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SECOND-ORDER CARDINAL CHARACTERIZABILITY

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Scott [**13**] has pointed out that a close relationship exists between the axiomatizations of set theory and of the theory of types. In this paper we examine the relationship between the model theory of set theory and the model theory of a particular piece of the theory of types, namely pure second-order logic. It will be seen that set-theoretical techniques play a key role in the study of second-order definability, while various considerations in this study raise problems concerning the foundations of set theory.

The axiom of choice is assumed throughout the following discussion, which can be formalized in Gödel-Bernays set theory. Greek letters $\alpha, \beta, \gamma, \delta$ range over ordinals; κ, λ, μ range over cardinals, which are initial ordinals; κ^+ is the cardinal successor of a cardinal κ, while $\kappa + 1$ is its ordinal successor. Since an ordinal α equals the set of its predecessors, the structure $\langle \alpha, \in \cap\, \alpha \times \alpha \rangle$ (hereafter denoted by "$\langle \alpha, \in \rangle$") is a well-order structure of type α. For any sets A and B, ${}^{A}B$ is the set of all functions from A into B, and card(A) is the cardinal equinumerous with A.

1. Second-order languages. We shall consider primarily pure second-order relational languages with identity. It is well known that every second-order sentence is logically equivalent to a *full prenex sentence*, i.e., to a prenex sentence in which all relation quantifiers precede any individual quantifier. Among the quantifier interchange rules which allow economical translations of sentences into full prenex sentences, the following consequence of the axiom of choice is particularly useful: for any formula φ, any individual variable $\boldsymbol{v}$, any n-ary

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relation variable $\boldsymbol{R}$, and any $(n+1)$-ary relation variable $\boldsymbol{S}$, $\bigwedge \boldsymbol{v} \bigvee \boldsymbol{R}\varphi$ is logically equivalent to $\bigvee \boldsymbol{S} \bigwedge \boldsymbol{v}\psi$, where ψ is obtained from φ by replacing each subformula of the form $\boldsymbol{R}\boldsymbol{v}_1 \ldots \boldsymbol{v}_n$ by $\boldsymbol{S}\boldsymbol{v}\boldsymbol{v}_1 \ldots \boldsymbol{v}_n$.

Prenex sentences fall into a natural prefix classification: a full prenex sentence is $\bigvee_n^1$ ("exists-one-n") or $\bigwedge_n^1$ ("all-one-n") iff it begins respectively with an existential or a universal relation quantifier and has n homogeneous blocks of relation quantifiers. By associating with each sentence its class of models we induce a classification of the second-order definable classes of structures. A class of structures is $\Diamond_n^1$ ("diamond-one-n") iff it is both $\bigvee_n^1$ and $\bigwedge_n^1$. Thus we have the following hierarchy of definable classes of structures:

$$
\begin{array}{ccccccccc}
 & & \bigvee_1^1 & & & & \bigvee_2^1 & & \cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
\Diamond_1^1 & & & & \Diamond_2^1 & & & & \Diamond_3^1 \;\cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
 & & \bigwedge_1^1 & & & & \bigwedge_2^1 & & \cdots
\end{array}
$$

We use the $\bigvee$-$\bigwedge$-$\Diamond$ notation to distinguish this hierarchy in pure logic from the analogous *analytical hierarchy*

$$
\begin{array}{ccccccccc}
 & & \Sigma_1^1 & & & & \Sigma_2^1 & & \cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
\Delta_1^1 & & & & \Delta_2^1 & & & & \Delta_3^1 \;\cdots \\
 & \subseteq & & \subseteq & & \subseteq & & \subseteq & \\
 & & \Pi_1^1 & & & & \Pi_2^1 & & \cdots
\end{array}
$$

of sets definable over the structure $\langle \omega, \in \rangle$ in an applied second-order language for number theory, since we feel that the Σ-Π-Δ notation is overworked and since we shall have occasion to refer to both hierarchies.

The influence of the axioms for set theory on the properties of the analytical hierarchy have been studied by several authors (for example, see Addison [**1**] or Moschovakis [**11**]). Their results carry over in a sense to the pure second-order hierarchy for languages which contain a binary relation parameter, since the class of isomorphs of $\langle \omega, \in \rangle$ is $\bigwedge_1^1$ definable. However, these results do not apply to weaker second-order languages, and in particular to the parameter-free language. It is this language which will be our primary object of study.

2. Characterizable cardinals. Since any model of a parameter-free sentence φ is determined up to isomorphism by the cardinality of its universe, instead of discussing the class of models of φ it suffices to discuss the *spectrum* of φ, i.e., the class of all cardinals κ such that the structure $\langle \kappa \rangle$ is a model of φ. In the case that the spectrum of φ contains a single cardinal κ, we say that φ *characterizes* κ.

Which cardinals are characterizable by sentences in the various second-order prefix classes? Using the compactness and Löwenheim-Skolem theorems, it is easy to see that the spectrum of a propositional combination of $\bigvee_1^1$ sentences contains an infinite cardinal iff it contains all infinite cardinals; hence the only cardinals characterizable by propositional combinations of $\bigvee_1^1$ sentences are the finite cardinals. On the other hand, we shall see that many infinite cardinals are $\Diamond_2^1$ characterizable. We note first that the following notions are definable using only one relation quantifier:

(1) for any set A, $\operatorname{card}(A) \geqq \aleph_0$ iff there is an unbounded ordering of A [an $\bigvee_1^1$ condition];

(2) for any sets A, B, $\operatorname{card}(A) = \operatorname{card}(B)$ iff there is a one-one function on A onto B [an $\bigvee_1^1$ condition]; and

(3) for any binary relation R, R is a well-ordering iff R is a linear ordering and

$$\forall X[X \neq \varnothing \Rightarrow \exists x \in X \forall y \in X \langle y, x\rangle \notin R]$$

[an $\bigwedge_1^1$ condition].

LEMMA 2.1. *If $\{\lambda : \lambda < \kappa\}$ is a $\Diamond_2^1$ spectrum, then so are $\{\lambda : \lambda \leqq \kappa\}$ and $\{\kappa\}$.*

PROOF. Note that, for any set A, $\operatorname{card}(A) \leqq \kappa$ iff

$$\exists R[\underbrace{R \text{ well orders } A}_{\bigwedge_1^1 \text{ by (3)}} \ \&\ \forall x(\underbrace{\operatorname{card}(R^{-1}\{x\}) = \operatorname{card}(A)}_{\bigvee_1^1 \text{ by (2)}} \text{ or } \underbrace{\operatorname{card}(R^{-1}\{x\}) < \kappa}_{\Diamond_2^1 \text{ by hypothesis}})]$$

$$\text{iff} \quad \forall B[B \subseteq A \quad \Rightarrow \quad \overbrace{\operatorname{card}(B) = \operatorname{card}(A)} \quad \text{or} \quad \overbrace{\operatorname{card}(B) < \kappa}].$$

Hence $\{\lambda : \lambda \leqq \kappa\}$ is a $\Diamond_2^1$ spectrum, as is $\{\kappa\}$ since it equals $\{\lambda : \lambda \leqq \kappa\} \sim \{\lambda : \lambda < \kappa\}$.

By (1), the set of finite cardinals is an $\bigwedge_1^1$ spectrum. Hence one can show by induction using Lemma 2.1 that, for any $n < \omega$, $\{\lambda : \lambda < \aleph_n\}$ is a $\Diamond_2^1$ spectrum and $\aleph_n$ is $\Diamond_2^1$ characterizable. What about $\aleph_\omega$? It too is $\Diamond_2^1$ characterizable. To see this, let $\Psi(A, R, S)$ be the following $\Diamond_2^1$ condition on a set A, a binary relation R, and a set S:

$$\begin{aligned} &R \text{ well-orders } A \ \&\ \forall x(\operatorname{card}(A) \neq \operatorname{card}(R^{-1}\{x\})) \\ &\qquad \&\ \forall x[x \in S \Leftrightarrow \operatorname{card}(R^{-1}\{x\}) \geqq \aleph_0 \\ &\qquad\qquad \&\ \forall y(\langle y, x\rangle \in R \Rightarrow \operatorname{card}(R^{-1}\{x\}) \neq \operatorname{card}(R^{-1}\{y\}))] \end{aligned}$$

(i.e., R and S "code" the set of infinite cardinals less than $\operatorname{card}(A)$). Then, for any set A, $\operatorname{card}(A) < \aleph_\omega$ iff

(4a) $\exists R \exists S[\Psi(A, R, S) \ \&\ \operatorname{card}(S) < \aleph_0]$ iff

(4b) $\forall R \forall S[\Psi(A, R, S) \Rightarrow \operatorname{card}(S) < \aleph_0]$.

Since (4a) and (4b) are $\bigvee_2^1$ and $\bigwedge_2^1$ conditions respectively, $\aleph_\omega$ is $\Diamond_2^1$ characterizable by Lemma 2.1. Continuing in this manner, one may show that a large number of cardinals are $\Diamond_2^1$ characterizable.

The extent of the class of $\Diamond_2^1$ characterizable cardinals is even more remarkable in view of the following theorem and its corollary.

THEOREM 2.2. *For any second-order spectrum $\mathscr{S}$ which contains no finite cardinal,*
(a) $\{2^\kappa : \kappa \in \mathscr{S}\}$ *is an* $\bigvee_2^1$ *spectrum, and*
(b) $\{(2^\kappa)^+ : \kappa \in \mathscr{S}\}$ *is a* $\diamondsuit_2^1$ *spectrum.*

COROLLARY 2.3. *For any second-order characterizable cardinal* κ, 2^κ *is* $\bigvee_2^1$ *characterizable and* $(2^\kappa)^+$ *is* $\diamondsuit_2^1$ *characterizable.*

Part (a) of the theorem was shown by Zykov [**17**]. We sketch here a proof of both (a) and (b). Let n be a positive integer. For any $(n+1)$-ary relation C and any x, let $C_x = \{\langle x_1, \dots, x_n\rangle : \langle x, x_1, \dots, x_n\rangle \in C\}$. We say that C codes a set of n-ary relations over a set B iff

$$\forall x[C_x \subseteq B \,\&\, \forall y(y \neq x \,\&\, C_x \neq \varnothing \Rightarrow C_x \neq C_y)].$$

This is clearly a first-order condition, and the condition that C codes all n-ary relations over B is an $\bigwedge_1^1$ condition. Note that if C codes all n-ary relations over a set of cardinality κ, then the domain of C has cardinality $2^{\kappa^n} - 1$, which equals 2^κ if κ is infinite.

