A GRAPHICAL SOLUTION OF 3XM GAME MATRICES

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A Graphical Solution of 3xm Game Matrices

Summary

Anyone seeking to win a game might well employ game theory to determine his best course of action. One means of attaining this end in certain types of simple games is presented in "A Graphical Solution of 3xm Game Matrices." This method is actually a combination of existing techniques put together in such a way as to afford a complete picture of the game. From this picture, the optimal course of action is readily determined.

Introduction

Game theory is a relatively new field of mathematics, originating with the publication of THE THEORY OF GAMES AND ECONOMIC BEHAVIOR by John von Neumann and Oskar Morgenstern in 1944. Since then, the theory has grown so much in importance as to be considered one of the major scientific contributions of the first half of the twentieth century.

The most important use of the theory of games to date has been its application to logistics or the planning of strategy in "games" of war. Even now, the RAND Corporation, in conjunction with the U. S. Air Force, is working to determine the full extent of its applications. Most likely, the strategies in any future conflict will be planned not by generals on the field but by game theorists and computers far behind the line of battle.

Game theory, however, is not limited in its use to waging war; it can also be used in the interpretation and application of experimental data and, as the name implies, in playing ordinary, everyday games. In short, wherever there is a decision to be made, game theory may be employed to determine the best course of action.

I was first introduced to the theory of games through the Inter-School Math Society of Buffalo, an organization devoted to the promotion of interest in mathematics. Our study of games was quite interesting, but I was soon disappointed when I discovered that we weren't actually learning the "theory" of games, but rather the mechanics involved in the solution of a game. I desired some reason for the methods we used, and therefore decided to launch my own investigation of the theory of games.

During my research, I became intrigued by a few methods of solving games graphically. I decided to experiment with these in order to see what sort of variations there might be. One day, my experimentation revealed what seemed to be a new and different means of solving one particular classification of games. I tested it on a few games, and, after some expansion and revision, I was able to develop my idea into a method that worked in all cases I could imagine.

At this point, I entered my project in the Physics Division of the Science Congress of Western New York, as there was no separate division for mathematics. Here I won second prize; in the final competition, I achieved third place. This award entitled me to compete in the State Science Congress at Vassar College on May 16, 1958, where I won second prize.

During the summer of 1958, I had the opportunity to discuss my project with Dr. John G. Kemeny of Dartmouth College. He offered some valuable hints concerning a remedy for the one major drawback of my report, namely the lack of an efficient method for selecting the optimal solution from the graph. Since that time, I have worked his suggestions into my project and have attempted an axiomatic proof of my method. This method, as it now stands, is presented in the remaining chapters of this report.

As of now, I have one theorem yet to establish concerning my method. (A statement of this theorem is presented at the end of the appendix.) Once I have accomplished this proof, I hope to secure the publication of my report in a mathematical magazine. After that, I may continue to do further research in game theory, or, as seems likely now, I will turn my attention to the application of Boolean algebra to circuit design. But whatever I decide to do, I know that I will continue my interest in mathematics, as my work on this project has helped me to better appreciate what can be done and what is being done by modern mathematics.

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An Illustration of a Graphical Solution of 3 xm Game Matrices

A game is essentially a conflict situation; one person opposing another in any sort of contest from a friendly game of cards to a deadly duel. Each participant has a certain number of plans which he may follow in playing the game, and any plan of his, when opposed by a definite plan of his opponent, will yield a certain result called the payoff. A complete course of action in playing the game is called a strategy, and a good strategy must tell the player exactly what to do under all circumstances. For instance, in a game of tick-tack-toe, a strategy must tell a player where to place his first mark, and then, according to what his opponent does, where to place his remaining marks. Each complete course of action, and there are about 400 of them, is properly termed a strategy.

The games which I have considered for solution are limited to two players, one of whom is further limited to three distinct strategies. I numbered the strategies of the first player, or player A, 1, 2 and 3; and those of the second player, or player B, 1, 2, 3, . . ., m. Since each player can choose any one of his strategies, the game can proceed in any one of 3m different ways, where m is the number of strategies possessed by player B. If player A's strategy is represented by i, and player B's by j, then an amount g_{ij} can be assigned as the payoff won by player A as a result of that particular course of the game.

All possible amounts g_{ij} can then be arranged in a rectangular array, or game matrix, with the rows representing the strategies available to A and the columns the strategies available to B.

Thus, in the game matrix G:

$$
\begin{array}{ccc|c|c|c|c|}
 & & \multicolumn{4}{c}{B} \\
 & & 1 & 2 & 3 & 4 \\
\hline
 & 1 & 5 & 2 & 3 & 0 \\
\hline
A & 2 & 3 & 4 & 5 & 6 \\
\hline
 & 3 & 1 & 6 & 1 & 4 \\
\hline
\end{array}
$$

player A has three strategies and player B has four. The amounts g_{ij} in the matrix represent the payoff to A for each combination of strategies. For instance, if A plays strategy 1 and B plays strategy 3, A wins three units. As can be seen from the matrix, player A will always win in this game. (If player B were to win, a negative entry would appear in the matrix.)

Obviously, A would like to be able to mix his strategies in such a manner as to win as much as possible. Similarly, B would like to be able to mix his strategies in such a manner as to lose as little as possible. It is in the accomplishment of these aims that game theory plays an important part.

Since player A has the fewest strategies, it is easiest to consider him first. A three component probability vector p^0 can be used to represent the optimal mixture of his strategies since all its entries are non-negative numbers having a sum of one. (The game is considered as being played only once.) It is convenient to label the components of this vector x, y, and 1-x-y, thereby limiting the number of unknowns to two. The next step in the solution is to multiply this vector by the matrix and obtain the average value of each of B's strategies. Thus, $p^0G = (v_1, v_2, v_3, v_4)$, and the following values result:

$$v_1)\quad 4x + 2y + 1 \tag{1}$$

$$v_2)\quad -4x - 2y + 6 \tag{2}$$

$$v_3)\quad 2x + 4y + 1 \tag{3}$$

$$v_4)\quad -4x + 2y + 4 \tag{4}$$

At this point, most texts introduce a third unknown, the vector $V = (v,v,v,v)$, to represent the desired value of the game. They state that $p^0G \geq V$, or that each of the four payoff values is either greater than or equal to v. They then can solve the resulting inequalities for the appropriate values of x and y.

This method, however, would require three dimensions if it were to be done graphically, and I felt that if the graph could be reduced to two dimensions, it would be much simpler to work with. In order to accomplish this reduction, I assumed that $p^0G=V$; that is, that all of the four payoff values were equal to v, and therefore equal to each other. This actually was to assume that each player would utilize all of his strategies with his optimal mix, as only in this case would all the payoffs be equal. Actually, not all the strategies might be active, and therefore, only two, three, or even none of the payoff values might be equal. However, I found that any of these cases would show up readily on the graph. Thus eliminating one unknown, I obtained the following equations:

$$v_1=v_2) \quad y = -2x + 5/4 \tag{5}$$

$$v_1=v_3) \quad y = x \tag{6}$$

$$v_1=v_4) \quad x = 3/8 \tag{7}$$

$$v_2=v_3) \quad y = -x + 5/6 \tag{8}$$

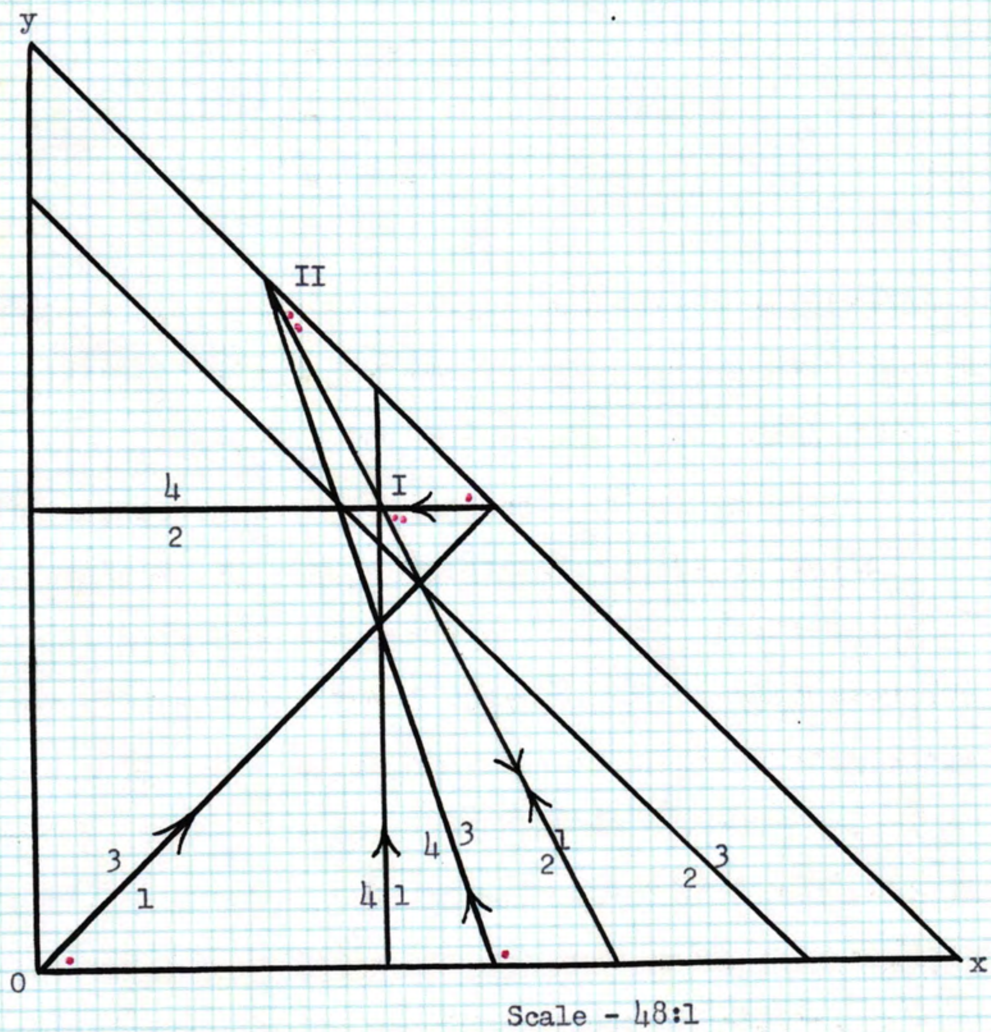
$$v_2=v_4) \quad y = 1/2 \tag{9}$$

$$v_3=v_4) \quad y = -3x + 3/2 \tag{10}$$

In order to graph these equations, I used a graph marked in fractions rather than in integers since, by the definition of a probability vector, neither x nor y can exceed 1. Also, by the same definition, no value can be less than zero, and all possible values of x and y are therefore limited to the area bounded by the two axes and the line $x+y=1$.

