Diamond_2^1$ spectrum. Some particular cardinals which can be shown to be $\Diamond_2^1$ characterizable are $\aleph_\alpha$ for $\alpha < \delta_2$, $\aleph_\Omega$, and the first fixed point of the aleph function.

The characterizability of various large cardinals is also investigated. Theorem. The class of inaccessible cardinals and the class of Mahlo cardinals are $\Diamond_2^1$ spectra. Theorem. The least weakly compact cardinal and the least Ramsey cardinal (if they exist) are Λ_2^1 but not V_2^1 characterizable.

Since the class of $\Diamond_2^1$ - characterizable cardinals is countable it must have an upper bound, though the above results show that this bound must be quite large. Theorem. There is no upper bound to the class of $\Diamond_2^1$ - characterizable cardinals which is arithmetical in the aleph function.