Sacks's construction gives a version of density for the r.e. degrees which is effective in the following sense: Given r.e. indices (Gödel numbers) for sets B, C of degree **b**, **c** respectively, there is a uniform method of finding a r.e. index for a set D of degree **d** so that $\mathbf{b} < \mathbf{d} < \mathbf{c}$ if $\mathbf{b} < \mathbf{c}$. In the construction, B is embedded directly in D so that $\mathbf{b} \leqq \mathbf{d}$ at once; then parts of C are selectively embedded in D so that $\mathbf{d} \not\leqq \mathbf{b}$ and $\mathbf{c} \not\leqq \mathbf{d}$ (assuming $\mathbf{b} < \mathbf{c}$). These conditions are represented by series of requirements which tend to conflict. Even when the requirements are assigned priorities, the conflict is so severe that a particular requirement may be interfered with by the satisfaction of requirements of higher priority infinitely often, yet in the end be satisfied. This feature has already appeared in Sacks's work (notably in the principal result of XXIX 204, and in XXIX 202, Theorems 3 and 4 of §6 and Theorem 1 of §7).

Sacks's approach to assuring $\mathbf{c} \not\leqq \mathbf{d}$ is similar to parts of these earlier theorems. The key combinatorial argument is contained in Lemma 2. The proof of Lemma 2 may be shortened considerably by observing that the crucial stages, of which the lemma asserts there are an infinity, can be taken to be the non-deficiency stages in the construction of D.

An exciting innovation is Sacks's handling of $\mathbf{d} \not\leqq \mathbf{b}$. His construction of D includes a process by which the embedding of parts of C in D is not permanent, but is effectively erasable. This degree of freedom is essential to the argument when B is non-recursive. For the embedding of B in D may lead to mistakes which need to be erased.

The remaining fact, that $\mathbf{d} \leqq \mathbf{c}$, is unexpectedly difficult. Sacks's proof, contained in Lemma 9, is incorrect. This is because in determining d(p(e, i, m)) from c Sacks inducts both on e and on p(e, i, m), which is not well-founded. Here p(e, i, m) is essentially a tripling function. The quickest rectification is to change line 25 of page 302 in the definition of d(s, p(e, i, m)) to read "$(j)_{j<i}[(Ef)_{f \leqq e}(Ek)(j = p(f, k)) \rightarrow d(s-1, j) = U(Y_b(s, j, e)))]$." The proof of Lemma 9 then needs only slight, obvious changes in order to proceed correctly by induction on e. Another possibility is to leave the definitions as they are and substantially revise Lemma 9 so that the induction is on p(e, i, m). This alternative was pointed out to the reviewer by James C. Owings, Jr. Owings reports that his observation was motivated by the dissertation of Driscoll, *Contributions to metarecursion theory*, Cornell University, 1965 (cf. XXXIV 115(4)). Driscoll's version of the density theorem in metarecursion theory generalizes this second alternative to the proof of Lemma 9. Interestingly enough, Owings has not been able to generalize our first alternative to the proof of Lemma 9 to metarecursion theory.

This paper is a landmark of recursion theory. The result is important, and the fresh ideas involved in its proof are very pleasing. In his writing Sacks tends to rely on equations more than words. To the reviewer this seems appropriate and inevitable in view of the difficulty of the material. ROBERT W. ROBINSON

GERALD E. SACKS. *A minimal degree less than O′.* ***Bulletin of the American Mathematical Society,*** vol. 67 (1961), pp. 416–419.

A degree a is minimal if $O < \mathbf{a}$ and $\mathbf{b} = O$ whenever $\mathbf{b} < \mathbf{a}$. Spector (XXII 374) constructed a minimal degree which is easily seen to be $\leqq O''$. Sacks applies the same basic method as Spector, but with more circumspection. It is harder to show that Sacks's degree is minimal, but it is $\leqq O'$ at once from its definition. Then $< O'$ follows from minimality, since Kleene and Post showed in XXI 407 that O' is not minimal. The style is compressed, which in this case enhances the understandability of the construction and proof. ROBERT W. ROBINSON

C. SPECTOR. *Inductively defined sets of natural numbers.* ***Infinitistic methods, Proceedings of the Symposium on Foundations of Mathematics, Warsaw, 2–9 September 1959,*** Państwowe Wydawnictwo Naukowe, Warsaw, and Pergamon Press, Oxford-London-New York-Paris, 1961, pp. 97–102.

A set S of natural numbers is said to be *inductively defined* with respect to a predicate Q for which $T_1 \subseteq T_2$ & $Q(x, T_1) \rightarrow Q(x, T_2)$, iff $S = S_c$, where for each ordinal a, $S_a = \{x: \exists b[b < a \;\&\; Q(x, S_b)]\}$, and c is the least ordinal for which $S_c = S_{c+1}$. The author shows that a set inductively defined with respect to a Π^1_1 set is itself Π^1_1 and that the ordinal of the induction is at most ω_1. For hyperarithmetic Q he also classifies the sets S_a according to their level in the hyperarithmetic hierarchy; for example, if Q is Π^0_1 then for all $a < \omega_1$ and $b < \omega$, $S_{\omega a}$ is Σ^0_{2a} and $S_{\omega a+b}$ is Π^0_{2a+1}. The classifications with a few exceptions are optimal in the sense that for certain Q all S_a are complete (universal) sets for their given levels in the hyperarithmetic hierarchy; for example, a Π_1^0 Q can be so chosen that for all $a < \omega_1$, $S_a = V_a$ (the set of Gödel numbers of recursive well-founded relations with ordinals less than a), in which case $S_{\omega a}$ is a complete Σ_{2a}^0 set and $S_{\omega a + b}$ is a complete Π_{2a+1}^0 set (for $a < \omega_1$ and $0 < b < \omega$). An unstated corollary of this result is the fact that V_a is recursive in V_b for $a \leqq b < \omega_1$.