As every game matrix is composed of two sets of strategies, different combinations of these strategies will result in different subgames, all of which are contained in the original matrix. If one of these subgames is made sufficiently small, all its strategies will be active, and therefore all the payoff values will be equal. This means that on the graph every intersection of three lines ($v_a=v_b$, $v_a=v_c$, and $v_b=v_c$) plus every intersection of a line $v_a=v_b$ with a boundary ($x=0$, $y=0$, $x+y=1$) indicates a solution to one subgame of the original matrix. Since every basic solution of the matrix is associated with at least one square submatrix[1], all that is necessary in order to solve the game is to determine which subgame solution is also the solution of the original matrix.

Graph for Player A



Note: The red dots pertain to the intersections of the lines they lie between.

If the game value at a point representing the solution to a subgame is considered to be v, then since $p^0G \geq V$, every value v_k of a strategy k not equal to v must be greater than v if that point is to be a solution of the matrix. This is to say that if a point is to be the solution of the matrix, all strategies not active at that point must have a greater value than the active ones.

On the graph, it can be determined that one strategy has a larger value than another on one side of the line representing their equality, and a lesser value on the other side of the line. To determine which strategy had a greater value on which side of the line, I compared their values at the point (0,0). Since A's strategy 3 is his only active one at that point, the value for each of B's strategies is the payoff in the third row. Therefore all I needed to do was to determine which payoff was larger and then record the results. If a line happened to pass through the origin, that is, if two of B's strategies had equal payoffs against A's strategy 3, I considered either the point (1,0) or the point (0,1) as my point of reference. To eliminate extra symbols on the graph, I indicated the equation of a line and its corresponding inequalities by writing the number of each component strategy on that side of the line at which it was greater than the other. Thus, at the origin in the sample game,

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[1] Dresher, Melvin, MATHEMATICAL THEORY OF ZERO SUM TWO PERSON GAMES, p. 16

$v_2 > v_1$, $v_2 > v_3$, $v_2 > v_4$, $v_4 > v_1$, and $v_4 > v_3$; and at the point (0,1), $v_3 > v_1$.

The next step in the solution is to eliminate those points where $p^0G \quad V$ is not satisfied. This is done by discarding all points whose active strategies have greater values than those of the inactive ones. For instance, at the intersection of (8), (9), and (10), $v_1 < v_3$, and the point therefore cannot be a solution to the matrix. However, at the intersection of (5), (7) and (9), $v_1 < v_3$, and the point can be a solution to the game. Thus, each intersection is checked separately in order to determine whether or not it can be a solution.

Since this can get to be quite a lengthy process, as all endpoints must also be checked, I have devised a way to shorten the procedure. First, I simply check all intersections within the boundaries (the two axes and the line x+y=1). Then, the only endpoints which I need to check are those lying on lines that pass through the points left by the first check. (For a proof of this method, see Theorem 16.) Thus, in the sample game, the endpoints of (8) need not be checked.

After this process has been completed in the sample matrix, only those points indicated by the red dots are left. Since there are still a number of them, however, it must be determined which of them give the highest game value and thereby the best solution.

To accomplish this, I took the value of y in terms of x from each of the six equations and substituted it in the expression for the payoff value of one of its component strategies. (If the y term was missing in the equation, I merely substituted the appropriate value of x in the expression.) Making such a substitution for each of the six equations gives the following results:

y in (5) substituted in (1) $7/2$

y in (6) substituted in (1) $6x + 1$

x in (7) substituted in (1) $2y + 11/8$

y in (8) substituted in (2) $-2x + 13/3$

y in (9) substituted in (2) $-4x + 5$

y in (10) substituted in (3) $-10x + 7$

From the signs of the literal in each expression, the exact manner in which the value along each line will vary can be determined. If the sign is positive, the value will vary in the same way as the literal. If the sign is negative, the value will vary in the opposite manner. If the literal is missing altogether, the value will remain constant.* Thus, the game value for (6) will increase as x increases, and the value for (8) will decrease as x increases. In (5), the value will remain constant as there is no literal. After following this procedure

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* As is evident, the constant has no effect on the way in which the value will vary. Therefore, it may be ignored in order to shorten this step.

for each equation, I recorded the results on the graph by indicating with an arrow the direction in which the value increased along a line. If the value remained constant, I used arrows pointing in opposite directions.

With this information on the graph, it can easily be seen that points I and II will give the highest game value. Therefore, A can use the mixture represented by either one of them, or, since they lie on a line of constant value, he can use the mixture represented by any point on the line between them. Thus, two possible mixtures for A are $p^0 = (1/4, 3/4, 0)$ and $p^1 = (3/8, 1/2, 1/8)$. The general solution for A may be represented by the strategy $p^0 = (x, 5/4 - 2x, x - 1/4)$ where x is greater than or equal to 1/4 but less than or equal to 3/8. The game value for any of these mixtures will be 7/2.

Player B, in his best mix, should naturally use only those strategies against which A wins the least. In this case, they are strategies 1, 2, and 4. To find their optimal mixture, I first reversed the payoff matrix by interchanging the rows and the columns and changing the signs of all payoffs. In doing so, I omitted B's strategy 3 as it gave A a higher value than the game value. Then, I found B's optimal mixture in the same manner I found A's: first, I assigned a probability vector to his optimal mixture; second, I multiplied the matrix by the vector to determine the payoff values against each of A's strategies; third, I set all of these values equal to each other and graphed the resulting equations; fourth, I eliminated those points which did not satisfy $p^0G \geq V$; and fifth, I determined which of the remaining points gave the optimal solution. This procedure gave B an optimal strategy of $q^0 = (1/2, 1/2, 0, 0)$.

This is the general method of solution at which I have arrived. Of course, different cases will arise with different matrices, and in the appendix to this report, I have indicated a few of these variations along with their solutions.

The general method which I have employed in solving these matrices is basically my own, and I have indicated a proof of its key features in the next section. In establishing this proof, I have used the same basic structure as used by Dr. John G. Kemeny in AN INTRODUCTION TO FINITE MATHEMATICS. My own theorems were then derived from this foundation. Definitions of the terms used in the proof will be found in the glossary.

A Graphical Solution of 3xm Game Matrices:
An Axiomatic Proof

Definition 1. A matrix is a rectangular array of numbers written in the form

$$G = \begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1n} \\ g_{21} & g_{22} & \cdots & g_{2n} \\ \cdot & \cdot & \cdots & \cdot \\ g_{m1} & g_{m2} & \cdots & g_{mn} \end{pmatrix}.$$

The letters g_{ij} stand for real numbers, and m and n are integers.[1]

Definition 2. If i represents a strategy of player A and j a strategy of player B, then the amount g_{ij} is the payoff to player A and the array of all values g_{ij} is the game matrix.[1]

Definition 3. An m component row vector p is a mixed strategy vector for A if it is a probability vector; similarly, an n component column vector is a mixed strategy vector for B if it is a probability vector. (A probability vector is a vector with non-negative entries whose sum is one.) Let V and V' be the vectors

[1]Kemeny, John G., et al, AN INTRODUCTION TO FINITE MATHEMATICS, Chapter V, Section 6.

$$V = \underbrace{(v,v,\ldots,v)}_{m \text{ components}} \text{ and } V' = \left.\begin{pmatrix} v \\ v \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix}\right\} n \text{ components}$$

where v is a number. Then v is the value of the game and p^0 and q^0 are optimal strategies if and only if the following inequalities hold:

$$p^0G \geq V,$$
$$Gq^0 \leq V'.$$[1]

Theorem 1. If G is a matrix game which has a value and optimal strategies, then the value of the game is unique.[1]

Theorem 2. If G is a matrix game with the value v and optimal strategies p^0 and q^0, then $v=p^0Gq^0$.[1]

Theorem 3. If G is a game with value v and optimal strategies p^0 and q^0, then v is the largest expectation A can assure for himself. Similarly, v is the smallest expectation B can assure for himself.[1]

Definition 4. A matrix game G is strictly determined if there is an entry g_{ij} in G that is the minimum entry in the ith row and the maximum entry in the jth column.[1]

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[1]Kemeny, John G., et al, AN INTRODUCTION TO FINITE MATHEMATICS, Chapter V, Section 6.