For part (a), suppose that φ is a second-order sentence whose spectrum $\mathscr{S}$ contains only infinite cardinals. Let l be the maximum of the ranks of relation variables occurring in φ. Then a set A has cardinality 2^κ for some κ in $\mathscr{S}$ iff there is a subset B of A and, for each $n \leqq l$, an $(n+1)$-ary relation C_n such that

(5) the domain of C_l has cardinality card(A),

(6) for all $n \leqq l$, C_n codes all n-ary relations over B, and

(7) φ is true when its individual variables are interpreted as ranging over B and its n-ary relation variables are interpreted as ranging over the relations coded by C_n.

Since (5) is an $\bigvee_1^1$ condition, (6) is $\bigwedge_1^1$, and (7) is first-order (being just a relativization of φ), $\{2^\kappa : \kappa \in \mathscr{S}\}$ is an $\bigvee_2^1$ spectrum.

Note that one cannot show in a similar fashion that $\{2^\kappa : \kappa \in \mathscr{S}\}$ is the $\bigwedge_2^1$ spectrum $\mathscr{T}$ which contains the cardinal of a set A iff for any relations $B, C_1, \dots, C_l$ over A, conditions (5) and (6) imply condition (7). Two things go wrong with such an attempt. In the first place, $\mathscr{T}$ contains all cardinals not of the form 2^κ for some cardinal κ. This defect could be remedied if $\{2^\kappa : \kappa \geqq 0\}$ were an $\bigwedge_2^1$ spectrum, but in general it is not, as will be seen below. Under certain strong assumptions, such as the generalized continuum hypothesis, this class is an $\bigwedge_2^1$ spectrum (e.g., it is the class of successor cardinals) and so is $\{2^\kappa : \kappa \in \mathscr{S}\}$. In the second place, $\mathscr{T}$ will omit a cardinal 2^κ for some κ in $\mathscr{S}$ if $2^\kappa = 2^\lambda$ for some other cardinal λ not in $\mathscr{S}$. This defect can be remedied, however, as we shall do below.

The difficulties encountered in attempting to show that $\mathscr{T} = \{2^\kappa : \kappa \in \mathscr{S}\}$ suggest a proof of part (b). Since, for any sets A and B,

(8) card$(A) < (2^{\operatorname{card}(B)})^+$ iff card$(A) \leqq 2^{\operatorname{card}(B)}$ iff there is a binary relation C with domain a subset of A which codes a set of unary relations over B [an $\bigvee_1^1$ condition],

the condition that card$(A) = (2^{\operatorname{card}(B)})^+$ is a $\diamondsuit_2^1$ condition by the methods in the proof of Lemma 2.1. Now, for any φ, $\mathscr{S}$, and l as above, a set A has cardinality

$(2^\kappa)^+$ for some κ in $\mathscr{S}$ iff there are relations $B, C_1, \ldots, C_l$ over A such that $\mathrm{card}(A) = (2^{\mathrm{card}(B)})^+$ and conditions (6) and (7) hold [an $\bigvee_2^1$ condition] iff for any well-ordering R of A there is an x in A such that $\mathrm{card}(A) = (2^{\mathrm{card}(R^{-1}\{x\})})^+$ and for any relations $B, C_1, \ldots, C_l$ over A, if $B = R^{-1}\{x\}$ and condition (6) holds, then (7) holds also [an $\bigwedge_2^1$ condition]. Hence $\{(2^\kappa)^+ : \kappa \in \mathscr{S}\}$ is a $\diamondsuit_2^1$ spectrum.

The hypothesis that $\mathscr{S}$ contain only infinite cardinals can be eliminated at the expense of replacing "2^κ" by "2^{κ^l}" in the conclusion, where l is any sufficiently large integer. We have not done so here in order to keep the notation simple and since this generalization is not needed for the corollary.

As noted above, under the assumption of the generalized continuum hypothesis, Corollary 2.3 can be strengthened to conclude that 2^κ is $\diamondsuit_2^1$ characterizable. This strengthening is not possible in general, since Kunen [**8**] has shown that it is consistent relative to Zermelo-Fraenkel set theory that $2^{\aleph_0}$ is not $\bigwedge_2^1$ characterizable. Hence, by our remarks above, it is also consistent relative to Zermelo-Fraenkel set theory that $\{2^\kappa : \kappa \geqq 0\}$ is not an $\bigwedge_2^1$ spectrum. Kunen's result is not surprising if one considers the fact that the "natural" characterizations of $2^{\aleph_0}$ as, for example, the cardinality of the power set of ω or the cardinality of a least-upper-bound complete linear ordering with a countable dense subset are all $\bigvee_2^1$ characterizations.

Several questions arise as a result of these considerations. We have noticed that it is consistent that there is a cardinal (e.g., $2^{\aleph_0}$) which is $\bigvee_2^1$ but not $\bigwedge_2^1$ characterizable. In § 3 we show that such a cardinal always exists, and that in fact there is a genuine prefix hierarchy of second-order characterizable cardinals. On the other hand, the general problem of which cardinals are second-order characterizable is in a sense reducible to the problem of which cardinals are $\diamondsuit_2^1$ characterizable by Corollary 2.3. Furthermore, if one makes the assumption that $2^{\aleph_0}$ possesses no "unnatural" characterizations, i.e., that $2^{\aleph_0}$ is not $\bigwedge_2^1$ characterizable, then our preliminary results about $\diamondsuit_2^1$ characterizability indicate that $2^{\aleph_0}$ is greater than $\aleph_\omega$. Hence we are led to a study of the $\diamondsuit_2^1$ characterizable cardinals in an attempt to gain further insight into both the class of second-order characterizable cardinals and the size of the continuum.

What is the extent of the class of $\diamondsuit_2^1$ characterizable cardinals? In the first place, there are only countably many such cardinals, so that the class has an upper bound. Also, if we replace "$\mathrm{card}(S) < \aleph_0$" by "$\mathrm{card}(S) < \aleph_1$" in (4a) and (4b), then we see that $\aleph_\Omega$ is $\diamondsuit_2^1$ characterizable, so that the class is not an initial segment of the sequence of cardinals. This situation leads us to ask the following two questions:

(I) What is the smallest cardinal which is not $\diamondsuit_2^1$ characterizable?

(II) What is the supremum of the class of $\diamondsuit_2^1$ characterizable cardinals?

In order to investigate question (I), we digress in § 4 to discuss $\diamondsuit_2^1$ *characterizable ordinals*, i.e., ordinals α such that the class of isomorphs of $\langle \alpha, \in \rangle$ is a $\diamondsuit_2^1$ class. The importance of these ordinals can be seen as follows: if in (4a) and (4b) we replace "$\mathrm{card}(S) < \aleph_0$" by

(9) $R \cap S \times S$ has order type α

then we can show that $\aleph_\alpha$ is $\Diamond^1_2$ characterizable for any $\Diamond^1_2$ characterizable ordinal α.

In § 5 we apply the results of § 4 to study the characterizability of cardinals in various $\Diamond^1_2$ spectra. Finally, in § 6, we examine the relationship between the extent of the class of $\Diamond^1_2$ characterizable cardinals and the model theory of set theory.

3. A hierarchy of characterizable cardinals. Standard universal set arguments show that the analytical hierarchy is indeed a true hierarchy, i.e., that for all $n > 0$ there are Σ^1_n sets which are not Π^1_n and Δ^1_n sets which are neither Σ^1_n nor Π^1_n (cf. [**14**, § 7.8]). These results also establish a hierarchy theorem for pure languages which contain, say, one individual and one binary relation parameter. In this section we strengthen this result to establish a hierarchy of second-order characterizable cardinals, employing in our proof a method which is similar to the use of universal set arguments in number theory.

We show first, by way of illustration, that for any $n > 0$ there is a $\Diamond^1_{n+1}$ characterizable cardinal which is not $\bigvee^1_n$ characterizable. Call a cardinal κ *weakly* $\bigvee^1_n$ *characterizable* iff κ is the smallest cardinal in some $\bigvee^1_n$ spectrum, and let κ_n be the smallest infinite cardinal which is not weakly $\bigvee^1_n$ characterizable. Then κ_n is obviously not $\bigvee^1_n$ characterizable, so it remains to show that κ_n is $\Diamond^1_{n+1}$ characterizable.

For any cardinal κ and any sentence φ, define the *κ-spectrum* of φ to be the set of all cardinals less than κ in the spectrum of φ.

LEMMA 3.1. *For any $n > 1$ there is an $\bigvee^1_n$ sentence ψ_n involving a binary relation parameter $\boldsymbol{<}$ and individual parameters $\boldsymbol{\lambda}$, $\boldsymbol{m}$ such that, for any infinite cardinal κ and any set S of cardinals, S is the κ^+-spectrum of an $\bigvee^1_n$ sentence iff there is an m in ω such that*

$$S = \{\lambda : \langle \kappa + 1, \in, \lambda, m\rangle \models \psi_n\}.$$

PROOF. The proof proceeds by an evaluation of the complexity of the truth definition for $\bigvee^1_n$ sentences. Assume that these sentences are in a language which has the connectives $\sim$ and $\wedge$, the quantifier $\bigvee$, the identity symbol $\approx$, the individual variables $\boldsymbol{v}_i$ for i in ω, and the j-ary relation variables $\boldsymbol{R}_{i,j}$ for i, j in ω. Let $\langle \varphi^0_m : m \in \omega\rangle$ be a recursive enumeration of all first-order formulas in this language and, for each $n > 0$, let $\langle \varphi^n_m : m \in \omega\rangle$ be a recursive enumeration of all prenex $\bigvee^1_n$ formulas which contain no free individual variables.

