1 Distributed Hypergraph Coloring

Design a single-round distributed vertex coloring algorithm for \( r \)-uniform hypergraphs with the smallest number of colors possible.

More precisely, the network is modeled as an \( r \)-uniform hypergraph \( H = (V, E) \) where \( n = |V| \) and for each hyperedge \( e \in E \), we have \( e \subset V \) and \( |e| = r \). Nodes have ids 1 to \( n \). The maximum node-degree is at most \( \Delta \), i.e., \( \forall v \in V, |\{e \in E | v \in e\}| \leq \Delta \). A legal \( k \)-coloring \( C \) of \( H \) is a mapping from \( V \) to \( \{1, 2, \ldots, k\} \) such that no hyperedge \( e \) is monochromatic, that is,

\[
\forall e \in E, |\{x | x \in \{1, 2, \ldots, k\}, u \in e, C(u) = x\}| \geq 2.
\]

Design a deterministic coloring function that for any \( r \)-uniform hypergraph \( H \) with \( n \) nodes and maximum degree \( \Delta \), receives the id of each node \( v \) and also (the ids of the other nodes in each of) the hyperedges incident on \( v \) in \( H \) and outputs the color of \( v \) in a way that the colors of all nodes together form a legal \( k \)-coloring of \( H \). Try to find the smallest possible \( k \). Note that \( k \) will be a function of \( n \), \( r \), and \( \Delta \).

**Bonus Points** Prove a non-trivial lower bound on \( k \) as a function of \( n \), \( r \) and \( \Delta \).