Theorem 4. If G is a strictly determined matrix game with an entry g_{ij} as indicated in Definition 4, the value of the game is $v=g_{ij}$. Moreover, A should chose the row that contains g_{ij} and B should choose the column that contains g_{ij}.[1]

Theorem 5. If p^0 and p^1 are two optimal strategies for A in a matrix G, then the strategy

$$p = ap^0 + (1 - a)p^1,$$

where a is any number satisfying $0 < a < 1$, is also an optimal strategy for A.

Similarly, if q^0 and q^1 are optimal strategies for B in G, then the strategy

$$q = aq^0 + (1 - a)q^1,$$

where a is any number satisfying $0 < a < 1$, is also an optimal strategy for B.[1]

Theorem 6. Let G be any mxn matrix game; then there exists a value v for G and optimal strategies p^0 for player A and q^0 for player B. In other words, every matrix game possesses a solution.[2]

Theorem 7. If a matrix game G is not strictly determined, there exists a square submatrix G' of G with all strategies active such that

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[1] Kemeny, John G., et al, AN INTRODUCTION TO FINITE MATHEMATICS, Chapter V, Section 6.

[2] Ibid., Chapter V, Section 7.

the optimal strategies p^0 and q^0 in G' are also optimal in G.[1]

Theorem 8. If a square submatrix G' is a 3x3 matrix with a, b, and c as B's active strategies, and an optimal strategy $p^0 = (x, y, 1-x-y)$ for A, one solution lies at the intersection of the lines representing the equations $v_a = v_b$, $v_a = v_c$, and $v_b = v_c$. This point may be anywhere in the polygon bounded by the two axes and the line $x+y=1$.

Proof: If all strategies are active, A's probability vector must contain only non-zero numbers. This limits the values of x and y to those in the polygon bounded by the two axes and $x+y=1$.

Also, if all of B's strategies are active, then each must give A the same payoff value. (If a strategy gave a higher value, B would not use that strategy in his optimal mix, and it would not be active. If it gave a lower value, A's mix would not be optimal, and again the strategy would not be active.) Thus $p^0G'=V$, and $v_a=v_b=v_c$.

Theorem 9. If a square submatrix is a 2x2 contained in a 3x2 matrix with a and b as B's active strategies, and an optimal strategy $p^0 = (x, y, 1-x-y)$ for A, then one solution lies at the intersection of $v_a=v_b$ with one of the axes or the line $x+y=1$.

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[1]Dresher, Melvin, MATHEMATICAL THEORY OF TWO PERSON ZERO SUM GAMES, p. 16.

Proof: If only two of A's strategies are active, one entry in the probability vector must be zero. Also, since a and b are active, $v_a=v_b$ must hold true. Hence the theorem.

Theorem 10. There exists at least one possible solution point P_i for each square submatrix G' of a matrix G.

Proof: By Theorem 6, every matrix game possesses a solution and optimal strategies p^0 and q^0. Since p^0 can be represented by a probability vector as in Theorem 8, it can also be represented on the graph by an ordered pair of numbers (x,y).

Theorem 11. If some P_i of a square submatrix G' is also to be a possible solution point for a game matrix G, then every active strategy of B must have a value v_a at P_i such that for any other strategy k possessed by B, v_a v_k.

Proof: By Definition 3, if v is the value of the matrix game G' and p^0 is A's optimal strategy, then $p^0G' \geq V$. But, by Theorem 7, $p^0G'=V$, and $p^0G \geq V$ must also be true. Hence the theorem.

Theorem 12. At any point p on a line $v_a=v_b$, the value v_a is less than or equal to the value v' at any point p' on one segment of the line formed by its division by p, and greater than or equal to the value v'' at any point p'' on the other segment.

Proof: Let the payoff matrix for strategies a and b be

$$G' = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \\ g_{31} & g_{32} \end{pmatrix}$$

and A's probability vector be $p^0 = (x, y, 1-x-y)$.

Then, the values v_a and v_b are

$$v_a = (g_{11}-g_{31})x + (g_{21}-g_{31})y + g_{31}, \text{ and}$$

$$v_b = (g_{12}-g_{32})x + (g_{22}-g_{32})y + g_{32}.$$

Setting v_a equal to v_b and solving for y,

$$y = \frac{g_{31}-g_{11}-g_{32}+g_{12}}{g_{21}-g_{31}-g_{22}+g_{32}}x + \frac{g_{32}-g_{31}}{g_{21}-g_{31}-g_{22}+g_{32}} .$$

Substituting this in v_a and combining terms,

$$v_a = \left[g_{11}-g_{31} + (g_{21}-g_{31}) \left(\frac{g_{31}-g_{11}-g_{32}+g_{12}}{g_{21}-g_{31}-g_{22}+g_{32}}\right)\right]x$$

$$+ (g_{21}-g_{31}) \left(\frac{g_{32}-g_{31}}{g_{21}-g_{31}-g_{22}+g_{32}}\right) + g_{32}.$$

Let k be the coefficient of the x term. Then, if

$k > 0$, v_a increases as x increases;

$k < 0$, v_a decreases as x increases; and

$k = 0$, v_a remains constant.

If the y term disappears in $v_a = v_b$, the theorem can be proved similarly by solving for x.

Theorem 13. The optimal P_i is that P_i which has passing through it no line $v_a=v_b$ with another point P_i' giving a greater value to v_a than does P_i.

Proof: If P_i' gave a greater value to v_a than did P_i, it would be optimal. Hence for P_i to be optimal, P_i' cannot have a greater value.

Theorem 14. If a line $v_a=v_b$ gives a constant value to v_a and passes through two optimal P_i, all points on that line between the two P_i also represent optimal solutions.

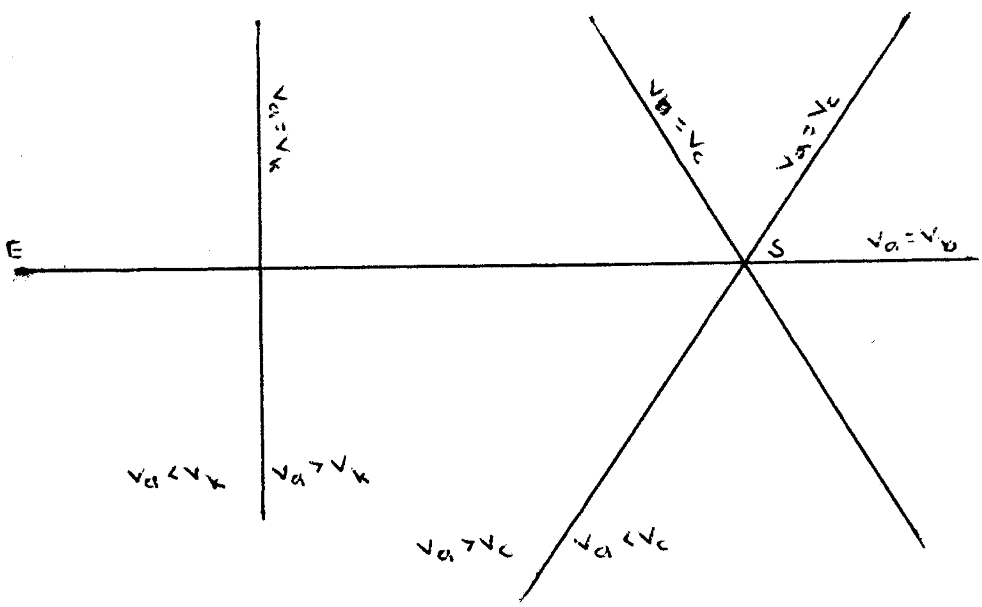
Proof: If the endpoints of the line are P_i, then the value v_k of any strategy k of B not active at both endpoints must be greater than or equal to the value v at both endpoints. Therefore, it follows that it must be greater than v for the length of the segment, and since the segment has a constant value, all points on it represent optimal solutions.

Theorem 15. All optimal P_i on the line $v_a=v_b$ described in Theorem 14 must lie between the two original P_i.

Proof: Let $v_a=v_b$ be the line of a constant value v with the endpoints P_1 and P_2. Then, any line $v_a=v_k$ passing through P_1 or P_2 must give a greater value to k on that segment of the line $v_a=v_b$ containing the other endpoint, and a lesser value to k on the other segment. Hence, all points on the external segments are excluded from representing optimal solutions as an inactive strategy has a lesser value than an active one.

Theorem 16. Any line $v_a = v_b$ passing through the intersection of $v_a = v_c$ and $v_b = v_c$ not containing a P_i in the interior of the polygon described in Theorem 8 has no P_i at tis intersection with a boundary of the polygon.

Proof: If the point S (see the diagram) at the intersection of $v_a = v_b$ and $v_b = v_c$ is not a P_i, then v_a must be greater than some value v_k at S. Then, if v_k is to be greater than v_a at one endpoint of $v_a = v_b$, the line $v_a = v_k$ must intersect $v_a = v_b$ between S and the endpoint. By the hypothesis, this point is not a P_i, and v_a must be greater than v_c at this intersection. But, since this intersection lies between S and the endpoint, v_a is also greater than v_c at the endpoint, and the endpoint cannot be a P_i. Hence the theorem.



Appendix:

Matrices and Their Solutions

Each example in this section is composed of the following:

1) A game matrix G.