We define first an $\bigwedge^1_1$ class $\mathscr{C}_0$ of structures such that for any infinite cardinal κ and any cardinal $\lambda \leqq \kappa$, $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle \in \mathscr{C}_0$ if and only if

(1) $w = \omega$, i.e., w is the $\in$-least element of $\kappa + 1$ with an $\in$-predecessor but no immediate $\in$-predecessor [a first-order condition],

(2) $+$ and $\cdot$ are the relations of addition and multiplication on ω [a first-order condition since $+, \cdot$ are implicitly definable over $\langle \omega, \in\rangle$],

(3) γ is the maximum of λ and w [a first-order condition],

(4) B is a subset of $\gamma \times \omega \times \lambda$ which codes all finite sequences of elements of λ,

i.e., if B_α is $\{\langle m, \beta\rangle : \langle \alpha, m, \beta\rangle \in B\}$ for any α in γ, then $\bigcup\{{}^n\lambda : n \in \omega\} = \{B_\alpha : \alpha \in \gamma\}$ [an $\bigwedge_1^1$ condition],

(5) R is a subset of $\omega \times \omega \times \gamma \times \omega \times \lambda$ which codes fixed relations $R_{i,j} \subseteq {}^j\lambda$ for any i, j in ω, where $R_{i,j} = \{\langle \alpha_0, \ldots, \alpha_{j-1}\rangle : \exists\beta < \gamma\forall k < j\langle i, j, \beta, k, \alpha_k\rangle \in R\}$ [a first-order condition], and

(6) T is a subset of $\omega \times \gamma$ such that, for any $m < \omega$ and any $\alpha < \gamma$, $\langle m, \alpha\rangle \in T$ iff φ_m^0 is satisfied in $\langle\lambda\rangle$ when $\boldsymbol{v}_i$ is interpreted as $B_\alpha(i)$ and $\boldsymbol{R}_{i,j}$ is interpreted as $R_{i,j}$, i.e., iff the domain of B_α includes every integer i such that $\boldsymbol{v}_i$ has a free occurrence in φ_m^0 and

(a) φ_m^0 is $\boldsymbol{v}_i \approx \boldsymbol{v}_j$ and $B_\alpha(i) = B_\alpha(j)$, or

(b) φ_m^0 is $\boldsymbol{R}_{i,j}\boldsymbol{s}$ for some sequence $\boldsymbol{s}$ of j variables $\boldsymbol{v}_{s(0)}, \ldots, \boldsymbol{v}_{s(j-1)}$ and $\langle B_\alpha(s(0)), \ldots, B_\alpha(s(j-1))\rangle \in R_{i,j}$, or

(c) φ_m^0 is $\sim\varphi_{m'}^0$ and $\langle m', \alpha\rangle \notin T$, or

(d) φ_m^0 is $\varphi_{m'}^0 \wedge \varphi_{m''}^0$ and $\langle m', \alpha\rangle$, $\langle m'', \alpha\rangle \in T$, or

(e) φ_m^0 is $\bigvee\boldsymbol{v}_i\varphi_{m'}^0$ and there is a β in γ such that $B_\alpha(j) = B_\beta(j)$ for all $j \neq i$, i is in the domain of B_β, and $\langle m', \beta\rangle \in T$ [a first-order condition since the recursive enumeration of the first-order formulas is definable over $\langle\omega, +, \cdot\rangle$].

Now for each $n > 0$ we define a $\diamondsuit_{n+1}^1$ class $\mathscr{C}_n$ such that for any λ, κ as above, $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle$ is in $\mathscr{C}_n$ if and only if conditions (1) through (5) hold and

(7) T is a subset of $\omega \times \gamma$ such that, for any $m < \omega$ and any $\alpha < \gamma$, $\langle m, \alpha\rangle \in T$ iff φ_m^n is satisfied in $\langle\lambda\rangle$ when $\boldsymbol{R}_{i,j}$ is interpreted as $R_{i,j}$, i.e., iff φ_m^n is $\bigvee\boldsymbol{R}_{i_1,j_1}\cdots\bigvee\boldsymbol{R}_{i_k,j_k} \sim \varphi_{m'}^{n-1}$ and there are R', T' such that $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R', T'\rangle \in \mathscr{C}_{n-1}$, $\langle m', \alpha\rangle \notin T'$, and $R_{i,j} = R'_{i,j}$ whenever $\langle i, j\rangle \notin \{\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle\}$.

Finally, for any $n > 1$, let ψ_n be an $\bigvee_n^1$ sentence such that, for any κ, λ as above and any m in ω, $\langle \kappa + 1, \in, \lambda, m\rangle$ is a model of ψ_n iff there are α, w, γ in $\kappa + 1$ and relations $+, \cdot, B, R, T$ such that

$\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle \in \mathscr{C}_{n-1}$,

φ_m^n is a sentence $\bigvee\boldsymbol{R}_{i_1,j_1}\ldots\bigvee\boldsymbol{R}_{i_k,j_k} \sim \varphi_{m'}^{n-1}$, and

$\langle m', \alpha\rangle \notin T$.

Using Lemma 3.1 we can construct a $\diamondsuit_{n+1}^1$ class $\mathscr{C}$ such that, for any cardinal κ, $\langle \kappa + 1, \in\rangle \in \mathscr{C}$ iff $\kappa = \kappa_n$, i.e., iff $\kappa \geqq \aleph_0$,

$$\forall m < \omega[\underbrace{\langle \kappa + 1, \in, \kappa, m\rangle \models \sim\psi_n}_{\bigwedge_n^1} \text{ or } \underbrace{\exists\lambda < \kappa\langle \kappa + 1, \in, \lambda, m\rangle \models \psi_n}_{\bigvee_n^1}] \tag{8}$$

and

$$\forall\lambda < \kappa\exists m < \omega[\underbrace{\langle \kappa + 1, \in, \lambda, m\rangle \models \psi_n}_{\bigvee_n^1} \& \underbrace{\neg\exists\mu < \lambda\langle \kappa + 1, \in, \mu, m\rangle \models \psi_n}_{\bigwedge_n^1}]. \tag{9}$$

Furthermore, the class $\mathscr{D}$ of all structures $\langle A, R\rangle$ isomorphic to $\langle \text{card}(A) + 1, \in\rangle$ is a $\diamondsuit_2^1$ class since $\langle A, R\rangle \in \mathscr{D}$ iff

(10) A is infinite, R well-orders A with a last element y, and, for any $x \neq y$ in A, $\text{card}(A) \neq \text{card}(R^{-1}\{x\})$.

Thus κ_n is $\Diamond^1_{n+1}$ characterizable as the cardinality of any set A such that

(11) for some R, $\langle A, R\rangle \in \mathscr{D} \cap \mathscr{C}$ [an $\bigvee^1_n$ condition] or

(12) A is infinite and, for any R, if $\langle A, R\rangle \in \mathscr{D}$ then $\langle A, R\rangle \in \mathscr{C}$ [an $\bigwedge^1_n$ condition].

Hence we have shown that there is a $\Diamond^1_{n+1}$ characterizable cardinal which is not $\bigvee^1_n$ characterizable.

The above construction can be modified to show that there is an $\bigwedge^1_n$ characterizable cardinal which is not $\bigvee^1_n$ characterizable. In order to do this, we first strengthen Lemma 3.1 by using the techniques in the proof of Theorem 2.2.

LEMMA 3.2. *For any $n > 1$ there is an $\bigvee^1_2$ sentence ψ'_n and an $\bigwedge^1_2$ sentence ψ''_n such that, for any infinite cardinal κ and any λ, if $2^\lambda \leqq \kappa$ then, for any m in ω, $\langle \kappa + 1, \in, \lambda, m\rangle$ is a model of ψ_n iff it is a model of ψ'_n iff it is a model of ψ''_n.*

PROOF. In a manner similar to that used in the proof of Lemma 3.1, we define an $\bigwedge^1_1$ class $\mathscr{C}_0$ of structures such that, for any infinite cardinal κ and any λ such that $2^\lambda \leqq \kappa$, $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle$ is in $\mathscr{C}_0$ iff conditions (1) through (4) hold and

(13) R is a subset of $\kappa \times \omega \times \omega \times \gamma \times \omega \times \lambda$ which codes all ω-sequences of finitary relations over λ (such an R exists since $2^\lambda \leqq \kappa$); i.e., if

$$R_{\beta,i,j} = \{\langle \alpha_0, \ldots, \alpha_{j-1}\rangle : \exists\delta < \gamma \forall k < j \langle \beta, i, j, \delta, k, \alpha_k\rangle \in R\},$$

then for any R' as in (5) there is a β in κ such that, for all i, j in ω, $R'_{i,j} = R_{\beta,i,j}$ [an $\bigwedge^1_1$ condition]; and

(14) T is a subset of $\omega \times \kappa \times \omega \times \gamma$ such that, for any m, n in ω, any $\beta < \kappa$, and any $\alpha < \gamma$, $\langle n, \beta, m, \alpha\rangle \in T$ iff

(a) $n = 0$ and φ^0_m is satisfied in $\langle \lambda\rangle$ when $\boldsymbol{v}_i$ is interpreted as $B_\alpha(i)$ and $\boldsymbol{R}_{i,j}$ is interpreted as $R_{\beta,i,j}$ [a first-order condition as in (6)], or

(b) $n > 0$ and φ^n_m is satisfied in $\langle \lambda\rangle$ when $\boldsymbol{R}_{i,j}$ is interpreted as $R_{\beta,i,j}$, i.e., φ^n_m is $\bigvee \boldsymbol{R}_{i_1,j_1} \ldots \bigvee \boldsymbol{R}_{i_k,j_k} \sim \varphi^{n-1}_{m'}$ and there is a β' in κ such that $\langle n-1, \beta', m', \alpha\rangle \notin T$ and $R_{\beta',i,j} = R_{\beta,i,j}$ whenever $\langle i, j\rangle \notin \{\langle i_1, j_1\rangle, \ldots, \langle i_k, j_k\rangle\}$ [a first-order condition].

