Advanced Algorithms	11/07, 2017
Lecture 8	
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This draft has not been checked by the lecturer yet.

1 Graph as a geometric object

Let us start with some structural insights regarding geometric structure of a graphs. In example, we might have a representation of cuts in G as a property of its geometric embedding. Ideas below are form Linial, London and Rabinovich [LLR95].

For a cut $S \subseteq V$ we define δS as a set of edges going "across" S, that is $\delta S = \{(u,v) : u \in S \wedge v \in V \setminus S\}$.

Definition 1. *An* elementary cut metric *associated with* S *is:* $d^S_{ij} = 1$*, if* $(i,j) \in \delta S$*,* $d^S_{ij} = 0$ *otherwise.*

Given edge capacities $c : E \to \mathbb{R}^+$, we write down an IP.

Variables: $d_{ij} \in \{0,1\}$ for $i,j \in V$.
Objective and Constraints:

$$\begin{aligned}
&\min \sum_{i,j} c_{ij} \cdot d_{ij} \\
&\text{s.t. } \forall_{i,j,k}\ d_{ij} \le d_{ik} + d_{kj} \\
&d_{st} = 1
\end{aligned}$$

A d^* being a solution to above is an elementary cut metric associated with a min s-t cut in G.

Relaxing this IP to LP, we obtain.

Variables: d_{ij} for $i,j \in V$.
Objective and Constraints:

$$\begin{aligned}
&\min \sum_{i,j} c_{ij} \cdot d_{ij} \\
&\text{s.t. } \forall_{i,j,k}\ d_{ij} \le d_{ik} + d_{kj} \\
&\forall_{i,j} d_{ij} \ge 0 \\
&d_{st} \ge 1
\end{aligned}$$

We obtain an object for which $d_{st} = 1$. Moreover, it can be shown that it is a linear combination of elementary cut metrics: there is a weight function $y : 2^V \to \mathbb{R}^+$ s.t.

$$d_{ij} = \sum_{S \subseteq V : ij \in \delta S} y(S).$$

Any linear combination of elementary cut metrics is called a *cut metric.*

2 Warm up: Min s-t cut via LP

Assume that d^* is a minimal solution to LP. How to extract the cut from d^*? Idea: plot the vertices according to $d^*(s,v)$, call their coordinates $x_0 = 0 \le x_1 \le ... \le x_n = 1$. Natural cuts are vertical lines, that is separating x_{i-1} (and smaller indices) from x_i (and larger indices).

Theorem 2. *Smallest vertical s-t cut is the smallest s-t cut.*

Proof. Let E_j be the said vertical cut and define its capacity $c_j = \sum_{e=(u,v)\in \delta E_j} c_e$, and take as weights $y_j = x_j - x_{j-1}$. It follows that $d^*_e \ge \sum_{j : e \in E_j} y_j$.

Consider cost of optimal solution of LP:

$$\mathrm{OPT(LP)} = \sum_e c_e d_e^* \geq \sum_e c_e \sum_{j:e\in E_j} y_j = \sum_j \sum_{e\in E_j} c_e y_j = \sum_j y_j \sum_{e\in E_j} c_e =$$

$$= \sum_j y_j c_j \geq \sum_j y_j \cdot c_{j_{\min}} = c_{j_{\min}} \sum_j (x_j - x_{j-1}) = c_{j_{\min}} d_{st} \geq c_{j_{\min}}$$

Since $\mathrm{OPT(IP)} \geq \mathrm{OPT(LP)}$, it follows that $c_{j_{\min}}$ is the min-cost s-t cut. □

3 Sparsest cut

Let us consider some non-trivial (NP-hard) cut properties. For example, define *sparsity* of the cut to be

$$\alpha(S) = \frac{\mathrm{cap}(\delta S)}{\min(|S|, |V \setminus S|)}.$$

We are interested in finding the sparsest cut, or the sparsity of the sparsest cut:

$$\alpha(G) = \min_{S\subseteq V} \alpha(S).$$

Other connected parameter is the *flux* of the graph (Leighton and Rao [LR88]):

$$\mathrm{flux}(G) = \min_{S\subseteq V} \frac{\mathrm{cap}(\delta S)}{|S| \cdot |V \setminus S|}.$$

flux is a 2-approximation of the sparsest cut.

We go into more general setting. Consider a pairwise demands dem_{ij} for sending flow, and a capacities cap_{ij}. Then

$$\min_{S\subseteq V} \frac{\mathrm{cap}(\delta S)}{\mathrm{dem}(\delta S)}$$

is a trivial upper bound on the ratio of demands we can satisfy with flows, while keeping the flows constrained by edge capacities. However, working with ratios is quite unwieldy, we like things to be linear.