2) An associated probability vector $p^0 = (x, y, 1-x-y)$ for A and a probability vector $q^0 = (a, b, 1-a-b)$ for B's active strategies. Since the payoff matrix is reversed in finding B's optimal mix, q^0 is written as a row vector instead of a column vector.

3) The equations derived from the matrix. For the sake of brevity v_1 is shortened to 1, $v_1=v_2$ to 1=2, etc.

4) The graph employed in the solution. On the graph is found the following information:

   a) The limiting boundaries ($x=0$, $y=0$, $x+y=1$) designated by the outer triangle on each graph.

   b) The equations derived from the matrix with their associated inequalities and changes of the payoff values.

   c) Red dots indicating possible solution points with two dots indicating the optimal solution. Unless otherwise indicated, the dot belongs to the intersection of the two lines it lies between.

5) The solution of the matrix.

6) Comments on the solution.

Matrix 1

		B			
		1	2	3	4
	1	1	7	3	4
A	2	5	6	4	5
	3	7	2	θ	3

This matrix is strictly determined; that is, the value g_{23} is simultaneously the minimum value in its row and the maximum in its column. Thus, by Theorem 4, A's optimal mixture is

$$p^0 = (0,1,0),$$

and B's optimal mixture is

$$q^0 = (0,0,1,0).$$

As the game is strictly determined, a graph is not necessary in its solution. The rest of the matrices in this section were checked for such a strictly determined solution before the graphical method was used.

Matrix 2

		B 1	B 2	B 3
A	1	1	1	3
A	2	1	3	2
A	3	3	2	2

I. Payoff values to A

1) $-2x - 2y + 3$

2) $-x + y + 2$

3) $x + 2$

II. Equal payoff values

1=2) $y = -x/3 + 1/3$
1=3) $y = -3x/2 + 1/2$
2=3) $y = 2x$

III. Solution

$p^0 = (1/7, 2/7, 4/7)$
$v = 15/7$

I. Payoff values to B

1) $2a + 2b - 3$

2) $a - b - 2$

3) $-a - 2$

II. Equal payoff values

1=2) $b = -a/3 + 1/3$
1=3) $b = -3a/2 + 1/2$
2=3) $b = 2a$

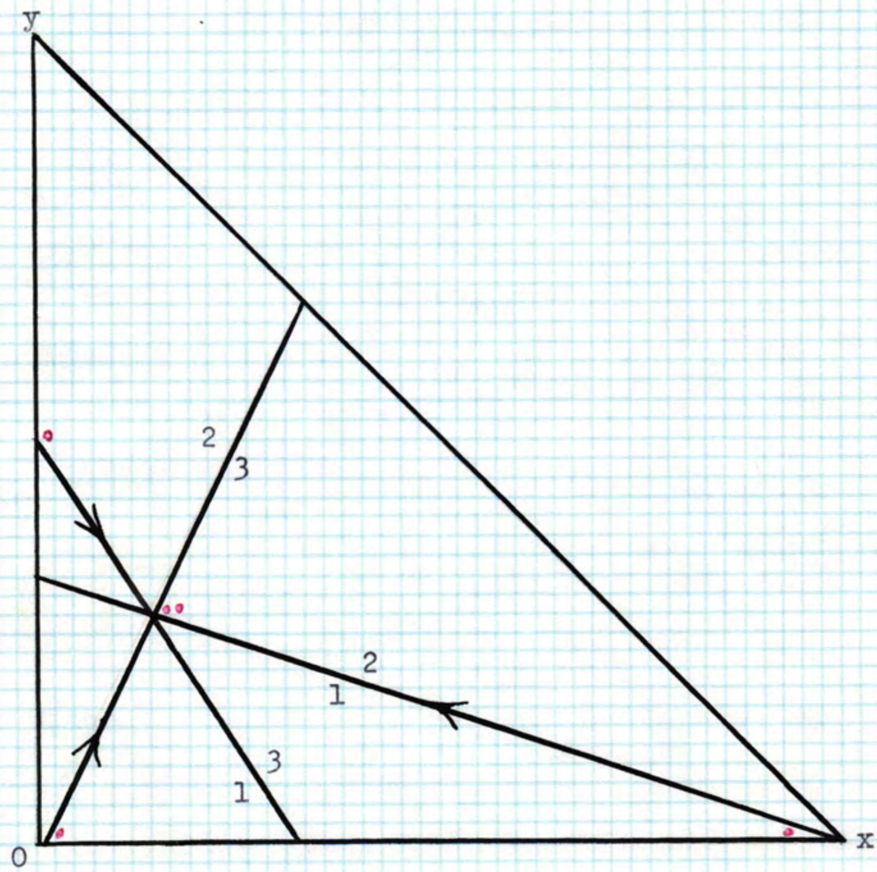
III. Solution

$q^0 = (1/7, 2/7, 4/7)$
$v' = -15/7$

Because of the symmetry of this matrix (the rows and the columns are interchangeable), the graphs for the solutions of the two players are the same. Since all strategies are active, the solution is found by Theorem 8.

Matrix 2

Players A and B



Scale - 42:1

Matrix 3

		B 1	B 2	B 3
A	1	203	403	103
A	2	303	3	103
A	3	3	103	303

I. Payoff values to A

1) $200x + 300y + 3$
2) $300x - 100y + 103$
3) $-200x - 200y + 303$

II. Equal payoff values

1=2) $y = x/4 + 1/4$
1=3) $y = -4x/5 + 3/5$
2=3) $y = -5x + 2$

III. Solution

$p^0 = (1/3, 1/3, 1/3)$
$v = 509/3$

I. Payoff values to B

1) $-100a - 300b - 103$
2) $-200a + 100b - 103$
3) $300a + 200b - 303$

II. Equal payoff values

1=2) $b = a/4$
1=3) $b = -4a/5 + 2/5$
2=3) $b = -5a + 2$

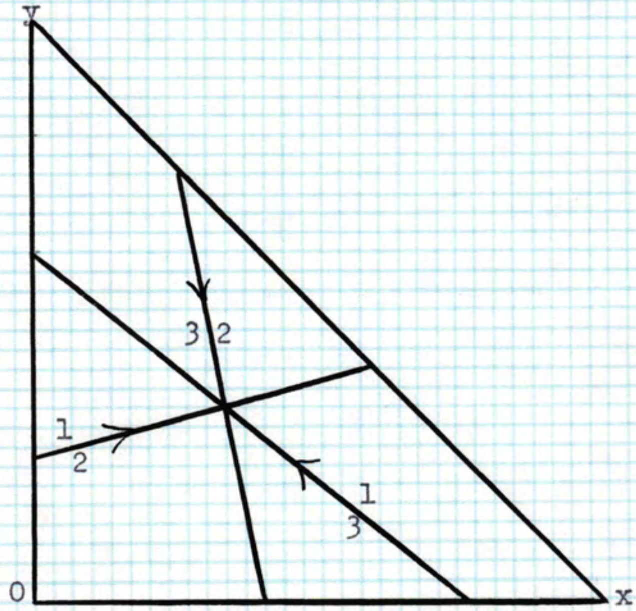
III. Solution

$q^0 = (8/21, 2/21, 11/21)$
$v' = -509/3$

Except for the fact that this matrix lacks symmetry, its solution is similar to that of Matrix 2 and is also derived by Theorem 8.