Now for any $n > 1$, let ψ'_n be an $\bigvee^1_2$ sentence and ψ''_n be an $\bigwedge^1_2$ sentence such that, for any κ, λ as above and any m in ω, $\langle \kappa + 1, \in, \lambda, m\rangle \models \psi'_n$ iff there are $\alpha, \beta, w, \gamma, +, \cdot, B, R, T$ such that $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle$ is in $\mathscr{C}_0$ and $\langle n, \beta, m, \alpha\rangle \in T$, and $\langle \kappa + 1, \in, \lambda, m\rangle \models \psi''_n$ iff, for any $\alpha, \beta, w, \gamma, +, \cdot, B, R, T$, if $\langle \kappa + 1, \in, \lambda, w, +, \cdot, \gamma, B, R, T\rangle \in \mathscr{C}_0$, then $\langle n, \beta, m, \alpha\rangle \in T$

In order to apply Lemma 3.2, let

$$\mathscr{S} = \{\kappa : \forall\lambda < \kappa (2^\lambda < \kappa)\}$$

and define κ'_n to be the smallest member of $\mathscr{S}$ which is not weakly $\bigvee^1_n$ characterizable. By § 2(8), $\mathscr{S}$ is a $\Diamond^1_2$ spectrum since for any set A, $\operatorname{card}(A) \in \mathscr{S}$ iff

(15) for any $B \subseteq A$, if $\operatorname{card}(B) \neq \operatorname{card}(A)$, then $2^{\operatorname{card}(B)} < \operatorname{card}(A)$ iff

(16) there is a well-ordering R of A such that for any x in A, if $\operatorname{card}(R^{-1}\{x\}) \neq \operatorname{card}(A)$, then $2^{\operatorname{card}(R^{-1}\{x\})} < \operatorname{card}(A)$.

Using Lemma 3.2 and the fact that $\mathscr{S}$ is a $\Diamond^1_2$ spectrum, we can replace the $\bigvee^1_n$ clauses in (8) and (9) by $\Diamond^1_2$ clauses to obtain an $\bigwedge^1_n$ class $\mathscr{C}'$ such that, for any cardinal κ, $\langle \kappa + 1, \in \rangle \in \mathscr{C}'$ iff $\kappa = \kappa_n'$. Hence, by (12), κ_n' is $\bigwedge_n^1$ characterizable.

By similar modifications of (8) and (9) using "universal" $\bigwedge_n^1$ sentences in addition to the "universal" $\bigvee_n^1$ sentences ψ_n, we obtain the following hierarchy theorem.

THEOREM 3.3. *For any $n > 1$ there is a $\Diamond_n^1$ characterizable cardinal which is neither $\bigvee_{n-1}^1$ nor $\bigwedge_{n-1}^1$ characterizable, an $\bigvee_n^1$ characterizable cardinal which is not $\bigwedge_n^1$ characterizable, and an $\bigwedge_n^1$ characterizable cardinal which is not $\bigvee_n^1$ characterizable.*

In conclusion, we mention an open problem concerning the notion of a κ-spectrum. It follows immediately from Theorem 3.3 that for any $n > 1$ there is a cardinal κ such that there is an $\bigvee_n^1$ κ-spectrum which is not an $\bigwedge_n^1$ κ-spectrum. What is the least such cardinal κ? In particular, is the least such κ always ω? Asser [**2**] raised this "spectrum problem" for $n = 1$ and $\kappa = \omega$; Bennett [**3**] considered it for arbitrary n, but could not show that there was any second-order ω-spectrum which was not already an $\bigvee_1^1$ ω-spectrum. Both authors were investigating ω-spectra in response to the problem raised by Scholz [**12**] of characterizing the class of $\bigvee_1^1$ ω-spectra.

4. Characterizable ordinals. As in our study of characterizable cardinals, we shall be interested in determining both the smallest ordinal not characterizable in a given manner and also the supremum of the set of ordinals characterizable in that manner. The first problem leads to a study of the characterizability of countable ordinals, while the second turns out to be equivalent to its version for cardinal characterizability (cf. Theorem 4.1).

For any linear ordering R, let $|R|$ be the order type of R, and for any x in the field of R, let $R \upharpoonright x$ be the initial segment of R determined by x, i.e., $\{\langle y, z\rangle : \langle y, z\rangle \in R \,\&\, \langle z, x\rangle \in R\}$. Note that the following conditions are $\bigvee_1^1$ definable:

(1) for any linear orderings R, S, $|R| = |S|$ iff there is an order-preserving map of the field of R onto the field of S, and

(2) for any linear ordering R and any well-ordering S, $|R| < |S|$ iff $|R| = |S \upharpoonright x|$ for some x in the field of S.

THEOREM 4.1. *For any $n > 1$ and any cardinal κ, κ is $\bigvee_n^1$ (or $\bigwedge_n^1$) characterizable iff the order structure $\langle \kappa, \in\rangle$ is $\bigvee_n^1$ (or $\bigwedge_n^1$) characterizable.*

PROOF. If κ is characterizable, then $\langle \kappa, \in\rangle$ is characterizable as that well-order structure $\langle A, R\rangle$ such that $\operatorname{card}(A) = \kappa$ and for any x, $\operatorname{card}(A) \neq \operatorname{card}(R^{-1}\{x\})$. Conversely, if $\langle \kappa, \in\rangle$ is $\bigvee_n^1$ characterizable, then κ is obviously $\bigvee_n^1$ characterizable; if $\langle \kappa, \in\rangle$ is $\bigwedge_n^1$ characterizable, then κ is $\bigwedge_n^1$ characterizable as the cardinality of a set A such that, for any well-ordering R of A, if $\operatorname{card}(A) \neq \operatorname{card}(R^{-1}\{x\})$ for any x in A, then $|R| = \kappa$.

By the compactness theorem, the only $\bigvee_1^1$ characterizable ordinals are the finite ordinals. Hence at least an $\bigwedge_1^1$ sentence is needed to characterize an infinite ordinal, and such a sentence characterizes $\langle \omega, \in\rangle$ (cf. § 2(3) and § 3(1)). Though we shall be interested primarily in the $\Diamond_2^1$ characterizability of countable ordinals, we shall return to the $\bigwedge_1^1$ characterizable ordinals later.

The characterizability of countable ordinals is closely related to the definability of sets of well-orderings of ω in second-order number theory, as the following theorem shows. For any countably infinite ordinal α, let WO_α be the set of all well-orderings of ω with order type α.

THEOREM 4.2. *For any $n > 1$ and any countably infinite ordinal α, α is $\bigvee_n^1$ (or $\bigwedge_n^1$) characterizable iff WO_α is a Σ_n^1 (or a Π_n^1) set.*

PROOF. If α is $\bigvee_n^1$ or $\bigwedge_n^1$ characterizable by a sentence φ, then φ also defines WO_α as a Σ_n^1 or a Π_n^1 set. Conversely, if WO_α is Σ_n^1, then $\langle \alpha, \in \rangle$ is $\bigvee_n^1$ characterizable as that well-order structure $\langle A, R \rangle$ such that there exist an x in A and relations $+, \cdot, S$ on A such that

(3) $|R \upharpoonright x| = \omega$ [a first-order condition by § 3(1)],

(4) $+$ and $\cdot$ are the relations of addition and multiplication on $R^{-1}\{x\}$ [a first-order condition by § 3(2)],

(5) S is an ordering of $R^{-1}\{x\}$ with $|S| = |R|$ [an $\bigvee_1^1$ condition], and

(6) S is "in" WO_α [an $\bigvee_n^1$ condition since WO_α is Σ_n^1].

Also, if WO_α is Π_n^1, then $\langle \alpha, \in \rangle$ is $\bigwedge_n^1$ characterizable as that well-order structure $\langle A, R \rangle$ such that $\mathrm{card}(A) = \aleph_0$ [an $\bigwedge_2^1$ condition] and, for any $x, +, \cdot, S$ satisfying conditions (3) through (5), condition (6) holds.

Later we shall extend Theorem 4.2 to $\bigwedge_1^1$ characterizability, but first we derive some of its consequences. For any $n > 0$ let δ_n be the least ordinal which is not the order type of any Δ_n^1 well-ordering of ω, and let $\delta_{(\omega)} = \sup \{\delta_n : n \in \omega\}$.

THEOREM 4.3. *For any $n > 0$ and any infinite ordinal α, if $\alpha < \delta_n$ then WO_α is a Δ_n^1 set.*

PROOF. Let S be a Δ_n^1 well-ordering of ω with order type α. Then

$$\begin{aligned} WO_\alpha &= \{R : \exists T (T = S \;\&\; |T| = |R|)\} \\ &= \{R : \forall T (T = S \Rightarrow |T| = |R|)\}, \end{aligned}$$

so that WO_α is a Δ_n^1 set.

The converse of Theorem 4.3 is true for $n = 1$ since WO_α is not a Σ_1^1 set for any $\alpha \geqq \delta_1$ (otherwise, by standard arguments, every Π_1^1 set would also be Σ_1^1; cf. [**14**, p. 184]); it is also true for $n \geqq 2$ whenever the set of Δ_n^1 relations is a basis for the set of all Σ_n^1 sets of relations.

COROLLARY 4.4. *For any countable ordinal α, α is $\diamondsuit_2^1$ characterizable iff α is $\bigvee_2^1$ characterizable iff $\alpha < \delta_2$.*

PROOF. If WO_α is a Σ_2^1 set, then it contains a Δ_2^1 well-ordering by the basis theorem for Σ_2^1 (cf. [**14**, p. 190]). Hence the corollary follows from Theorems 4.2 and 4.3.

Having determined which countable ordinals are $\bigvee_1^1$ and $\bigvee_2^1$ characterizable, we now turn to $\bigwedge_1^1$ and $\bigwedge_2^1$ characterizability. We shall show that the $\bigwedge_1^1$ characterizable ordinals are cofinal with the $\bigvee_2^1$ characterizable ordinals and that δ_2 is $\bigwedge_2^1$ characterizable. In order to do this, we shall use several facts relating Δ_2^1

definability to implicit Π_1^1 definability. For any $n > 0$, call a subset A of ω a Π_n^1 (or a Σ_n^1) *singleton* iff $\{A\}$ is a Π_n^1 (or a Σ_n^1) set. It is easy to see that A is a Σ_n^1 singleton iff A is a Δ_n^1 set (cf. [**14**, p. 188] where the proof of a weaker result actually establishes this fact). Suzuki [**16**] showed that a subset of ω is Δ_2^1 iff it is hyperarithmetical (i.e., Δ_1^1-definable) in a Π_1^1 singleton.

