1 Randomized Ruling Set

In lecture 4, we saw a deterministic algorithm for computing a \((k, k \log n)\)-Ruling Set in \(O(k \log n)\) rounds. Here, we analyze a simpler and faster but randomized algorithm.

Let each node \(v\) pick a random \(6 \log n\)-bit number \(r_v \in \{0, 1\}^{6 \log n}\). Then, make \(v\) join the set \(S\) if the random number \(r_v\) of \(v\) is strictly larger than the random number \(r_u\) of all neighbors \(u\) of \(v\).

Clearly \(S\) is an independent set. Show that with probability at least \(1 - \frac{1}{n}\), for each node \(v\), there is at least one node \(s \in S\) within \(O(\log n)\) hops of \(v\). Also explain how to extend this idea to an algorithm for generating a \((k, O(k \log n))\)-Ruling Set.

2 Minimum Spanning Tree (MST) of a Subgraph

In lecture 5 we showed that a MST of a graph \(G\) can be computed in time \(\tilde{O}(\sqrt{n} + D)\). Now assume there is a connected subgraph \(H\) of \(G\) and we want to compute an MST \(T_H\) of \(H\), but are allowed to use communication in \(G\) (not only communication in \(H\)). How fast can you compute an MST \(T_H\)? The argument should take only a few lines.

Hint: Let \(D_H\) be the diameter of \(H\). The best time is one of the following:

\[
\tilde{O}(\sqrt{n} + D_H) , \quad \tilde{O}(\sqrt{n \cdot D_H} + D) , \quad \tilde{O}(\sqrt{n} + D).
\]

3 A Lower Bound for Minimum Cut Approximation

In lecture 6 we showed approximation lower bounds for MST and Single Source Shortest Paths (SSSP). Now we consider the problem of approximating a Minimum Cut (see definition below). Prove that \(\alpha(n)\)-approximating a minimum cut takes \(\Omega(\sqrt{n} + D)\) rounds.

Definition: A set of edges \(E' \subseteq E\) is a cut of \(G\) if \(G\) is not connected when we delete \(E'\). The minimum cut problem is to find a cut of minimum weight. A cut \(E''\) is an \(\alpha(n)\)-approximation to a Minimum Cut \(E'\), if

\[
\omega(E') \leq \omega(E'') \leq \alpha(n) \cdot \omega(E').
\]