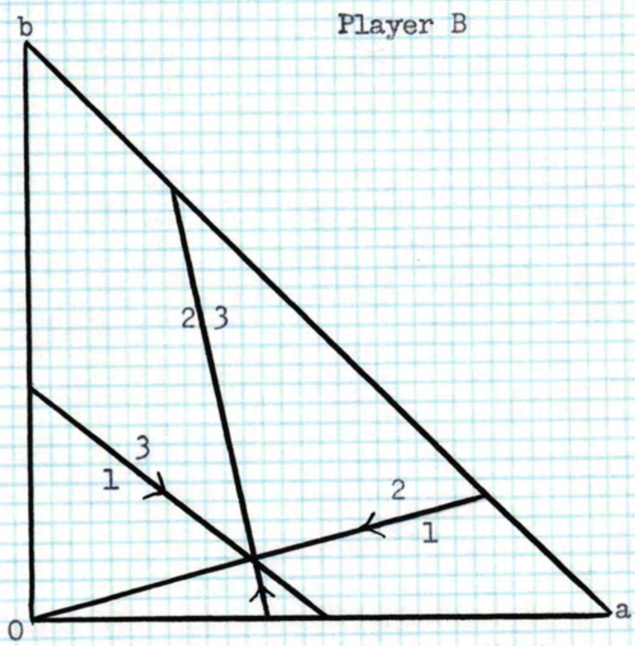
Matrix 3

Player A



Scale - 30:1

Player B



Scale - 30:1

Matrix 4

		B 1	2	3
	1	4	3	2
A	2	3	4	3
	3	2	3	4

I. Payoff values to A

1) $2x + y + 2$
2) $y + 3$
3) $-2x - y + 4$

II. Equal payoff values

1=2) $x = 1/2$
1=3) $y = -2x + 1$
2=3) $y = -x + 1/2$

III. Solution

$p^0 = (x, 1-2x, x)$
$v = 3$

I. Payoff values to B

1) $-2a - b - 2$
2) $-b - 3$
3) $2a + b - 4$

II. Equal payoff values

1=2) $a = 1/2$
1=3) $b = -2a + 1$
2=3) $b = -a + 1/2$

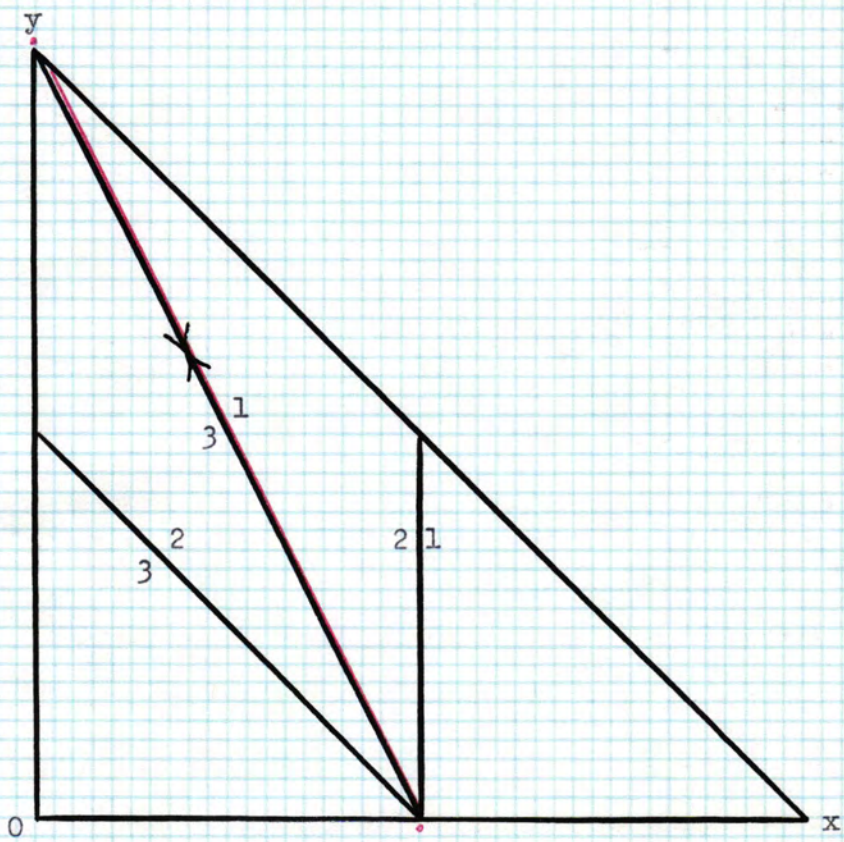
III. Solution

$q^0 = (a, 1-2a, a)$
$v' = -3$

In this matrix, the value along the line representing $v_1 = v_3$ is constant and, since both endpoints of the line segment are possible solution points, every point along the line represents a solution. Thus, a general instead of a specific solution results. To transform this general solution to a specific one, values can be substituted for x and a in p^0 and q^0 as long as x and a are greater than or equal to zero and less than or equal to one-half. Theorems 8, 14, and 15 apply to this solution.

Matrix 4

Players A and B



Scale - 40:1

Matrix 5

$$\begin{array}{cc|c|c|c|}
 & & & B & \\
 & & 1 & 2 & 3 \\
\hline
 & 1 & 3 & 1 & 6 \\
\hline
A & 2 & 2 & 2 & 0 \\
\hline
 & 3 & 8 & 0 & 3 \\
\hline
\end{array}$$

I. Payoff values to A

1) $-5x - 6y + 8$
2) $x + 2y$
3) $3x - 3y + 3$

II. Equal payoff values

1=2) $y = -3x/4 + 1$
1=3) $y = -8x/3 + 5/3$
2=3) $y = 2x/5 + 3/5$

III. Solution

$p^0 = (2/7, 5/7, 0)$
$v = 12/7$

I. Payoff values to B

1) $3a + 5b - 6$
2) $-2a - 2b$
3) $-5a + 3b - 3$

II. Equal payoff values

1=2) $b = -5a/7 + 6/7$
1=3) $b = -4a + 3/2$
2=3) $b = 3a/5 + 3/5$

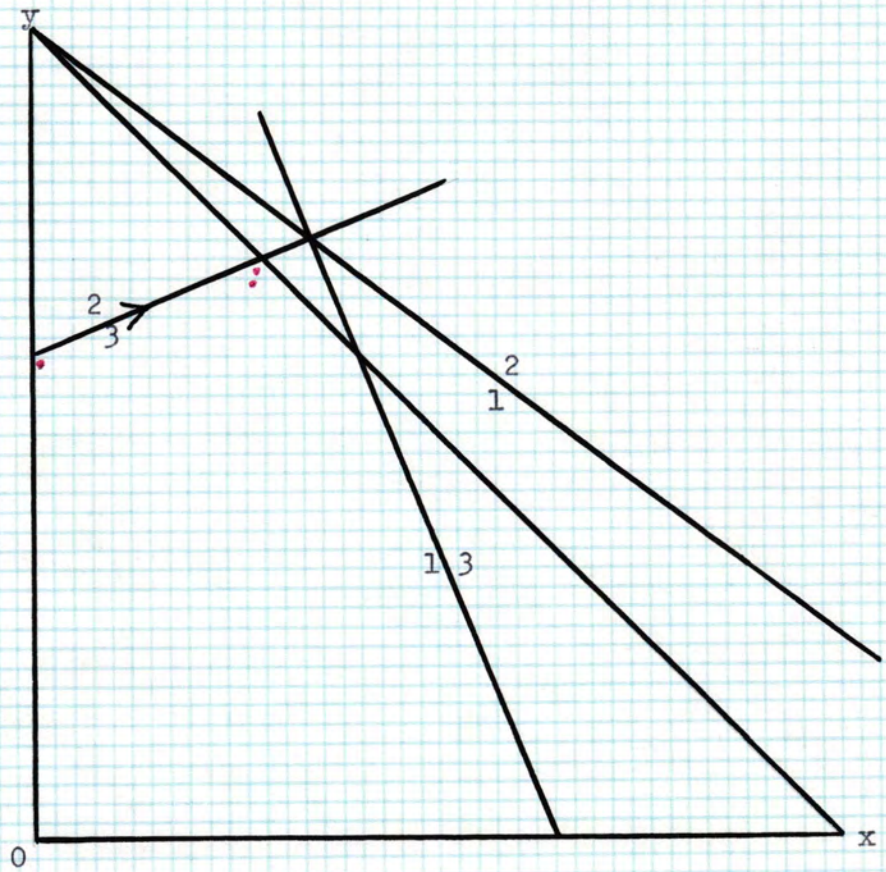
III. Solution

$q^0 = (0, 6/7, 1/7)$
$v' = -12/7$

Not all strategies in this matrix are active, and its solution is therefore identical to that of a square submatrix contained within it. It may be noted that the original assumption of all payoff values being equal is false. However, as was mentioned in the text, this situation is obvious on the graph as, in one instance, the equations intersect without the range of possible values for x and y, and in the other instance, an endpoint of a line affords a better solution than the intersection of the three lines. Theorem 9 was used to find the solution.

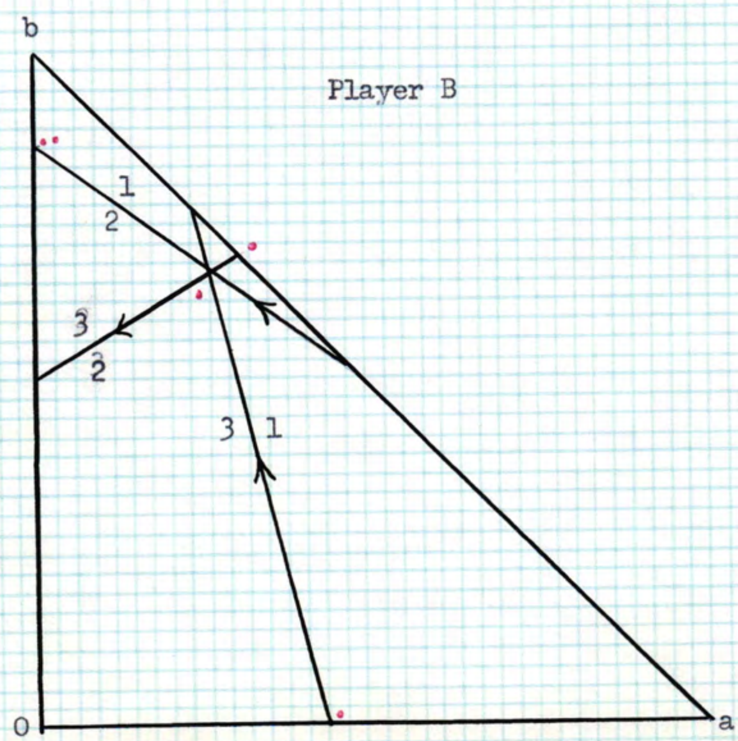
Matrix 5

Player A



Scale - 42:1

Player B



Scale - 35:1

Matrix 6

$$\begin{array}{cc|c|c|c|} & & \multicolumn{3}{c}{B} \\ & & 1 & 2 & 3 \\ \hline & 1 & 3 & 2 & 4 \\ \hline A & 2 & 0 & 4 & 1 \\ \hline & 3 & 1 & 3 & 4 \\ \hline \end{array}$$