Indices can be assigned to Π_1^1 singletons by applying the Novikov-Kondô-Addison uniformization theorem [**14**, p. 188] to a universal Π_1^1 set P_1^1 which enumerates all Π_1^1 sets of subsets of ω [**14**, p. 175] to obtain a Π_1^1 set P such that a subset A of ω is a Π_1^1 singleton iff, for some e in ω, $\langle e, A\rangle \in P$. We note in passing that the set of Π_1^1 singletons is thereby a Π_1^1 set and that, by the basis theorem for Σ_2^1, there is a Δ_2^1 set which is not a Π_1^1 singleton; hence we obtain an alternate proof of Suzuki's [**16**], Corollary 2. More to the point, the set $\{e : \exists A\langle e, A\rangle \in P\}$ of indices of Π_1^1 singletons is a Σ_2^1 set. To each index e in this set I we assign an ordinal $|e|$ as follows. Since P is Π_1^1, there is a recursive set Q such that for any e, A the set

$$Q_{e,A} = \{\langle m, n\rangle : \langle e, A, m, n\rangle \in Q\}$$

is a linear ordering and is furthermore a well-ordering iff $\langle e, A\rangle \in P$ (cf. Kleene [**6**, XXII]). For any e in I, let $|e|$ equal $|Q_{e,A}|$ for the unique A such that $\langle e, A\rangle \in P$.

LEMMA 4.5. *$\{|e| : e \in I\}$ is cofinal with δ_2.*

PROOF. For any e in I, the unique well-ordering R such that $R = Q_{e,A}$ for some A has order type $|e|$ and, being a Σ_2^1 singleton, is a Δ_2^1 well-ordering; hence $|e| < \delta_2$. Conversely, suppose that $\{|e| : e \in I\}$ is bounded by $|R|$ for some Δ_2^1 well-ordering R. Then, for any Π_1^1 singleton A and any index e of A,

$$\{A\} = \{B : |Q_{e,B}| < |R|\},$$

so that A is a Σ_1^1 singleton relative to R and hence hyperarithmetical in R. Now the hyperjump of R is also a Δ_2^1 set, and so by Suzuki's theorem it must be hyperarithmetical in some Π_1^1 singleton and thereby hyperarithmetical in R, which is impossible. Hence no such R exists.

We now extend Theorem 4.2 to $\bigwedge_1^1$ characterizability.

THEOREM 4.6. *For any countably infinite ordinal α, α is $\bigwedge_1^1$ characterizable iff WO_α is a Π_1^1 set.*

PROOF. Necessity is proved as in Theorem 4.2. Sufficiency is also proved in the same manner once we replace the $\bigwedge_2^1$ condition on $\langle A, R\rangle$ that $\operatorname{card}(A) = \aleph_0$ by membership of $\langle A, R\rangle$ in an $\bigwedge_1^1$ class of countable well-order structures containing $\langle \alpha, \in\rangle$. Since WO_α is Π_1^1, α is less than δ_2 by the basis theorem for Σ_2^1. By the lemma, there is an e in I such that $\alpha < |e|$. Then an infinite well-order structure $\langle A, R\rangle$ has order type less than $|e|$ iff there do not exist any $x, +, \cdot, S, B$ satisfying (3) and (4) such that $|Q_{e,B}| \leqq |S| \leqq |R|$, so that the class of all such structures is an $\bigwedge_1^1$ class as desired.

In order to apply Theorem 4.6 to describe the class of $\bigwedge_1^1$ characterizable ordinals, we first prove several lemmas.

LEMMA 4.7. *For any e in I, $WO_{|e|}$ is a Π_1^1 set.*

PROOF. For any e in I and any R, R is in $WO_{|e|}$ iff R is a well-ordering and there exists an A such that $|R| = |Q_{e,A}|$. But there exists an A such that $|R| = |Q_{e,A}|$ iff there exists a unique such A iff there exists such an A with $\{A\}$ Σ_1^1 in R iff there exists an A hyperarithmetical in R [a Π_1^1 condition by the methods of Kleene [7]] such that $Q_{e,A}$ is a well-ordering and neither $|R| < |Q_{e,A}|$ nor $|Q_{e,A}| < |R|$ [a Π_1^1 condition by (2)]. Hence $WO_{|e|}$ is a Π_1^1 set.

For any set W of countably infinite ordinals and any $n > 0$, call W a Π_n^1 (or a Σ_n^1) set of ordinals iff $\{R : |R| \in W\}$ is a Π_n^1 (or a Σ_n^1) set.

LEMMA 4.8. *$\{|e| : e \in I\}$ is a Σ_2^1 set of ordinals.*

PROOF. Let $W = \{|e| : e \in I\}$. Then $\{R : |R| \in W\}$ equals $\{R : \exists e \exists A\, (R$ is a well-ordering and $|R| = |Q_{e,A}|)\}$.

LEMMA 4.9. *$\{\alpha : \omega \leqq \alpha < \delta_2$ & WO_α is not $\Pi_1^1\}$ is a Σ_2^1 set of ordinals cofinal with δ_2.*

PROOF. Let R_0 be any Δ_2^1 well-ordering of ω. The following three conditions are all Σ_2^1 conditions on any well-ordering R:

(7) $WO_{|R|}$ is not Π_1^1 iff $\neg \exists e \forall S(|R| = |S| \Leftrightarrow \langle e, S\rangle \in P_1^1)$;

(8) $|R_0| < |R|$ iff $\exists S(S = R_0 \,\&\, |S| < |R|)$;

(9) $|R| < \delta_2$ iff $\exists e \exists A(\langle e, A\rangle \in P \,\&\, |R| < |Q_{e,A}|)$.

By (7), (8), and the basis theorem for Σ_2^1, for any $\alpha < \delta_2$ there is a β such that $\alpha < \beta < \delta_2$ and WO_β is not Π_1^1, so the set of ordinals in question is cofinal with δ_2. It is a Σ_2^1 set by (7) through (9) and § 2(3).

Noting that condition (7) is still Σ_2^1 if P is replaced by any Δ_2^1 enumerating set, we see that the lemma can be generalized to show, for example, that the set of infinite $\alpha < \delta_2$ such that WO_α is not a Boolean combination of Π_1^1 sets is a Σ_2^1 set of ordinals cofinal with δ_2. Theorem 4.6 together with Lemmas 4.5, 4.7, and 4.9 shows that the set of Δ_2^1 ordinals is split into two cofinal subsets, one containing precisely the ordinals α such that α is $\wedge_1^1$ characterizable. The following theorem establishes the order types of these cofinal subsets. Call a countably infinite ordinal α Σ_2^1 *regular* iff any cofinal Σ_2^1 subset of α has order type α.

THEOREM 4.10. *δ_2 is Σ_2^1 regular.*

PROOF. Let W be a cofinal Σ_2^1 subset of δ_2 and let $W' = \{R : |R| \in W\}$. Suppose that the order type of W is $|R_0|$ for some Δ_2^1 well-ordering R_0. For any ternary relation S and any n, let $S_n = \{\langle i,j\rangle : \langle n,i,j\rangle \in S\}$ and let Z be

$$\{S : \forall n S_n \in W' \,\&\, \forall m \forall n(\langle m, n\rangle \in R_0 \Rightarrow |S_m| < |S_n|)\}.$$

Then Z is a Σ_2^1 set and hence contains a Δ_2^1 member S. By construction, $\{|S_n| : n \in \omega\}$ is cofinal in W and hence also in δ_2. But then the ordering

$$\langle a, b\rangle < \langle c, d\rangle \Leftrightarrow a < c \quad \text{or} \quad (a = c \,\&\, \langle b, d\rangle \in S_a)$$

of pairs of integers is a Δ_2^1 well-ordering with order type at least δ_2, which is impossible.

By the remarks preceding Theorem 4.10, and by Lemmas 4.8, 4.9, and Theorem 4.10, we obtain the desired description of the class of $\bigwedge_1^1$ characterizable ordinals.

THEOREM 4.11. *The countable $\bigwedge_1^1$ characterizable ordinals form a cofinal subset of δ_2 with order type δ_2, and the complement of this set relative to δ_2 is also a cofinal subset of δ_2 with order type δ_2.*

Finally, we show that there is a countable $\bigwedge_2^1$ characterizable ordinal which is not $\bigvee_2^1$ characterizable.

THEOREM 4.12. *δ_2 is $\bigwedge_2^1$ characterizable.*

PROOF. By (9), $\{\alpha : \alpha \geqq \delta_2\}$ is a Π_2^1 set of ordinals. Since the set of countable $\bigwedge_1^1$ characterizable ordinals is cofinal in δ_2, the set of limits of countable sequences of these ordinals contains δ_2. But this set is a Π_2^1 set, since it equals $\{R : \forall n WO_{|R \upharpoonright n|}$ is $\Pi_1^1\}$, which is Π_2^1 by (7). Hence WO_{δ_2} is a Π_2^1 set, and δ_2 is $\bigwedge_2^1$ characterizable by Theorem 4.2.

5. Characterizability of cardinals in $\diamondsuit_2^1$ spectra. In the manner indicated at the end of § 2, our results about ordinal characterizability enable us to show that $\aleph_\alpha$ is $\diamondsuit_2^1$ characterizable for any $\alpha < \delta_2$. The same techniques also enable us to establish a more general result: by replacing the $\aleph$ function, which enumerates the class of all infinite cardinals, by the enumerating function F of an arbitrary $\diamondsuit_2^1$ spectrum, we shall show that F_α is a $\diamondsuit_2^1$ characterizable cardinal for any $\diamondsuit_2^1$ characterizable ordinal α. Consequently, by showing that several common classes of cardinals are $\diamondsuit_2^1$ spectra, we shall obtain $\diamondsuit_2^1$ characterizations for many cardinals which are not "too far out" in these spectra.

While ordinal characterizability will play a key role in what follows, in some cases it can be replaced by a weaker notion, for if F enumerates a class of cardinals, then in general F_α will be much larger than card(α) and the "extra room" in a structure of cardinality F_α may allow us to define a $\diamondsuit_2^1$ well-ordering with order type α even though α is not a $\diamondsuit_2^1$ characterizable ordinal. Accordingly, for any ordinal α and any cardinal κ, call α *$\diamondsuit_2^1$-definable over universes of power at least κ* iff there is an $\bigvee_2^1$ sentence ψ and an $\bigwedge_2^1$ sentence ψ' such that for any structure $\langle A, R\rangle$ with card$(A) \geqq \kappa$, $\langle A, R\rangle \models \psi$ iff $\langle A, R\rangle \models \psi'$ iff $|R| = \alpha$. It is easy to see that an ordinal α is $\diamondsuit_2^1$ characterizable iff it is $\diamondsuit_2^1$ definable over all universes of power at least card (α). Using the techniques of § 2, one can extend this result as follows.

