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| **Advanced Algorithms** | 09/22, 2017 |

Exercise 01

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# 1 Cut Counting

(A) Use Karger's Random Contraction Algorithm to prove that in each graph with edge connectivity $k$, for any $\alpha \geq 1$, the number of cuts of size at most $\alpha k$ is at most $(2n)^{2\alpha}$.

(B) A result of Tutte and Nash-Williams from 1960s shows that every graph with edge connectivity $k$ contains at least $k/2$ edge-disjoint spanning trees. Use this result to argue that the number of cuts of size at most $\alpha k$ is at most $\Theta(kn^{2\alpha})$.

# 2 Balanced Coloring

Consider a ground set $B$ of $n$ elements and $m$ susbets $S_1, \ldots, S_m \subseteq B$ of this ground set. Prove that there exists a way to color the elements red or blue such that for each of the given $m$ sets, the number of red and blue elements in this set differ by at most $O(\sqrt{n \log m})$.

# 3 Random Sampling in Graphs with Good Expansion

Consider an undirected unweighted graph $G = (V, E)$, for each subset $S \subset V$ of vertices, let $E(S, V \setminus S)$ denote set of edges connecting $S$ to $V \setminus S$, i.e., the edges with exactly one endpoint in $S$. Suppose that for some $\alpha = \Omega(\log n)$, we have that $\frac{|E(S,V \setminus S)|}{|S|} \geq \alpha$, for each subset $S \subset V$.

Prove that if we subsample the edges of $G$ with probability $p = \Omega(\frac{\log n}{\alpha \varepsilon^2})$, then all cuts are concentrated around their expectation, with high probability. That is, with probability $1 - 1/n$, for each cut $(S, V \setminus S)$, the number of sampled edges of this cut is in $(1 \pm \varepsilon)p|E(S, V \setminus S)|$.

**Remark:** You do not need the cut-counting arguments that we saw in the class.

# 4 Sparsification for Hypergraphs

In this exercise, we derive a sparsification for hypergraphs of rank $r$. The rank of a hypergraph is the maximum number of vertices in one hyperedge, that is, in a hypergraph of rank $r$ each hyperedge contains at most $r$ endpoints. We prove that we can sparsify each such hypergraph to merely $O(nr \log n/\varepsilon^2)$ weighted hyperedges[1], while maintaining all cut sizes up to $(1 \pm \varepsilon)$. Notice that a priori, such a hypergraph might have up to $\sum_{i=2}^{r} \binom{n}{i} \gg nr \log n$ hyperedges. For a non-trivial partition of the vertices $(S, V \setminus S)$—where $S \neq \emptyset$ and $S \neq V$—the corresponding cut in the hypergraph is defined as the set of all hyperedges that have at least one endpoint in each side of the cut.

(A) Using an extension of the contraction algorithm to hypergraphs, prove that the number of $\alpha$-min cuts is at most $n^{O(r\alpha)}$.

[1] This is $\tilde{O}(n^2)$ edges, in the worst case of $r = n$. However, as far as we know, $\tilde{O}(n)$ edges may suffice. Obtaining such a sparsfication for hypergraphs remains a (major) open question.

(B) Prove that a uniform sampling of each hyperedge with probability $p = \Omega(\frac{r \log n}{\varepsilon^2 k})$ preserves the size of each cut around its expectation, up to a $(1 \pm \varepsilon)$ factor. Here, $k$ denotes the size of the minimum cut.

(C) Follow the steps of what we did in the class for graphs to construct a non-uniform sampling that produces a sparsifier with $O(nr \log n/\varepsilon^2)$ edges.