I. Payoff values to A

1) $2x - y + 1$
2) $-x + y + 3$
3) $-3y + 4$

II. Equal payoff values

1=2) $y = 3x/2 - 1$
1=3) $y = -x + 3/2$
2=3) $y = x/4 + 1/4$

III. Solution

$p^0 = (4/5, 1/5, 0)$
$v = 12/5$

I. Payoff values to B

1) $a + 2b - 4$
2) $a - 3b - 1$
3) $3a + b - 4$

II. Equal payoff values

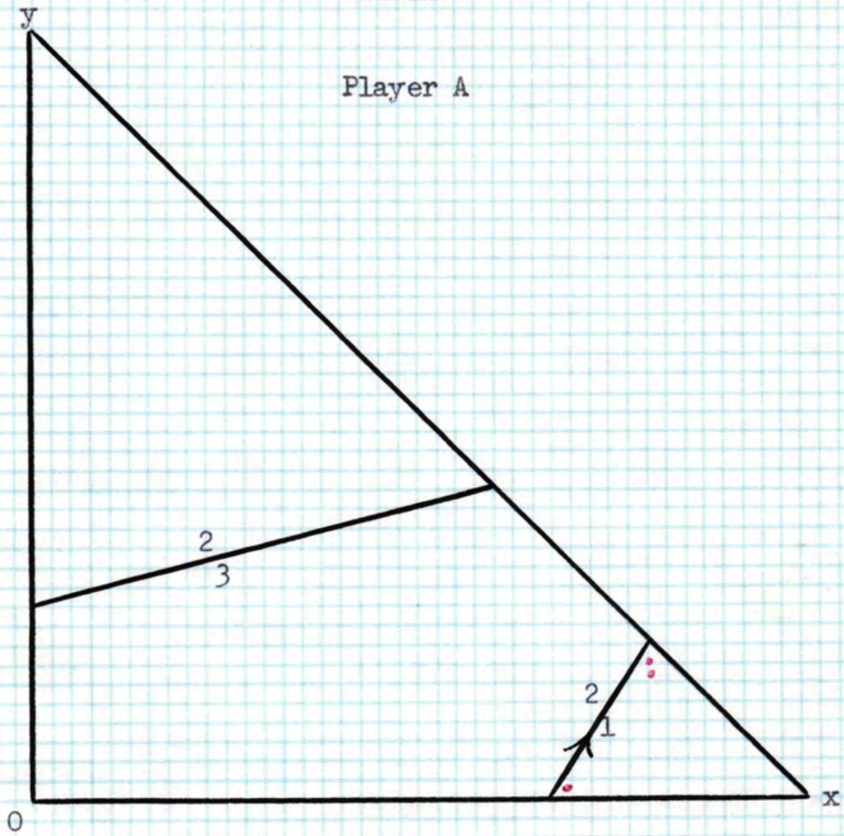
1=2) $b = 3/5$
1=3) $b = 2a$
2=3) $b = -a/2 + 3/4$

III. Solution

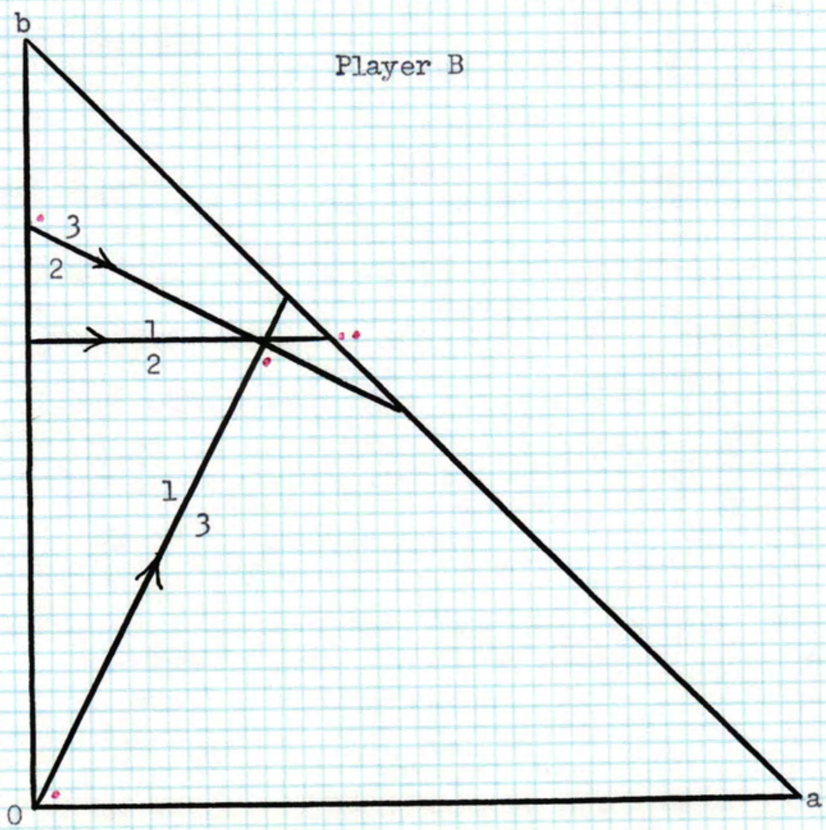
$q^0 = (2/5, 3/5, 0)$
$v' = -12/5$

This matrix is similar to Matrix 5 in that not all its strategies are active. From the matrix it is evident that B's strategy 3 is inferior to strategy 1, as he loses more no matter what strategy A plays. Therefore, the payoff values for these two strategies cannot be equal without giving a negative value to an entry in the probability vector. Since this is impossible, Theorem 9must be employed to find the solution.

Matrix 6



Scale - 40:1



Scale - 40:1

Matrix 7

		B 1	2	3	4
	1	1	6	2	5
A	2	5	1	6	2
	3	2	5	1	6

I. Payoff values to A

1) $-x + 3y + 2$
2) $x - 4y + 5$
3) $x + 5y + 1$
4) $-x - 4y + 6$

II. Equal payoff values

1=2) $y = 2x/7 + 3/7$
1=3) $y = -x + 1/2$
1=4) $y = 4/7$
2=3) $y = 4/9$
2=4) $x = 1/2$
3=4) $y = -2x/9 + 5/9$

III. Solution

$p^0 = (1/18, 4/9, 1/2)$
$v = 59/18$

I. Payoff values to B

1) $a - 4b - 2$
2) $a + 5b - 6$
3) $-a - 4b - 1$

II. Equal payoff values

1=2) $b = 4/9$
1=3) $a = 1/2$
2=3) $b = -2a/9 + 5/9$

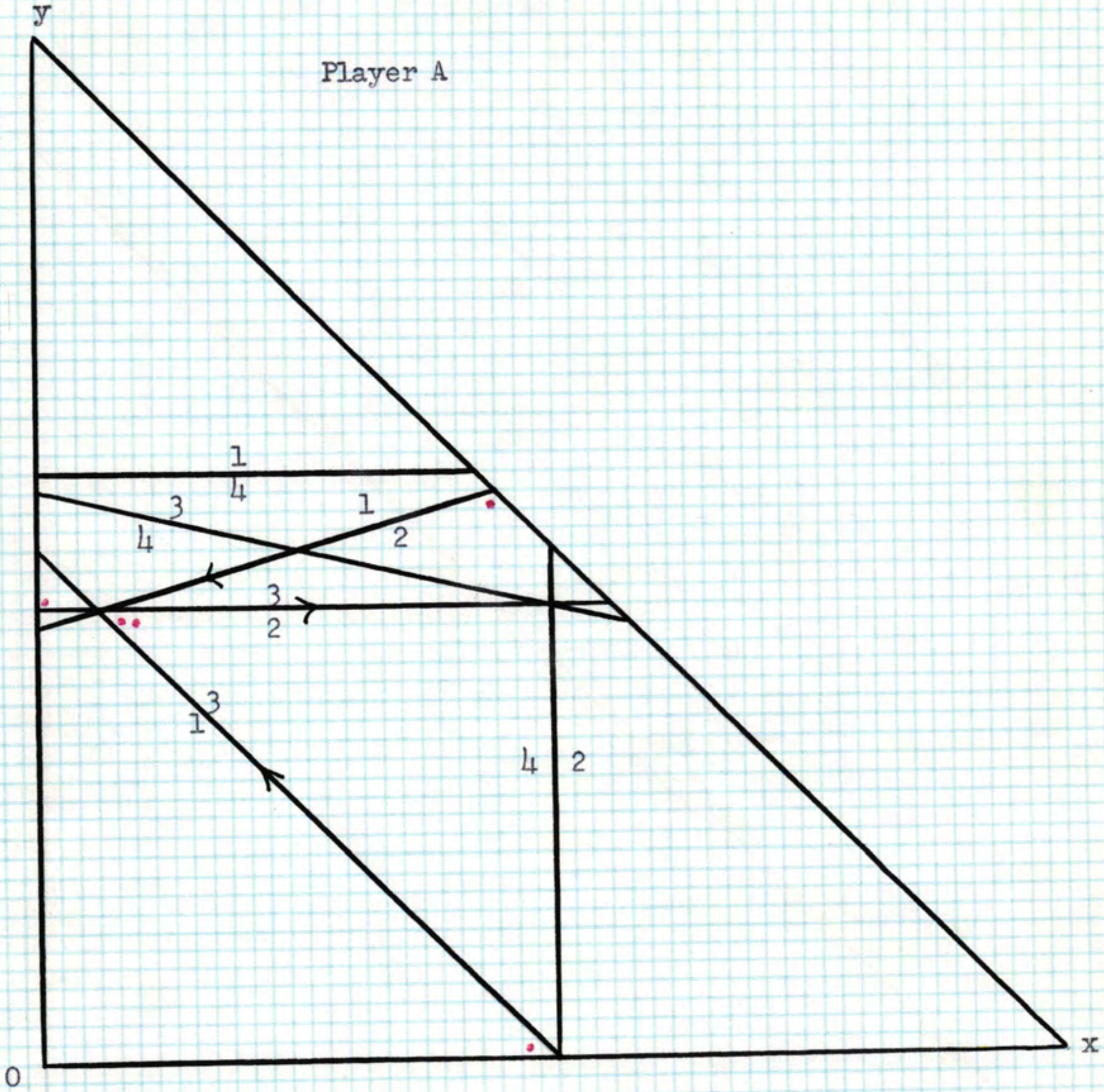
III. Solution

$q^0 = (1/2, 4/9, 1/18, 0)$
$v' = -59/18$

The method of solution of this matrix is similar to that used in the text of the report. Note that by Theorem 16, the endpoints of $v_1=v_4$, $v_2=v_4$, and $v_3=v_4$ do not have to be checked for possible solution points.

