



On Limits of Sequences of Hyperarithmetical Functionals and Predicates. by Hisao Tanaka; A
Note on the Effective Descriptive Set Theory. by Tosiyyuki Tugué; Hisao Tanaka
Review by: Stephen J. Garland
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a new development in Kleene's theory of recursion in objects of finite type. The details are to appear in four separate papers. Although a detailed review is inappropriate, early attention should be drawn to this development, since (it is hoped) it heralds a renewal of activity in a neglected but important area.

It can best be summarized in the author's own words: "Most of the results of this paper have the following form: a structure B associated with some generalization of recursion theory is given; then an object U of type n is constructed such that the members of B coincide with the objects of type $< n$ that are recursive in U . Since Kleene's definition of relative recursiveness is inductive, it follows that B can be defined by an induction based on U . If enough results of the above form can be found, it may be possible (as Kreisel has suggested) to prove theorems about structures occurring in generalizations of recursion theory by thinking of them as having been built up by inductive definitions based on objects of finite (or higher) type."

A sample result is: If α is a countable admissible ordinal, then there exists an F of type 2 such that $L_\alpha \cap 2^\omega$ is the 1-section of F , and such that for every G of type 2 and of lower degree than F , $L_\alpha \cap 2^\omega$ is not the 1-section of G .

D. A. CLARKE

C. E. M. YATES. *On the degrees of index sets*. *Transactions of the American Mathematical Society*, vol. 121 (1966), pp. 309–328.

This interesting paper gives a precise classification of the index sets of classes of recursively enumerable (r.e.) sets corresponding to many-one degrees and Turing degrees, and then uses these classifications to derive theorems about the r.e. degrees. The paper has its roots in work of Rogers (XXV 363), whose methods are much strengthened in this paper by the use of "infinite-injury" priority arguments. Let $\{R_e\}$ be an indexing of all r.e. sets, and if \mathcal{A} is a class of r.e. sets let $G(\mathcal{A}) = \{e \mid R_e \in \mathcal{A}\}$. The following results are proved: (i) If \mathcal{A} is any many-one degree containing an infinite r.e. set with non-empty complement, then $G(\mathcal{A})$ has highest isomorphism type for sets in Σ_3^0 ; (ii) if b is any r.e. degree, then $G(b)$ has highest isomorphism type for sets Σ_3^0 in b . As lemmas for obtaining the upper bounds, useful representations are given for sets in Σ_3^0 and Σ_4^0 in b . By an ingenious use of the recursion theorem, the author then uses (i) to derive the Sacks density theorem for the r.e. degrees. The exposition of this paper and its sequel (reviewed below) is very lucid (if somewhat terse).

LOUISE HAY

C. E. M. YATES. *On the degrees of index sets. II*. *Ibid.*, vol. 135 (1969), pp. 249–266.

In the notation of the previous review, let $G(\leq a) = \{e \mid R_e \text{ has degree } \leq a\}$, $G(\geq a) = \{e \mid R_e \text{ has degree } \geq a\}$, $G(\mid a) = \{e \mid R_e \text{ has degree incomparable with } a\}$. The author uses the techniques of his previous paper to prove: (i) If a is a recursively enumerable (r.e.) degree $< 0'$, then $G(\leq a)$ has highest isomorphism type for sets Σ_3^0 in a ; (ii) if $0 < a \leq 0'$ then $G(\geq a)$ has highest isomorphism type for sets in Σ_4^0 ; (iii) if $0 < a < 0'$ then $G(\mid a)$ has highest isomorphism type for sets in Π_4^0 . The recursion theorem is then combined with (i) to obtain extensions of several theorems of Sacks on existence of r.e. degrees satisfying given properties. As a final demonstration of the usefulness of index set techniques, the author applies his classifications to show that the r.e. degrees (unlike the r.e. sets) cannot be recursively enumerated without repetition, and he gives necessary and sufficient conditions for a class of r.e. sets of degree $\leq a$ r.e. $< 0'$ to be uniformly recursively enumerable.

LOUISE HAY

HISAO TANAKA. *On limits of sequences of hyperarithmetical functionals and predicates*. *Commentarii mathematici Universitatis Sancti Pauli*, vol. 14 no. 2 (1966), pp. 105–121.

TOSIYUKI TUGUÉ and HISAO TANAKA. *A note on the effective descriptive set theory*. *Ibid.*, vol. 15 no. 1 (1966), pp. 19–28.

Addison (*Bulletin of the American Mathematical Society*, vol. 61 (1955), pp. 75, 171) first observed that the arithmetic, hyperarithmetical, and analytic hierarchies developed by Kleene (XXI 409, 410, and 411) could be regarded as effective versions of the finite Borel, Borel, and projective hierarchies in descriptive set theory, and that results about these hierarchies could be established by "effectivizing" classical proofs. These two papers provide details in support of that observation.