The key tools for the above results are developed, though complete proofs are not given. In this respect, and also in view of Addison's unified approach to hierarchy theory (cf. XXIX 60(3)), it is interesting to note that alternative proofs of several results (in particular the results concerning V_a) can be obtained by "effectivizing" techniques of Lusin and Sierpiński (Lusin, *Sur les classes des constituantes des complémentaires analytiques*, ***Annali della Scuola Normale Superiore di Pisa,*** ser. 2, vol. 2 (1933), pp. 269–282).

The author poses the following open problem. Suppose that a set S is defined as the intersection of all sets satisfying a condition C and that S itself satisfies C; can S be inductively defined in a non-trivial way? STEPHEN J. GARLAND

T. G. McLAUGHLIN. *Some counterexamples in the theory of regressive sets.* ***Zeitschrift für mathematische Logik und Grundlagen der Mathematik,*** vol. 13 (1967), pp. 81–87.

A set α of natural numbers is regressive if there is a listing $\{a_0, a_1, \cdots\}$ of α and a partial recursive function p such that $p(a_0) = a_0$ and for each n, $p(a_{n+1}) = a_n$. (See Dekker XXXI 652(1) and Appel and McLaughlin XXXIII 621.) If $a_n < a_{n+1}$ for all n, then α is retraceable.

The main results of this paper are the following. (1) There exist r.e. sets $W_f \subseteq W_e$ such that the difference $W_e - W_f$ is retraceable, immune, but not hyperimmune. This contrasts with the result that if W is r.e., $\overline{W}$ immune and retraceable, then $\overline{W}$ is hyperimmune. Call a partial function f special if $\text{range}(f) \subseteq \text{domain}(f)$ and $x \in \text{domain}(f) \Rightarrow (\exists k)(f^k(x) = f^{k+1}(x))$. (2) There exist sets β such that β is regressed by a general recursive function but by no special general recursive function. The β can be constructed to be r.e., or to be immune and not hyperimmune. The latter result contrasts with the known result that any set retraceable by a general recursive function is retraceable by a special general recursive function. The author observes that proofs of (2) also show the existence of a recursive set γ such that $\beta \cap \gamma$ is not regressed by any general recursive function. This again contrasts with the familiar result that if β is retraceable by a general recursive function then so is $\beta \cap \gamma$ for any recursive γ.

It seems to the reviewer that the proof of Theorem 3(i) can be made clearer by observing that the definition of *potential preimage* at the beginning is equivalent to the following: Define $\Sigma(-1) = -1$, $\Sigma(0) = 0$, $\Sigma(s+1) = \Sigma_{k \leqq \Sigma(s)} r(k)$. The potential preimage of $(1, 0)$ is $R\hat{\psi}(0) \cup \{(1, 0)\}$, that of $(1, l)$ is $R\hat{\psi}_{(l)}$ for $l > 0$, and that of $\bigcup_{\Sigma(s-1) < k \leqq \Sigma(s)} R\hat{\psi}_{(k)}$ is $\{(1, x) \mid \Sigma(s) < x \leqq \Sigma(s+1)\}$ for $s \geqq 0$. This observation allows minor simplifications in the definitions of h and γ later in the proof. A. B. MANASTER

V. D. VUCKOVIC. *Creative and weakly creative sequences of r.e. sets.* ***Proceedings of the American Mathematical Society,*** vol. 18 (1967), pp. 478–483.

By generalising the notion of doubly weakly creative pair as used by Smullyan (*Theory of formal systems,* XXX 88(1)), the author introduces a notion of weakly creative sequence (of recursively enumerable sets) which is then shown to be equivalent to "creative sequence" as defined by Cleave (*Creative functions,* XXIX 102) and Lachlan (*Standard classes of recursively enumerable sets,* ***Zeitschrift für Mathematische Logik und Grundlagen der Mathematik,*** vol. 10 (1964), pp. 23–42). The paper is lucidly written and follows the Post-Myhill tradition of recursive function theory. J. P. CLEAVE

M. P. SCHÜTZENBERGER. *On the definition of a family of automata.* ***Information and control,*** vol. 4 (1961), pp. 245–270.

M. P. SCHÜTZENBERGER. *Finite counting automata.* Ibid., vol. 5 (1962), pp. 91–107.

M. P. SCHÜTZENBERGER. *Certain elementary families of automata.* ***Proceedings of the Symposium of Mathematical Theory of Automata, New York, N.Y.,*** Microwave Research Symposia series vol. 12, Polytechnic Press of the Polytechnic Institute of Brooklyn, New York 1963, pp. 139–153.

A notion of an automaton which is a generalization of the one-way one-tape finite automata