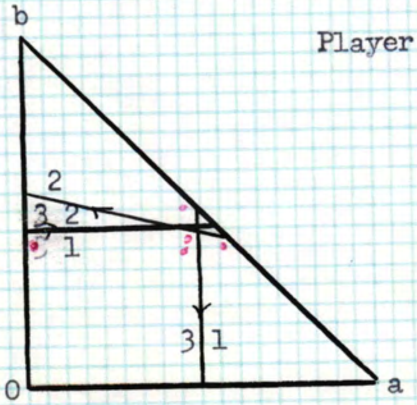
Matrix 7

Player A



Scale - 54:1

Player B



Scale - 18:1

Matrix 8

		B 1	2	3	4	5
	1	1	3	5	3	1
A	2	4	2	0	0	2
	3	1	2	1	4	4

I. Payoff values to A

1) $3y + 1$
2) $x + 2$
3) $4x - y + 1$
4) $-x - 4y + 4$
5) $-3x - 2y + 4$

II. Equal payoff values

1=2) $y = x/3 + 1/3$
1=3) $y = x$
1=4) $y = -x/7 + 3/7$
1=5) $y = -3x/5 + 3/5$
2=3) $y = 3x - 1$
2=4) $y = -x/2 + 1/2$
2=5) $y = -2x + 1$
3=4) $y = -5x/3 + 1$
3=5) $y = -7x + 3$
4=5) $y = x$

III. Solution

$p^0 = (3/8, 3/8, 1/4)$
$v = 17/8$

As can be seen from this matrix, the number of equations to be graphed increases with the number of strategies available to Player B. In general, when B has m strategies, $1 + 2 + \cdots + m-1$ equations result. This fact does present a drawback to the method when B possesses a number of strategies, but equally complex situations arise with the use of other methods of solution.

As A wins the same amount against B's strategies 1, 3, 4, and 5, there are a total of four possible sub-matrices for use in calculating B's optimal strategy. However, only two of these submatrices are actually optimal; these being the ones containing strategies 1, 3, and 4, and 1, 4, and 5. The reason the other two combinations of the strategies do not offer optimal solutions is that only two of B's strategies are active in them while three of A's are.

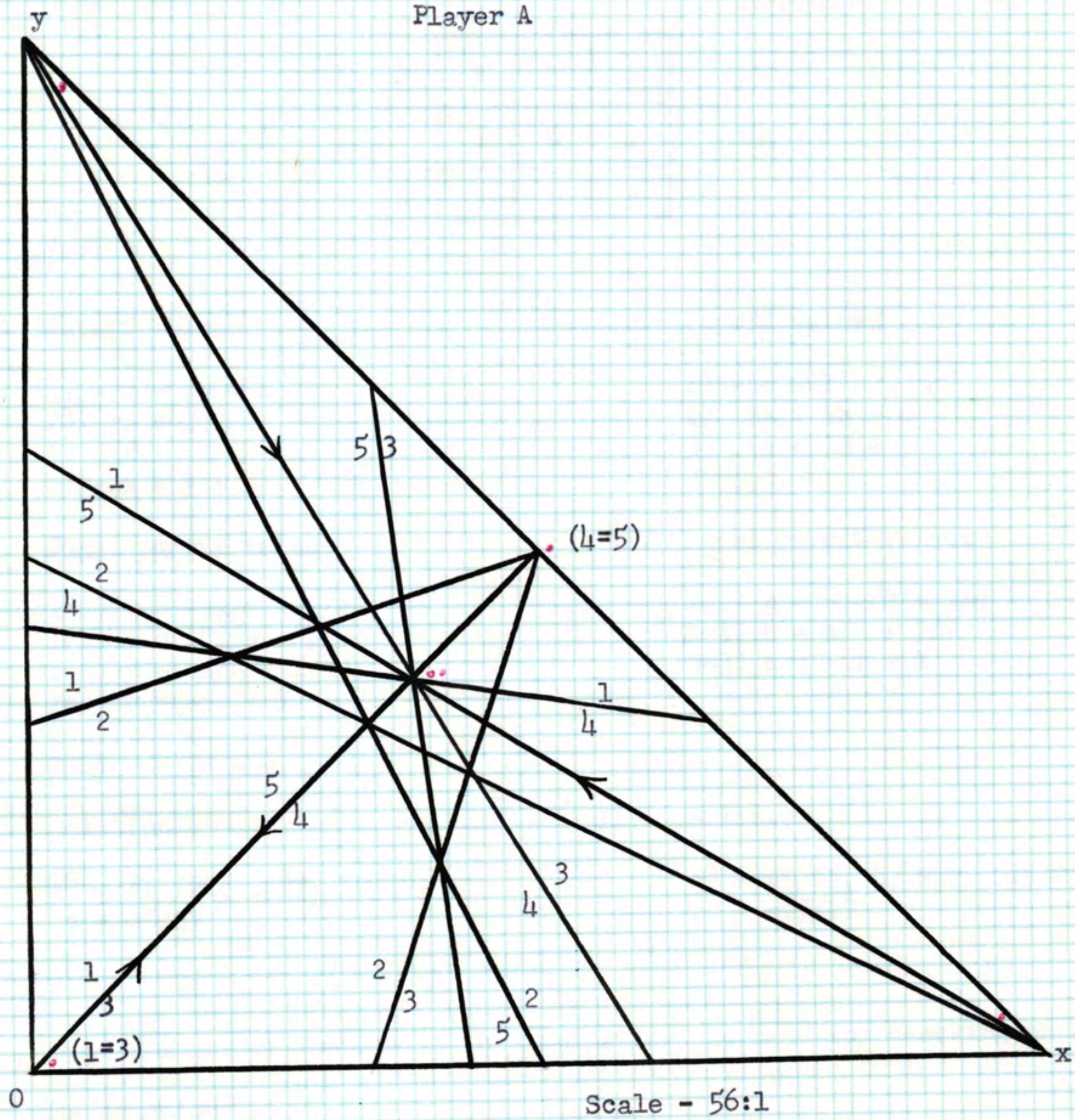
Since the solutions for the two optimal strategies are similar to others presented in this appendix, they themselves are not included. However, the results of these solutions give B optimal strategies of

$$q^0 = (17/32,\ 0,\ 3/32,\ 3/8,\ 0) \text{ and}$$
$$q^1 = (11/32,\ 0,\ 9/32,\ 0,\ 3/8).$$

The game value for B is $v' = -17/8$.

Matrix 8

Player A



Scale - 56:1

Matrix 9

		B 1	2	3	4
	1	1	3	5	1
A	2	1	2	0	4
	3	4	3	0	3

I. Payoff values

1) $-3x - 3y + 4$
2) $-y + 3$
3) $5x$
4) $-2x + y + 3$

II. Equal payoff values

1=2) $y = -3x/2 + 1/2$
1=3) $y = -8x/3 + 4/3$
1=4) $y = -x/4 + 1/4$
2=3) $y = -5x + 3$
2=4) $y = x$
3=4) $y = 7x - 3$

III. Solution

$p^0 = (13/29, 4/29, 12/29)$
$v = 65/29$

I. Payoff values

1) $-4b - 1$
2) $3a + 4b - 4$
3) $-a + 3b - 3$

II. Equal payoff values

1=2) $b = -3a/8 + 3/8$
1=3) $b = a/7 + 2/7$
2=3) $b = -4a + 1$

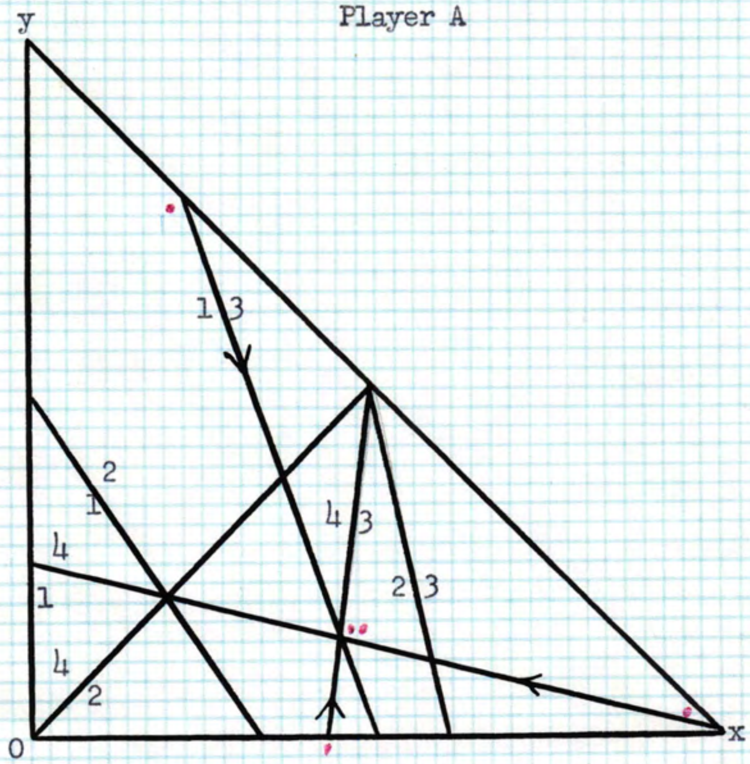
III. Solution

$q^0 = (5/29, 0, 9/29, 15/29)$
$v' = -65/29$

Note the use of Theorem 16 in eliminating the need for checking three of the equations.