THEOREM 5.1. *For any second-order characterizable ordinal α, α is $\diamondsuit_2^1$ definable over universes of power at least $2^{\operatorname{card}(\alpha)}$.*

PROOF. Suppose that φ is a second-order sentence which is true in a structure $\langle B, R\rangle$ iff $\langle B, R\rangle$ is isomorphic to $\langle \alpha, \in\rangle$. As in the proof of Theorem 2.2, let l be the maximum of the ranks of relation variables occurring in φ, and let ψ be an $\bigvee_2^1$ sentence such that $\langle A, R\rangle \models \psi$ iff there are relations $B, C_1, \ldots, C_l$ over A

such that B is the field of R and conditions (6) and (7) of § 2 hold; likewise, let ψ' be an $\bigwedge_2^1$ sentence such that $\langle A, R\rangle \models \psi'$ iff, for all relations $B, C_1, \ldots, C_l$ over A such that B is the field of R and condition (6) of § 2 holds, condition (7) also holds. Then ψ and ψ' define α over universes of power at least $2^{\mathrm{card}(\alpha)}$.

COROLLARY 5.2. *For any $\alpha < \delta_{(\omega)}$, α is $\diamondsuit_2^1$ definable over universes of power at least $2^{\aleph_0}$.*

PROOF. The corollary follows immediately from Theorems 4.2, 4.3, and 5.1.

Theorem 5.1 together with our results on ordinal characterizability will be the principal tools for obtaining $\diamondsuit_2^1$ characterizations of cardinals. Before proceeding in this direction, we obtain a useful "converse" to Lemma 2.1.

LEMMA 5.3. *For any $\diamondsuit_2^1$ characterizable cardinal κ, $\{\lambda : \lambda < \kappa\}$ and $\{\lambda : \lambda \leqq \kappa\}$ are $\diamondsuit_2^1$ spectra.*

PROOF. For any set A, card$(A) < \kappa$ iff there does not exist a subset B of A with cardinality κ [an $\bigwedge_2^1$ condition since κ is $\bigvee_2^1$ characterizable] iff there is a well-ordering of A in which no proper initial segment has cardinality κ [an $\bigvee_2^1$ condition since κ is $\bigwedge_2^1$ characterizable]. The rest of the lemma follows from Lemma 2.1.

THEOREM 5.4. *Let F enumerate a $\diamondsuit_2^1$ spectrum in increasing order. For any ordinal α in the domain of F, if there is a $\diamondsuit_2^1$ characterizable cardinal $\kappa \leqq F_\alpha$ such that α is $\diamondsuit_2^1$ definable over universes of power at least κ, then F_α is a $\diamondsuit_2^1$ characterizable cardinal.*

PROOF. Let F enumerate a $\diamondsuit_2^1$ spectrum $\mathscr{S}$. In the definition of the $\diamondsuit_2^1$ condition $\Psi(A, R, S)$ in § 2, replace "card$(R^{-1}\{x\}) \geqq \aleph_0$" by "card$(R^{-1}\{x\}) \in \mathscr{S}$"; also, in (4a) and (4b) of § 2, replace "card$(S) < \aleph_0$" by condition (9) of § 2. By Lemma 5.3, $\{\lambda : \lambda \geqq \kappa\}$ is a $\diamondsuit_2^1$ spectrum, and hence F_α is $\diamondsuit_2^1$ characterizable as that cardinal in $\{\lambda : \lambda \geqq \kappa\}$ satisfying the modified conditions (4a) or (4b).

COROLLARY 5.5. *Let F enumerate a $\diamondsuit_2^1$ spectrum in increasing order. Then, for any ordinal α in the domain of F,*

(a) *if α is $\diamondsuit_2^1$ characterizable, then so is F_α;*
(b) *if $\alpha < \delta_2$, then F_α is $\diamondsuit_2^1$ characterizable; and*
(c) *if $2^{\aleph_0} < F_\alpha$ and $\alpha < \delta_{(\omega)}$, then F_α is $\diamondsuit_2^1$ characterizable.*

PROOF. Part (a) is immediate, (b) follows from (a) and Corollary 4.4, and (c) follows from the theorem and Corollaries 2.3 and 5.2.

In order to apply Corollary 5.5, we now show that several classes of cardinals are $\diamondsuit_2^1$ spectra.

LEMMA 5.6. *For any class $\mathscr{S}$ of infinite cardinals, $\mathscr{S}$ is a $\diamondsuit_2^1$ spectrum iff there is an $\bigvee_2^1$ class $\mathscr{C}$ of structures $\langle A, R, S\rangle$ such that, for any ordinal α and any $S \subseteq \alpha$, $\langle \alpha, \in, S\rangle \in \mathscr{C}$ iff $S = \mathscr{S} \cap \alpha$.*

PROOF. If $\mathscr{S}$ is a $\diamondsuit_2^1$ spectrum, let $\mathscr{C}$ be the $\diamondsuit_2^1$ class of structures $\langle A, R, S\rangle$ such that, for any x, $x \in S$ iff card$(R^{-1}\{x\}) \in \mathscr{S}$ and, for any y, if $\langle y, x\rangle \in R$, then

$\mathrm{card}(R^{-1}\{y\}) < \mathrm{card}(R^{-1}\{x\})$. Conversely, if $\mathscr{C}$ is an $\vee_2^1$ class, then $\mathscr{S}$ is $\diamondsuit_2^1$ since for any set A, $\mathrm{card}(A) \in \mathscr{S}$ iff there are relations R, S and an $x \in A$ such that R well-orders A, $\mathrm{card}(A) = \mathrm{card}(R^{-1}\{x\})$, $\langle A, R, S\rangle \in \mathscr{C}$, and $x \in S$ [an $\vee_2^1$ condition] iff A is infinite and for any R, S, x, if R well-orders A, x is the R-least element of A such that $\mathrm{card}(A) = \mathrm{card}(R^{-1}\{x\})$, and $\langle A, R, S\rangle \in \mathscr{C}$, then $x \in S$ [an $\wedge_2^1$ condition].

THEOREM 5.7. *The classes of infinite cardinals, limit cardinals, regular cardinals, successors of beths, weakly inaccessible cardinals, and inaccessible cardinals are all* $\diamondsuit_2^1$ *spectra.*

PROOF. The infinite cardinals form an $\vee_1^1$ spectrum by § 2(1). Let $\mathscr{C}$ be the class of structures $\langle A, R, S\rangle$ such that, for any x, $x \in S$ iff $R^{-1}\{x\}$ is infinite and

$$\forall y[\langle y, x\rangle \in R \Rightarrow \exists z(\langle y, z\rangle, \langle z, x\rangle \in R \;\&\; \mathrm{card}(R^{-1}\{z\}) < \mathrm{card}(R^{-1}\{x\}))];$$

$\mathscr{C}$ is a $\diamondsuit_2^1$ class and hence the class of limit cardinals is a $\diamondsuit_2^1$ spectrum by the lemma. Next let $\mathscr{C}$ be the class of structures $\langle A, R, S\rangle$ such that, for any x, $x \in S$ iff $R^{-1}\{x\}$ is infinite and for every subset C of $R^{-1}\{x\}$, if

$$\forall y[\langle y, x\rangle \in R \Rightarrow \exists z(\langle y, z\rangle, \langle z, x\rangle \in R \;\&\; z \in C)],$$

then $\mathrm{card}(C) < \mathrm{card}(R^{-1}\{x\})$; $\mathscr{C}$ is a $\diamondsuit_2^1$ class and hence the class of regular cardinals is a $\diamondsuit_2^1$ spectrum by the lemma. Using § 2(8) and the definition

$$\beth_\alpha = \sup\{2^{\beth_\beta} : \beta < \alpha\} \cup \aleph_0$$

of the beth function, one can show that there is an $\vee_2^1$ class $\mathscr{C}$ of structures $\langle A, R, S\rangle$ such that $\langle \alpha, \in, S\rangle \in \mathscr{C}$ iff there is a binary relation T such that S is the range of T and $T = \{\langle \beta, \beth_\beta^+\rangle : \beth_\beta^+ \leqq \alpha\}$; hence the successors of the beths form a $\diamondsuit_2^1$ spectrum by the lemma. The rest of the theorem follows now, since a cardinal κ is weakly inaccessible iff it is an uncountable regular limit cardinal, and is inaccessible iff it is weakly inaccessible and $\forall \lambda < \kappa\, (2^\lambda < \kappa)$ [a $\diamondsuit_2^1$ condition by § 3(15), (16)].

COROLLARY 5.8. (a) *For any* $\alpha < \delta_2$, $\aleph_\alpha$ *and the* α*th weakly inaccessible cardinal (if it exists) are* $\diamondsuit_2^1$ *characterizable.*

(b) *For any* $\diamondsuit_2^1$ *characterizable cardinal* κ, $\aleph_\kappa$ *is* $\diamondsuit_2^1$ *characterizable.*

(c) *For any* $\alpha < \delta_{(\omega)}$, *if* $2^{\aleph_0} < \aleph_\alpha$ *then* $\aleph_\alpha$ *is* $\diamondsuit_2^1$ *characterizable.*

(d) *For any* $\alpha < \delta_{(\omega)}$, $\beth_\alpha^+$ *and the* α*th inaccessible cardinal (if it exists) are* $\diamondsuit_2^1$ *characterizable.*

PROOF. The proof is immediate from Theorem 4.1, Corollary 5.5, and Theorem 5.7.