The first paper develops an effective version of the Baire-de la Vallée Poussin hierarchy of

Borel sets and relates it to the Kleene hierarchy of hyperarithmetical sets. Separation theorems for the various levels of the effective Borel hierarchy are established.

The second paper proves that the effective hierarchy introduced in the first does in fact exhaust the Δ_1^1 sets. More importantly, it establishes an effective version of the theorem of Luzin on the projection of the points of the unicity of a Borel set (*Leçons sur les ensembles analytiques et leurs applications*, Gauthier-Villars, Paris 1930, p. 259). Specifically, for any Δ_1^1 relation A there is a recursive relation R such that $\exists! \alpha A(\alpha, \beta)$ is equivalent to $\forall \alpha \exists x R(\alpha, \beta, x)$. The idea behind the proof is surprisingly simple and in fact had been used earlier by Grzegorzczuk, Mostowski, and Ryll-Nardzewski (see 3.3.E in XXVII 80): If there is a unique α such that $A(\alpha, \beta)$, then that α must be hyperarithmetical in β ; but then, by a result of Kleene (XXVII 82(1)), the existence of such an α can be expressed in Π_1^1 fashion, as can the uniqueness of α , so that the theorem is proved. Considering that Luzin's theorem is an immediate corollary of its effective version, this proof is interesting in that it is a rare example of a distinctly non-Addisonian phenomenon: An effective version of the classical proof, while possible, is considerably more complicated than the above recursion-theoretic proof which has no classical analogue at all.

STEPHEN J. GARLAND

A. H. LACHLAN. *On some games which are relevant to the theory of recursively enumerable sets.* *Annals of mathematics*, ser. 2 vol. 91 (1970), pp. 291–310.

The theory of recursively enumerable (r.e.) sets, $T(\mathcal{R})$, is the set of those first-order sentences in the language \mathcal{L} with function symbols, \cap , \cup , $'$, and unary predicates, *finite* and $= \emptyset$, which are true of the r.e. sets. A basic game is one in which the player and opponent alternately place a natural number into one of a countable sequence of sets, W_0, W_2, W_4, \dots for player and W_1, W_3, W_5, \dots for opponent. A requirement is a quantifier-free sentence whose atomic formulas are of the form " β is finite" or " $|\beta| = n$ " where β is a Boolean combination of some of the W_i 's. The player wins if the W_i 's satisfy a given r.e. sequence of requirements when play terminates after ω steps. A sentence of \mathcal{L} is deducible from the game if it is true for any sequence of W_i 's which represent a win. The author notes that any sentence deducible from a game for which there exists an effective winning strategy is in $T(\mathcal{R})$. He then proves that the deducible sentences are closed under logical consequence, and by showing that certain fundamental theorems (e.g. the reduction principle and the existence of a maximal set) are deducible he demonstrates that "every currently known theorem of $T(\mathcal{R})$ is deducible from a game with an effective strategy."

The author shows that for games with a single restricted requirement there is a decision procedure for the existence of an effective winning strategy; also that by restricting requirements to contain only one type of formula, " β is finite" or " $|\beta| = n$," some theorems of $T(\mathcal{R})$ become non-deducible. Finally he describes an alternative game whose single requirement is a first-order sentence with a binary enumeration predicate (e.g. $\{e\}(x) = 0$) and indicates that "any question of elementary recursion theory can be put in the form 'Is σ deducible for all standard enumerations?'"

The author makes a strong argument for such games as a heuristic tool in studying $T(\mathcal{R})$ and shows how games either produced or were easily derived from the proofs of well-known theorems. His general style is informal and discursive, appropriate for the veteran student of $T(\mathcal{R})$. A newcomer would be wise to have at hand the articles by Friedberg, Lachlan, and Robinson that are given as references.

Some errata: Page 297, line 2, read R_{2i} for R_0 , R_0 is satisfied trivially; page 298, equation 8, $W_0 \cap W_2 = \emptyset$ for $W_0 \cup W_2 = \emptyset$; page 298, equation 9, $W_2 \cap W_j \neq \emptyset$ for $W_1 \cap W_j \neq \emptyset$; page 299, line 19, W_j for W . The reviewer never succeeded in finding out the definition of L and the significance of n on page 303.

GREGORY W. JONES

JOHN CLEAVE. *The primitive recursive analysis of ordinary differential equations and the complexity of their solutions.* *Journal of computer and system sciences*, vol. 3 (1969), pp. 447–455.

Beginning with Turing, workers in recursive function theory have sought to apply their notions of computability to real analysis. In this paper Cleave attempts to assess the complexity of the classical Cauchy-Lipschitz construction of solutions for the initial-value problem