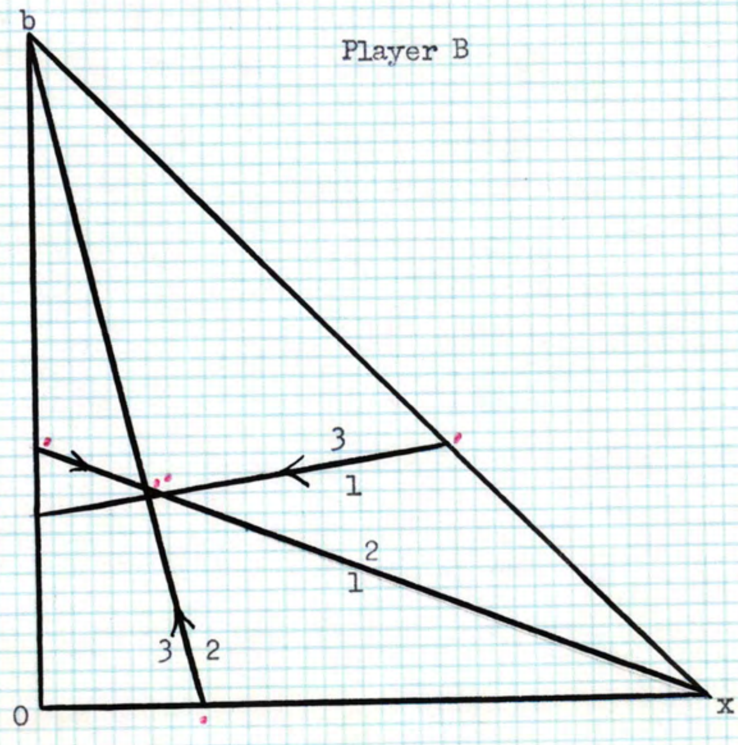
Matrix 9

Player A



Scale - 36:1

Player B



Scale - 35:1

Matrix 10

		B		
		1	2	3
	1	2	0	4
A	2	1	2	4
	3	2	1	0

I. Payoff values to A

1) $-y + 2$
2) $-x + y + 1$
3) $4x + 4y$

II. Equal payoff values

1=2) $y = x/2 + 1/2$
1=3) $y = -4x/5 + 2/5$
2=3) $y = -5x/3 + 1/3$

III. Solution

$p^0 = (0, 1/2, 1/2)$
$v = 3/2$

I. Payoff values to B

1) $2a + 4b - 4$
2) $3a + 2b - 4$
3) $-2a - b$

II. Equal payoff values

1=2) $b = a/2$
1=3) $b = -4a/5 + 4/5$
2=3) $b = -5a/3 + 4/3$

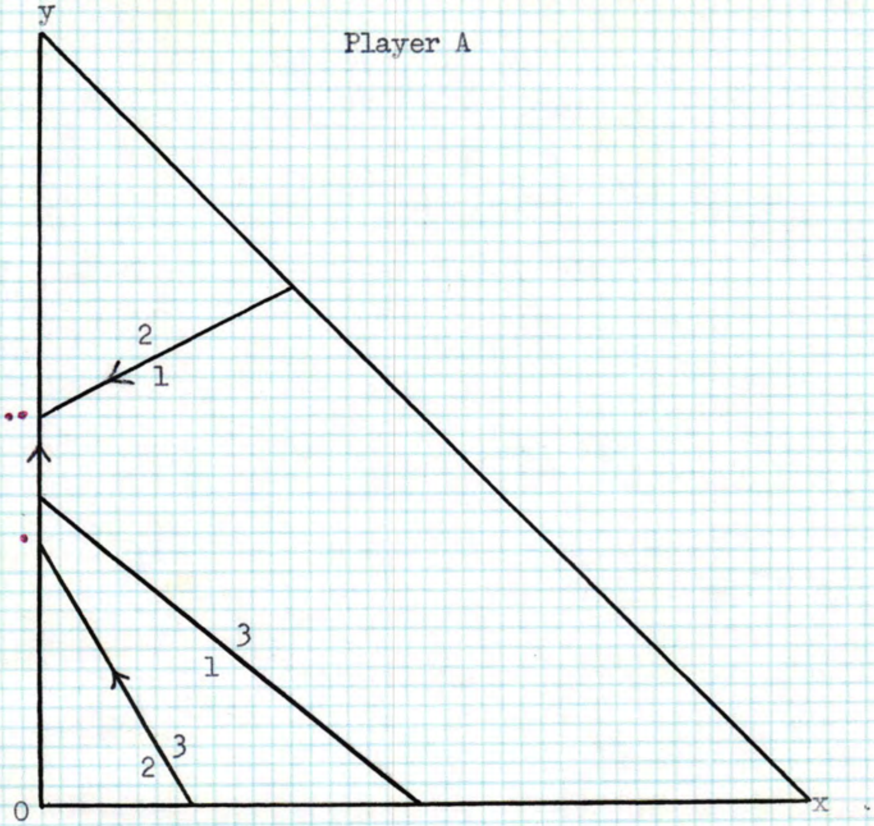
III. Solution

$q^0 = (1/2, 1/2, 0)$
$v' = -3/2$

This matrix shows the need for one more theorem. This theorem would state that if two or more possible solution points not lying on the same or intersecting line segments existed, each one must have at least one strategy in common with one of the others. Once the truth of this theorem is established, matrices such as this one may be solved by finding the equation of the line joining the two points and determining how the payoff value varies along it. Another theorem seems to suggest itself that would simplify this case even more; namely, that if two such possible solution points exist, they must lie on either the axes or the line x+y=1.

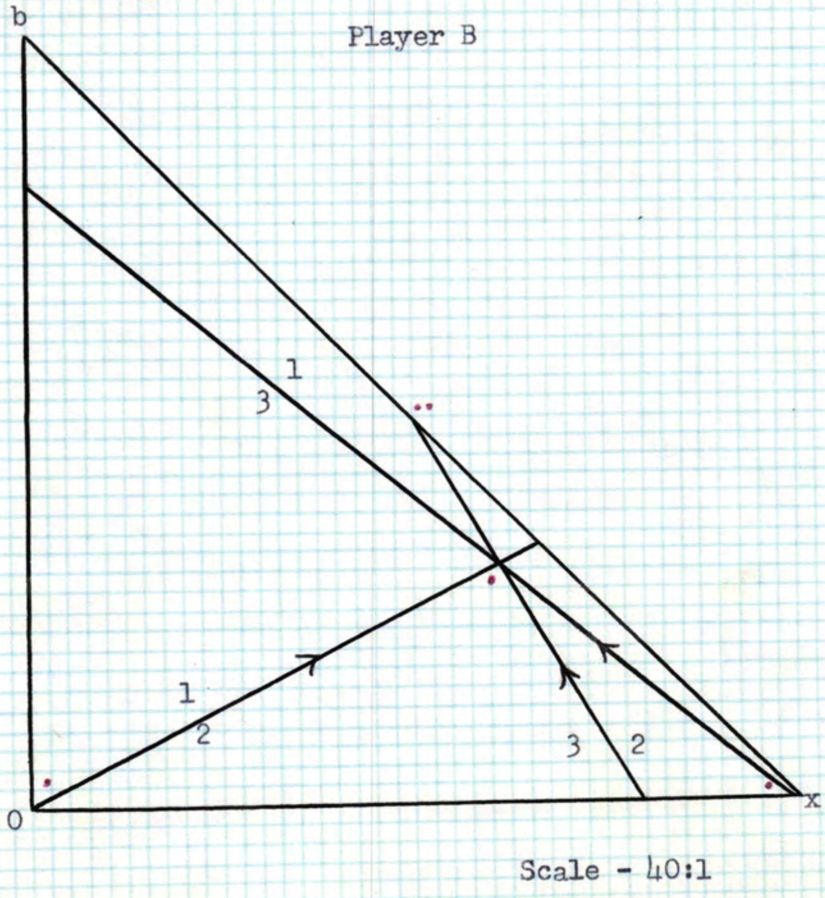
Matrix 10

Player A



Scale - 40:1

Player B



Scale - 40:1

Glossary

Active strategy - a strategy that is utilized in a player's optimal mix

Game theory - The mathematical theory dealing with decision making

Mixed strategy - a strategy composed of other strategies combined in a certain ratio

Optimal - best; an optimal mixture of strategies is that mixture which wins more for its user than any other

p^0, q^0 - optimal strategies for A and B respectively

Payoff - the amount won by a player as a result of a certain course of action in a game

Probability vector - a vector having only non-negative entries with a sum of one

Strategy - a complete set of rules as to how one player should make his decisions in a game

Subgame or submatrix - the game or matrix obtained by deleting one or more strategies from either player's set of strategies in a particular game

V, V' - the desired payoff value in a game in vector form

Vector - an ordered collection of numbers written in a row or a column

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