As noted before, Corollary 5.8 shows that cardinals which are not "too far out" in several $\diamondsuit_2^1$ spectra are $\diamondsuit_2^1$ characterizable. Characterizations of cardinals "farther out" in these spectra can be obtained also by utilizing various operators to "throw away" large numbers of cardinals in these spectra in such a manner that the remaining cardinals still form $\diamondsuit_2^1$ spectra to which Corollary 5.5 can be applied. We shall examine two such operators: the fixed point operator F and the

Mahlo operator M (cf. Lévy [9]), where, for any class $\mathscr{F}$ of ordinals,

$$F\mathscr{F} = \{\alpha \in \mathscr{F} : \mathscr{F} \cap \alpha \text{ has order type } \alpha\},$$

and

$$M\mathscr{F} = \{\alpha \in \mathscr{F} : \text{every closed, unbounded subset of } \alpha \text{ intersects } \mathscr{F}\}.$$

Call an operator T on the collection of all classes of ordinals into itself a *thinning* iff, for any class $\mathscr{F}$ of ordinals and any ordinal α,

$$T\mathscr{F} \cap \alpha = T(\mathscr{F} \cap \alpha) \subseteq \mathscr{F}.$$

Obviously both F and M are thinnings. For any thinning T, define the *graph* of T to be the class of all structures $\langle A, R, C, D\rangle$ such that

(1) R well-orders A, and

(2) "$D = TC$", i.e., $\{|R \upharpoonright x| : x \in D\} = T\{|R \upharpoonright x| : x \in C\}$.

A thinning is said to be an $\bigvee^1_2$ thinning if its graph is an $\bigvee^1_2$ class. We shall show that F and M are $\bigvee^1_2$ thinnings, and that $\bigvee^1_2$ thinnings preserve $\Diamond^1_2$ spectra.

THEOREM 5.9. *The fixed point operator and the Mahlo operator are $\bigvee^1_2$ thinnings.*

PROOF. It suffices in each case to show that condition (2) is an $\bigvee^1_2$ condition. First, "$D = FC$" holds iff

$$\forall x(x \in D \Leftrightarrow x \in C \;\&\; |R \upharpoonright x| = |(R \cap C \times C) \upharpoonright x|),$$

which is a $\Diamond^1_2$ condition by § 4(1). Second, a subset X of an ordinal α is closed iff $\forall\beta < \alpha \sup(X \cap \beta) \in X$ and is unbounded iff $\sup\{\beta + 1 : \beta \in X\} = \alpha$. Since these are both first-order conditions, "$D = MC$" is a $\Diamond^1_2$ condition.

THEOREM 5.10. *For any $\bigvee^1_2$ thinning T and any $\Diamond^1_2$ spectrum $\mathscr{S}$ containing only infinite cardinals, $T\mathscr{S}$ is a $\Diamond^1_2$ spectrum.*

PROOF. By Lemma 5.6 there is an $\bigvee^1_2$ class $\mathscr{C}$ such that a structure $\langle A, R, D\rangle$ is in $\mathscr{C}$ iff there exists a subset C of A such that $\langle A, R, C, D\rangle$ is in the graph of T and

$$\forall x(x \in C \Leftrightarrow |R \upharpoonright x| \in \mathscr{S} \cap |R|).$$

Hence, for any ordinal α and any $S \subseteq \alpha$, $\langle \alpha, \in, S\rangle \in \mathscr{C}$ iff $S = T(\mathscr{S} \cap \alpha) = T\mathscr{S} \cap \alpha$, so that $T\mathscr{S}$ is a $\Diamond^1_2$ spectrum by Lemma 5.6.

It follows from Theorem 5.10 and previous results, for example, that the first fixed point of the $\aleph$ function is $\Diamond^1_2$ characterizable, as is the first fixed point in the enumeration of fixed points of the $\aleph$ function, and so on. In general, as we shall now show, iterating an $\bigvee^1_2$ thinning α times, where α is a $\Diamond^1_2$ characterizable ordinal, still preserves $\Diamond^1_2$ spectra. For any thinning T and any class $\mathscr{F}$ of ordinals, let

$$\begin{aligned} T^\alpha\mathscr{F} &= \mathscr{F} && \text{if } \alpha = 0, \\ &= T\bigcap\{T^\beta\mathscr{F} : \beta < \alpha\} && \text{if } \alpha > 0. \end{aligned}$$

LEMMA 5.11. *For any thinning T, any class $\mathscr{F}$ of ordinals, and any ordinals α, β, if $\alpha \leqq \beta$, then $T^\alpha(\mathscr{F} \cap \alpha) = T^\beta(\mathscr{F} \cap \alpha)$.*

PROOF. Suppose not, and let α be the least ordinal such that for some $\beta \geqq \alpha$ and some $\gamma < \alpha$, γ is in $T^\alpha(\mathscr{F} \cap \alpha)$ but not $T^\beta(\mathscr{F} \cap \alpha)$. Then $T^{\gamma+1}(\mathscr{F} \cap \gamma + 1) =$ [since $\gamma \in T^\alpha(\mathscr{F} \cap \alpha) \subseteq T^{\gamma+1}(\mathscr{F} \cap \alpha)$] $T^{\gamma+1}(\mathscr{F} \cap \gamma) \cup \{\gamma\} =$ [by the minimality of α] $T^\gamma(\mathscr{F} \cap \gamma) \cup \{\gamma\} =$ [since $\gamma \in T^\alpha(\mathscr{F} \cap \alpha) \subseteq T^\gamma(\mathscr{F} \cap \alpha)$] $T^\gamma(\mathscr{F} \cap \gamma + 1)$. Hence

$$T^\alpha(\mathscr{F} \cap \gamma + 1) = T^\gamma(\mathscr{F} \cap \gamma + 1) = T^\beta(\mathscr{F} \cap \gamma + 1),$$

contradicting the choice of γ.

THEOREM 5.12. *For any $\bigvee^1_2$ thinning T and any $\diamondsuit^1_2$ characterizable ordinal α, T^α is an $\bigvee^1_2$ thinning.*

PROOF. Again we must show that condition (2) is an $\bigvee^1_2$ condition. But "$D = T^\alpha C$" holds iff there is an $S \subseteq A \times A$ such that

(a) $\forall x(|R \restriction x| = 0 \Rightarrow S^{-1}\{x\} = C)$,

(b) $\forall x(|R \restriction x| > 0 \Rightarrow \langle A, R, \bigcap\{S^{-1}\{y\} : \langle y, x\rangle \in R\}, S^{-1}\{x\}\rangle$ is in the graph of T),

and either

(c) $\exists x(|R \restriction x| = \alpha \;\&\; D = S^{-1}\{x\})$, i.e., $\alpha < |R|$ and "$D = T^\alpha C$," or

(d) $\neg \exists x|R \restriction x| = \alpha \;\&\; \langle A, R, \bigcap\{S^{-1}\{y\} : y \in A\}, D\rangle \in$ graph T, i.e., $\alpha \geqq |R|$ and "$D = T^{|R|}C = T^\alpha C$."

For any cardinal κ and any ordinal α, call κ a *fixed point of order* α of the $\aleph$ function iff κ is in $F^\alpha\{\lambda : \lambda \geqq \aleph_0\}$, and a *Mahlo cardinal of order* α iff $\kappa \in M^\alpha\{\lambda : \lambda \text{ regular}\}$. The following result is a direct consequence of Corollary 5.5 and Theorems 5.7, 5.10, and 5.12.

COROLLARY 5.13. *For any ordinals α and β, if α and β are $\diamondsuit^1_2$ characterizable (in particular, if $\alpha, \beta < \delta_2$), then the αth fixed point of order β of the $\aleph$ function and the αth Mahlo cardinal of order β (if it exists) are $\diamondsuit^1_2$ characterizable.*

By considering definability of ordinals over universes of at least a given power (cf. Theorem 5.4) one can strengthen Corollary 5.13 to show that, for any α, $\beta < \delta_{(\omega)}$, the αth Mahlo cardinal of order β (if it exists) is $\diamondsuit^1_2$ characterizable provided that the continuum is not extraordinarily large. Even more is possible, since it should be apparent that many results in this section and in the previous sections can be strengthened or generalized considerably if one considers not only second-order definability but also arbitrary higher-order definability. We have avoided such considerations primarily because they complicate the statement and proofs of our results without adding much in the way of ideas or consequences not already present in our study of second-order logic. However, it is probably appropriate at this point to indicate the form such generalizations take.

For any $n < \omega$, an *n*th *order logic* is that fragment of the simple theory of types which utilizes variables of type at most n, and $\delta^n_{(\omega)}$ is the least ordinal which is not the order type of a well-ordering of ω definable in *n*th order arithmetic. In the same manner in which Montague [**10**] extended Zykov's theorem to higher types, one can extend Theorem 2.2 to show that, for any nth order characterizable cardinal κ, $(2^{\kappa,n})^+$ is $\diamondsuit^1_2$ characterizable, where

$$2^{\kappa,0} = \kappa \quad \text{and} \quad 2^{\kappa,n+1} = 2^{2^{\kappa,n}}.$$

Among the many possible generalizations of other results are those of Corollaries 5.8 and 5.13 which show that, for any n and any $\alpha, \beta < \delta^n_{(\omega)}$, $\aleph_\alpha$ is $\diamondsuit^1_2$ characterizable if $\beth_n < \aleph_\alpha$, $\beth^+_\alpha$ is $\diamondsuit^1_2$ characterizable, and the αth Mahlo cardinal of order β (if it exists and is larger than $\beth_n$) is $\diamondsuit^1_2$ characterizable.

6. Characterizability and the model theory of set theory. We have already seen in § 2 that the $\diamondsuit^1_2$ characterizability of a particular cardinal (namely, $2^{\aleph_0}$) is both consistent and independent relative to Zermelo-Fraenkel set theory. What can be said about the size of the continuum if $2^{\aleph_0}$ is not $\wedge^1_2$ characterizable? Kunen's result [**8**] shows that $2^{\aleph_0}$ may still be less than $\aleph_\Omega$, while Corollary 5.8 shows that it must be at least $\aleph_{\delta_2}$. Since δ_2 is an $\wedge^1_2$ characterizable ordinal by Theorem 4.12, an examination of the proof of Theorem 5.4 shows that $\aleph_{\delta_2}$ is $\wedge^1_2$ characterizable, and hence $2^{\aleph_0}$ must be greater than $\aleph_{\delta_2}$ if it is not $\wedge^1_2$ characterizable. In fact, the results of § 4 and § 5 can be extended to show that if $2^{\aleph_0}$ is not $\wedge^1_2$ characterizable, then not only is it not among the first δ_2 members of any $\diamondsuit^1_2$ spectrum, but also it is not among the first σ_2 members of any $\wedge^1_2$ spectrum, where $\sigma_2 > \delta_2$ is the least ordinal which is not the order type of a Σ^1_2 well-ordering of a subset of ω. Thus if the power of the continuum does not satisfy a fairly weak describability condition, then it is already excluded from being equal to a large number of cardinals.

