1 Deterministic Spanner Construction

Devise a deterministic algorithm that constructs a 2-additive spanner with $\tilde{O}(n^{3/2})$ edges.

2 Monotone Submodular Maximization

Consider a set $U$ of $n$ elements that we can buy and a function $f : 2^U \to \mathbb{R}^+$, where for each subset $S \subseteq U$, the value $f(S)$ determines our profit if we buy exactly the elements of set $S$.

We assume two properties about this profit function: (A) Function $f$ is monotone in the sense that $f(S) \leq f(T)$ for any two sets $S, T$ such that $S \subseteq T$, and (B) Function $f$ is submodular in the sense that $f(S \cup i) - f(S) \geq f(T \cup i) - f(T)$ for any $i \in U$ and any two sets $S, T$ such that $S \subseteq T$. In simple words, the submodularity means that the marginal gain that we have by adding $i$ to our purchase set diminishes as we move from one purchase set $S$ to a superset of it $T$. That is, roughly speaking, the more that we already have in the purchase set, the less extra gain by adding an element to it.

Devise an algorithm that purchases a set $S$ of (approximately) maximum profit, subject to the constraint that $|S| \leq k$, for some given value $k \in \{1, 2, 3, \ldots, n\}$. What approximation factor do you get?

3 Connected Dominating Set

Given an $n$-node graph $G = (V, E)$, a set $S \subseteq V$ of vertices is called a Connected Dominating Set (CDS) if the following two properties are satisfied: (1) each node $v \in V$ is either in $S$ or has a neighbor in $S$, (2) the subgraph $G[S]$ induced by $S$—i.e., the one made of $S$-vertices and all edges whose both endpoints are in $S$—is connected. Devise an approximation algorithm for finding a minimum-cardinality CDS.

4 Max-weight Matroid Base (Williamson-Shmoys 2.12)

A matroid $(E, I)$ is defined by a ground set $E$ of elements and a collection $I = \{S_1, S_2, \ldots, S_\ell\}$ of independent subsets $S_i \subseteq E$, subject to the following conditions:

1. For any two subsets $S$ and $S'$ such that $S \subseteq S'$, if $S'$ is independent, then so is $S$, i.e., $(S' \in I) \Rightarrow (S \in I)$.

2. For any two independent sets $S$ and $T$ such that $|S| < |T|$, there exists an element $e \in T \setminus S$ such that $(S \cup \{e\}) \in I$.

We call an independent set $S$ a base if there is no $T \in I$ such that $S \subseteq T$. That is, $S$ is in some sense maximal with regard to independence.

Suppose that each element $e \in E$ has a weight $w_e \geq 0$. Devise an algorithm that finds a maximum-weight base of the matroid.
5 Walking on the Hypercube

Consider the $d$-dimensional hypercube, which has vertex set $\{0, 1\}^d$ and where every two vertices that differ in exactly one coordinate are connected by an edge. Suppose that each edge has a random length drawn from an exponential distribution with mean 1, and the lengths of different edges are independent. Devise an algorithm that finds a walk from the vertex $(0, 0, \ldots, 0)$ to the vertex $(1, 1, \ldots, 1)$ with expected length $O(\log d)$.

6 $k$-Center

Consider a set of $n$ points $P = \{p_1, p_2, \ldots, p_n\}$ and a distance metric $d : P \times P \to [0, \infty)$ where $d(p_i, p_j)$ indicates the distance between the two points $p_i, p_j \in P$. For a set $S \subseteq P$, and an arbitrary point $p' \in P$, the distance of $p'$ to $S$ is equal to $\min_{p'' \in S} d(p', p'')$. The objective is to find a set $S \subseteq P$ of $k$ points, called centers, such that the maximum distance of any point $p' \in P$ to the set $S$ is minimized. Devise a 2-approximation algorithm for this problem.