Advanced Algorithms 09/28, 2018

Exercise 01

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1 Monotone Submodular Maximization

Consider a set $\mathcal{U}$ of n elements that we can buy and a function $f : 2^{\mathcal{U}} \to \mathbb{R}^+$, where for each subset $S \subseteq \mathcal{U}$, the value $f(S)$ determines our profit if we buy exactly the elements of set S.

We assume two properties about this profit function: (A) Function f is monotone in the sense that $f(S) \leq f(T)$ for any two sets S, T such that $S \subseteq T$, and (B) Function f is submodular in the sense that $f(S \cup i) - f(S) \geq f(T \cup i) - f(T)$ for any $i \in \mathcal{U}$ and any two sets S, T such that $S \subseteq T$. In simple words, the submodularity means that the marginal gain that we have by adding i to our purchase set diminishes as we move from one purchase set S to a superset of it T. That is, roughly speaking, the more that we already have in the purchase set, the less extra gain by adding an element to it.

Devise an algorithm that purchases a set S of (approximately) maximum profit, subject to the constraint that $|S| \leq k$, for some given value $k \in \{1, 2, 3, \ldots, n\}$. What approximation factor do you get?

2 Connected Dominating Set

Given an n-node graph $G = (V, E)$, a set $S \subset V$ of vertices is called a *Connected Dominating Set (CDS)* if the following two properties are satisfied: (1) each node $v \in V$ is either in S or has a neighbor in S, (2) the subgraph $G[S]$ induced by S—i.e., the one made of S-vertices and all edges whose both endpoints are in S—is connected. Devise an approximation algorithm for finding a minimum-cardinality CDS.

3 Max-weight Matroid Base (Williamson-Shmoys 2.12)

A matroid $(\mathcal{E}, \mathcal{I})$ is defined by a ground set $\mathcal{E}$ of elements and a collection $\mathcal{I} = \{S_1, S_2, \ldots, S_\ell\}$ of *independent* subsets $S_i \subseteq \mathcal{E}$, subject to the following conditions:

1. For any two subsets S and S' such that $S \subseteq S'$, if S' is independent, then so is S, i.e., $(S' \in \mathcal{I}) \Rightarrow (S \in \mathcal{I})$.
2. For any two independent sets S and T such that $|S| < |T|$, there exists an element $e \in T \backslash S$ such that $(S \cup \{e\}) \in \mathcal{I}$.

We call an independent set S a *base* if there is no $T \in \mathcal{I}$ such that $S \subsetneq T$. That is, S is in some sense maximal with regard to independence.

Suppose that each element $e \in \mathcal{E}$ has a weight $w_e \geq 0$. Devise an algorithm that finds a maximum-weight base of the matroid.

4 Walking on the Hypercube

Consider the d-dimensional hypercube, which has vertex set $\{0, 1\}^d$ and where every two vertices that differ in exactly one coordinate are connected by an edge. Suppose that each edge has a random length drawn from an exponential distribution with mean 1, and the lengths of different edges are independent. Devise an algorithm that finds a walk from the vertex $(0, 0, \ldots, 0)$ to the vertex $(1, 1, \ldots, 1)$ with expected length $O(\log d)$.

5 k-Center

Consider a set of n points $P = \{p_1, p_2, \ldots, p_n\}$ and a distance metric $d : P \times P \to [0, \infty)$ where $d(p_i, p_j)$ indicates the distance between the two points $p_i, p_j \in P$. For a set $S \subseteq P$, and an arbitrary point $p' \in P$, the distance of p' to S is equal to $\min_{p'' \in S} d(p', p'')$. The objective is to find a set $S \subseteq P$ of k points, called centers, such that the maximum distance of any point $p' \in P$ to the set S is minimized. Devise a 2-approximation algorithm for this problem.