The answers to questions (I) and (II) raised in § 2 are also influenced by the model theory of set theory. We have seen that the smallest cardinal which is not $\diamondsuit^1_2$ characterizable lies between $\aleph_{\delta_2}$ and $\aleph_\Omega$, and that if the continuum hypothesis holds then this cardinal is at least as large as $\aleph_{\delta_{(\omega)}}$. How large can this cardinal be? Scott and Kunen observed in a conversation with the author that given any countable standard model $\mathfrak{M}$ of Zermelo-Fraenkel set theory (ZF) and any ordinal $\alpha < \Omega^{\mathfrak{M}}$ there is a countable standard model $\mathfrak{N}$ of ZF with the same ordinals as $\mathfrak{M}$ such that cardinals are absolute from $\mathfrak{M}$ to $\mathfrak{N}$ and, for any $\beta < \alpha$, $\aleph_\beta$ is $\diamondsuit^1_2$ characterizable in $\mathfrak{N}$ (the "trick" is to have $2^{\aleph_n} = \aleph_{f(n)}$ in $\mathfrak{N}$, where f is an increasing function in $\mathfrak{M}$ such that $\{\langle x, y\rangle : \exists n 2^x 3^y = f(n+1) - f(n)\}$ is a well-ordering of ω with order type α). Thus the answer seems to be "as large as possible." How small can the cardinal be? We do not know if it is consistent relative to ZF that $\aleph_{\delta_2}$ is not $\vee^1_2$ characterizable.

Turning now to question (II), it appears that we must answer it in the same fashion. For given any countable standard model $\mathfrak{M}$ of ZF and any cardinal κ in $\mathfrak{M}$, there is a countable standard model $\mathfrak{N}$ of ZF with the same ordinals as $\mathfrak{M}$ such that cardinals are absolute from $\mathfrak{M}$ to $\mathfrak{N}$ and $(2^{\aleph_0})^{\mathfrak{N}} > \kappa$ (cf. Cohen [**4**, § 8]); thus the supremum of the cardinals which are $\diamondsuit^1_2$ characterizable in $\mathfrak{N}$ is greater than κ. It follows that an upper bound to the countable class of $\diamondsuit^1_2$ characterizable cardinals is not definable in any fashion which is absolute for models of ZF with the same cardinals; e.g., it is not arithmetical in the aleph function, that is, definable in a first-order language with variables ranging over the ordinals and with constants denoting the aleph function and the operations of ordinal addition and multiplication.

The answer to question (II) is affected by still other considerations. Suppose that we start with a model of ZF in which there is no inaccessible cardinal and extend it to one containing an inaccessible. Then we have changed the supremum of the class of $\Diamond_2^1$ characterizable cardinals by Corollary 5.8(d). If we add cardinals whose existence cannot be proved from the existence of an inaccessible cardinal (e.g., Mahlo cardinals), then we may raise the supremum still further. Even though Mahlo cardinals are the largest "large cardinals" which we can show to be $\Diamond_2^1$ characterizable, we can add larger cardinals yet and still affect the supremum, as the following examples show.

Hanf and Scott [**5**] call a cardinal κ $\bigwedge_n^1$ *describable* iff there is an $\bigwedge_n^1$ class $\mathscr{C}$ and an $R \subseteq \kappa \times \kappa$ such that $\langle \kappa, R \rangle \in \mathscr{C}$ but $\langle \lambda, R \cap \lambda \times \lambda \rangle \notin \mathscr{C}$ for all $\lambda < \kappa$; they call κ $\bigwedge_n^1$ *indescribable* iff it is inaccessible and not $\bigwedge_n^1$ describable. They show that any $\bigwedge_1^1$ indescribable cardinal is weakly compact, and that the existence of $\bigwedge_{n+1}^1$ indescribable cardinals does not follow from the existence of $\bigwedge_n^1$ indescribable cardinals for any n. The smallest $\bigwedge_n^1$ indescribable cardinal is not $\bigvee_{n+1}^1$ characterizable since a straightforward coding argument shows that any $\bigvee_{n+1}^1$ characterizable cardinal is $\bigwedge_n^1$ describable. However, by the techniques of § 3, the class of $\bigwedge_n^1$ indescribable cardinals is an $\bigwedge_{n+1}^1$ spectrum, and so the smallest $\bigwedge_n^1$ indescribable cardinal κ, if it exists, is at least second-order characterizable as the smallest member of a second-order spectrum; hence $(2^\kappa)^+$ is $\Diamond_2^1$ characterizable, so that $\{\lambda : \lambda \leqq 2^\kappa\}$ is a $\Diamond_2^1$ spectrum by Lemma 5.3 and κ is $\bigwedge_{n+1}^1$ characterizable as the unique $\bigwedge_n^1$ describable cardinal in $\{\lambda : \lambda \leqq 2^\kappa\}$.

An alternative proof of the $\bigwedge_2^1$ characterizability of the first $\bigwedge_1^1$ indescribable cardinal κ is as follows. For any set A and any n, let $[A]^n = \{B : B \subseteq A \;\&\; \mathrm{card}(B) = n\}$. Silver [**15**] showed that κ is the smallest cardinal λ such that $\lambda \to (\lambda)^2$, i.e., such that for every function f from $[\lambda]^2$ into $\{0, 1\}$ there is a subset A of λ such that $\mathrm{card}(A) = \lambda$ and f is constant on $[A]^2$; from this definition, it is apparent that $\{\lambda : \lambda \to (\lambda)^2\}$ is an $\bigwedge_2^1$ spectrum, and that κ is $\bigwedge_2^1$ characterizable as above. The same method shows that the class of Ramsey cardinals, i.e., cardinals κ such that $\kappa \to (\kappa)^{<\omega}$, is also an $\bigwedge_2^1$ spectrum, and that the smallest Ramsey cardinal, if it exists, is $\bigwedge_2^1$ but not $\bigvee_2^1$ characterizable.

It is even possible to add cardinals which are not second-order characterizable at all (e.g., the first measurable cardinal) to a model of set theory and still affect the supremum of the class of $\Diamond_2^1$ characterizable cardinals, since, as noted at the end of § 5, $(2^{2^\kappa})^+$ is $\Diamond_2^1$ characterizable for any third-order characterizable cardinal κ. Is there a cardinal so large that adding it to a model of ZF does not affect the supremum of the class of $\Diamond_2^1$ characterizable cardinals? This question is admittedly rather vague, though we would like to know whether, in view of the above results, there is any precise sense in which an "absolute" upper bound can be placed on the class of $\Diamond_2^1$ characterizable cardinals.

In conclusion, we note that the richness of the class of $\Diamond_2^1$ characterizable cardinals suggests that efforts towards a better understanding of higher-order model theory might begin most profitably with an attempt to isolate natural subclasses of $\Diamond_2^1$ and to analyze the structure of these classes.

REFERENCES

1. J. W. Addison, *Hierarchies and the axiom of constructability*, Summaries of talks presented at the Summer Institute of Symbolic Logic in 1957 at Cornell University, second edition, Communications Research Division, Institute for Defense Analyses, 1960, pp. 355–362.

2. G. Asser, *Das Repräsentantenproblem im Prädikatenkalkül der ersten Stufe mit Identität*, Z. Math. Logik Grundlagen Math. **1** (1955), 252–263. MR **17**, 1038.

3. J. H. Bennett, *On spectra*, Doctoral Dissertation, Princeton University, Princeton, N.J., 1962.

4. P. J. Cohen, *Set theory and the continuum hypothesis*, Benjamin, New York, 1966. MR **38** #999.

5. W. P. Hanf and D. S. Scott, *Classifying inaccessible cardinals*, Notices Amer. Math. Soc. **8** (1961), 445. Abstract #61T-240.

6. S. C. Kleene, *Hierarchies of number theoretic predicates*, Bull. Amer. Math. Soc. **61** (1955), 193–213. MR **17**, 4.

7. ———, *Quantification of number-theoretic functions*, Compositio Math. **14** (1959), 23–40. MR **21** #2586.

8. K. Kunen, *Indescribability and the continuum*, Axiomatic Set Theory, Proc. Sympos. Pure Math., vol. 13, part I, Amer. Math. Soc., Providence, R.I., 1971, pp. 199–203. MR **44** #63.

9. A. Lévy, *The sizes of the indescribable cardinals*, Axiomatic Set Theory, Proc. Sympos. Pure Math., vol. 13, part I, Amer. Math. Soc., Providence, R.I., 1971, pp. 205–218. MR **43** #7321.

10. R. M. Montague, *Reductions of higher-order logic*, The Theory of Models (Proc. 1963 Internat. Sympos., Berkeley), edited by J. W. Addison, L. Henkin, and A. Tarski, North-Holland, Amsterdam, 1965, pp. 251–264. MR **34** #4114.

11. Y. N. Moschovakis, *Uniformization in a playful universe*, Bull. Amer. Math. Soc. **77** (1971), 731–736.

12. H. Scholz, *Ein ungelöstes Problem in der symbolischen Logik*, J. Symbolic Logic **17** (1952), 160.

13. D. S. Scott, *Axiomatizing set theory*, Lecture notes from the 1967 UCLA Summer Institute on Axiomatic Set Theory.

14. J. R. Shoenfield, *Mathematical logic*, Addison-Wesley, Reading, Mass., 1967.

15. J. H. Silver, *Some applications of model theory in set theory*, Doctoral Dissertation, University of California, Berkeley, Calif., 1966.

16. Y. Suzuki, *A complete classification of the Δ_2^1 functions*, Bull. Amer. Math. Soc. **70** (1964), 246–253. MR **28** #2042.

17. A. A. Zykov, *The spectrum problem in the extended predicate calculus*, Izv. Akad. Nauk SSSR Ser. Mat. **17** (1953), 63–67; English transl., Amer. Math. Soc. Transl. (2) **3** (1956), 1–14. MR **14**, 936; **17**, 1038.

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