Variables: d_{ij} for $i, j \in V$.
Objective and Constraints:

$$\begin{aligned}
&\min \sum \mathrm{cap}_{ij} d_{ij} \\
&\text{s.t. } \forall_{i,j,k}\ d_{ij} \leq d_{ik} + d_{kj} \\
&\forall_{i,j} d_{ij} \geq 0 \\
&\sum_{ij} \mathrm{dem}_{ij} d_{ij} = 1
\end{aligned}$$

Our plan in general:

1. solve LP above to obtain d^*,
2. we embed d^* into L_1^p for some large enough dimension p,
3. from L_1 embedding we obtain cut metric,
4. which decomposes to small number of elementary cut metrics,
5. we take the best of those cuts.

Step (2) follows from

Theorem 3. *Any n point metric has embedding: $\phi : V \to L_1^{\mathcal{O}(\log^2 n)}$ such that*

$$\frac{d_{ij}}{\mathcal{O}(\log n)} \leq |\phi(i) - \phi(j)|_1 \leq d_{ij}$$

where $|x|_1 = \sum_i |x_i|$

result from Bourgain 80s, proof by Linial London Rabinovich [LLR95] (in the next section).

Step (3) follows from Any L_1-metric on n points with dimension $k = \mathcal{O}(\log^2 n)$ can be written as a cut metric which is a linear combination of $(n-1)k$ elementary cuts, denote them C.

Say

$$S^* = \arg\min_{S \in C} \frac{\text{cap}(\delta S)}{\text{dem}(\delta S)}$$

$$\alpha_{S^*} = \frac{\text{cap}(\delta S^*)}{\text{dem}(\delta S^*)} \le \frac{\sum_{S \in C} \text{cap}(\delta S) y(S)}{\sum_{S \in C} \text{dem}(\delta S) y(S)} = \frac{\sum_{S \subseteq V} \text{cap}(\delta S) y(S)}{\sum_{S \subseteq V} \text{dem}(\delta S) y(S)} \le$$

$$\le \frac{\sum_{ij} \text{cap}_{ij} \cdot |\phi(i) - \phi(j)|_1}{\sum_{ij} \text{dem}_{ij} \cdot |\phi(i) - \phi(j)|_1} \le \frac{\sum_{ij} \text{cap}_{ij} \cdot d^*(i,j)}{\sum_{ij} \text{dem}_{ij} \cdot \frac{d^*(i,j)}{\Theta(\log n)}} = \Theta(\log n) \frac{\sum_{ij} \text{cap}_{ij} \cdot d^*(i,j)}{\sum_{ij} \text{dem}_{ij} \cdot d^*(i,j)}$$

Thus the best cut we find is $\mathcal{O}(\log n)$ factor away from fractional solution from the LP formulation, thus $\mathcal{O}(\log n)$ approximation to the OPT.

4 L1 Metric Embedding

Assume n-point metric space with metric d. We will embed it into a $k = \mathcal{O}(\log^2 n)$-dimensional L_1 space.

Take some $S \subseteq V$ and let $\phi(v) = d(v, S) = \min_{s \in S} d(v, s)$. By triangle inequality: $|\phi(v) - \phi(u)| \le d(u, v)$. Set S_i to be a random subset of V by including each $v \in V$ with probability $1/2^i$. Set $\phi_i(v) = d(v, S_i)/L$ (where $L = \log n$) the i-th coordinate of ϕ. We need to show only that distances are not reduced too much, as they are always at most than $d(u, v)$.

We define

$$\rho_t = \min_{\rho}(|B(v, \rho)| \ge 2^t \text{ and } |B(u, \rho)| \ge 2^t).$$

Obviously, they are non-decreasing: $\rho_0 \le \rho_1 \le \ldots$

Let us focus on t s.t. $\rho_t < d(u, v)/2$, i.e. $\rho_t = \min(d(u, v)/2, \rho_t)$. Then, w.l.o.g., $|B^o(v, \rho_t)| \le 2^t$ and $B|(u, \rho_{t-1})| \ge 2^{t-1}$ (we denote by B^o an open ball, and $|\cdot|$ counts the *number of points*).

There is

$$\Pr[S_t \cap B^o(v, \rho_t) = \emptyset] \ge e^{-1}$$

and

$$\Pr[S_t \cap B(u, \rho_{t-1}) \ne \emptyset] \ge 1 - e^{-0.5}$$

and those events are independent. If this happens, then it contributes at least $\rho_t - \rho_{t-1}$ to the distance. Then for some constant c,

$$\mathbb{E}[|\phi_t(v) - \phi_t(u)|] \ge c \cdot \frac{\rho_t - \rho_{t-1}}{L}$$

$$\mathbb{E}[|\phi(v) - \phi(u)|_1] = c \cdot d(u, v)/L,$$

To get w.h.p. instead of expectation, we increase number of dimensions to $\mathcal{O}(\log^2 n)$ to get w.h.p. of success for the ball intersections.

References

[LLR95] Nathan Linial, Eran London, and Yuri Rabinovich. The geometry of graphs and some of its algorithmic applications. *Combinatorica*, 15(2):215–245, 1995.

[LR88] Frank Thomson Leighton and Satish Rao. An approximate max-flow min-cut theorem for uniform multicommodity flow problems with applications to approximation algorithms. pages 422–431, 1988.