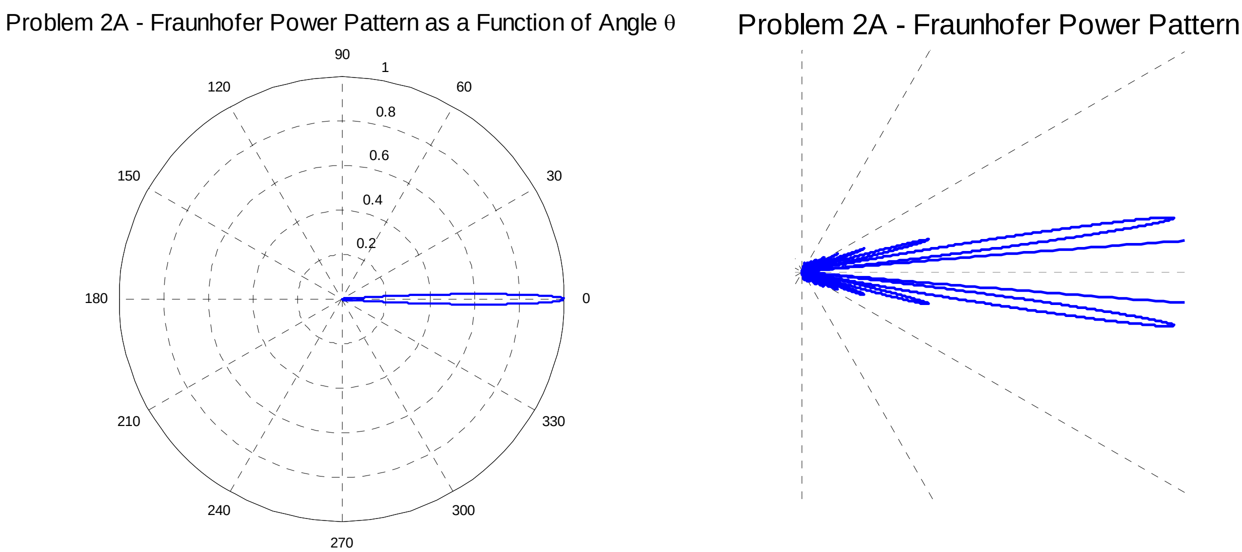
Problem Set V – The Two-Dimensional Convolution Theorem

Problem #2 – Fraunhofer Intensity Pattern

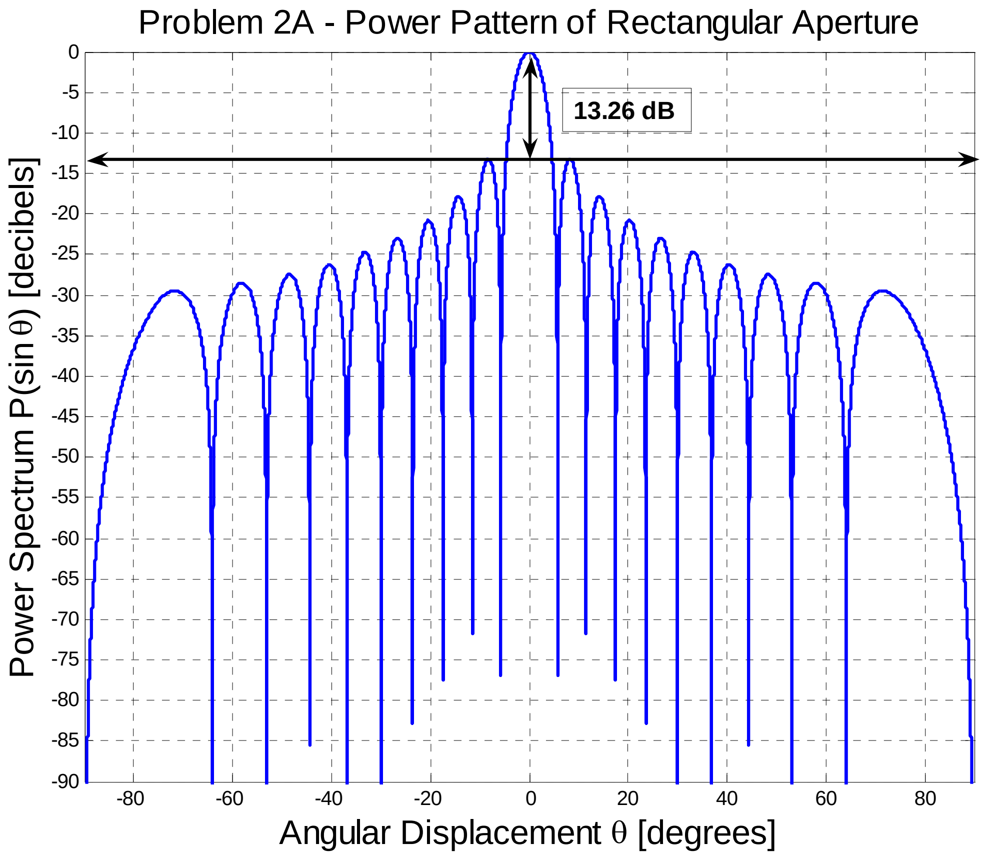
(a.) For a rectangular aperture of 1 meter across $\Pi(x)$ and illumination of wavelength 10 cm, the Fraunhofer intensity pattern is proportional to a squared sinc, according to the magnitude-squared Fourier transform of the rect function:

$$\boldsymbol{I \propto \left| \int_{-\infty}^{\infty} p(x) e^{-j2\pi x \frac{\sin\theta}{\lambda}} dx \right|^2 = sinc^2\left(\frac{\sin\theta}{\lambda}\right)}$$

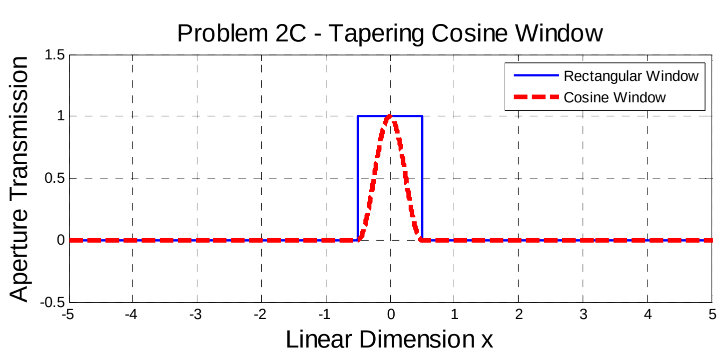


In either case, we obtain the squared sinc as our spectral power pattern.

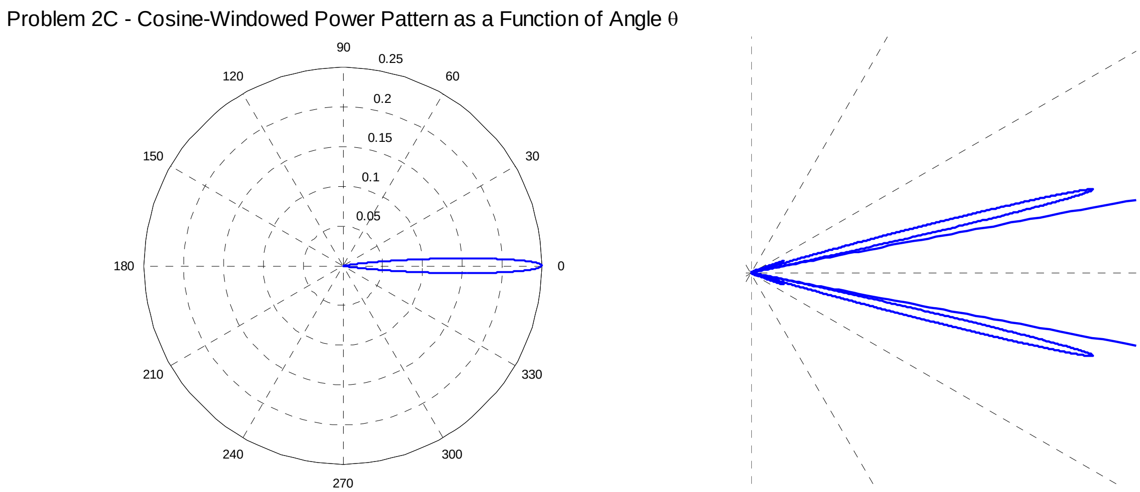
(b.) On a decibel scale, the ratio of the highest part of the pattern to the greatest sidelobe is approximately **13.261472 decibels**, as pictured below:



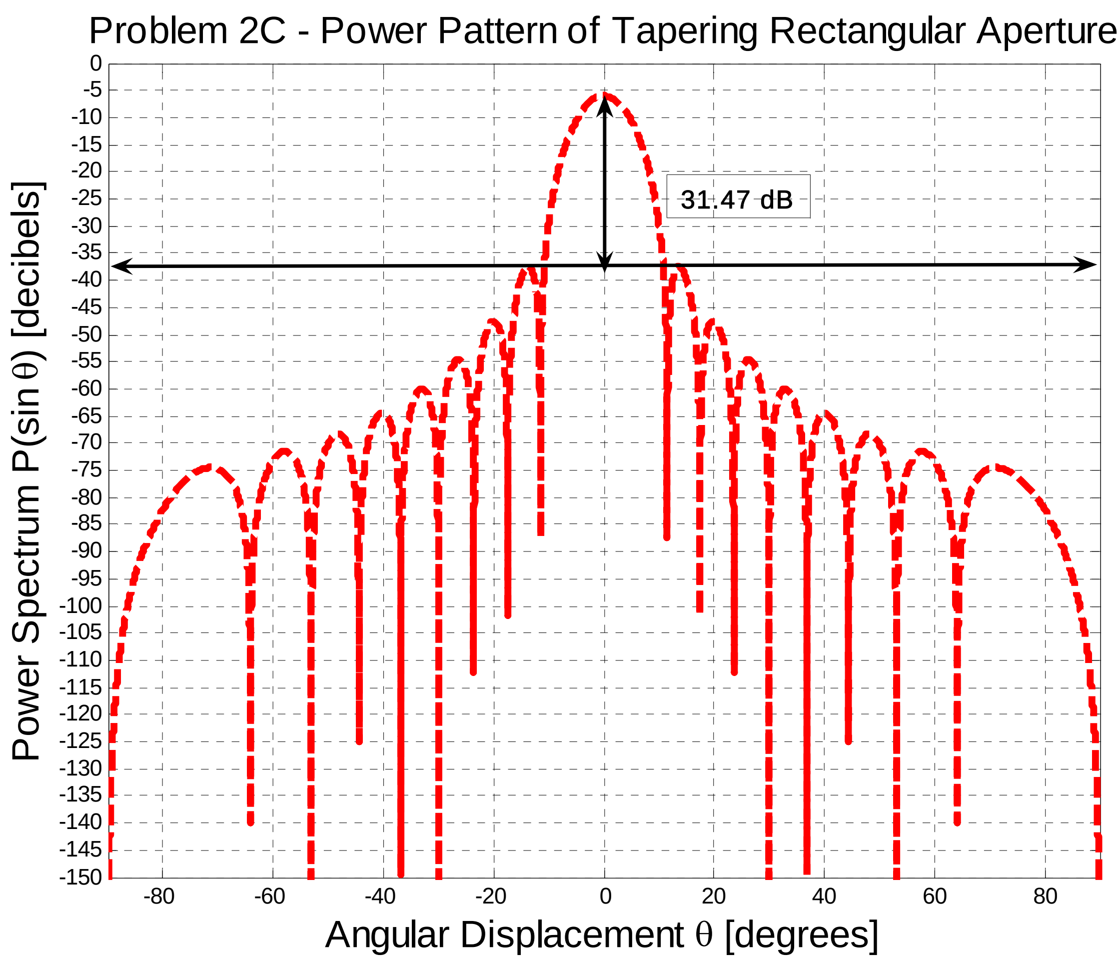
(c.) However, we can improve the sidelobe ratio by tapering the aperture smoothly, essentially smoothening the sharp transition in the ideal window so that we need fewer high frequency components to represent it and hence lower sidelobes in the Fraunhofer intensity pattern. In other words, because the Fourier transform of a sharp transition such as the rectangular aperture requires high frequency components, the spectral sinc pattern supports high sidelobes that spread the energy across several different spatial frequencies. By multiplying our rectangular aperture by a smoothly tapering cosine, we can temper some of the high sidelobes. We use the following window:



Mathematically, we express this window as $\frac{1+\cos 2\pi x}{2}$. Applying this cosine taper to the window prior to computing the Fourier transform, we obtain the following Fraunhofer intensity pattern:

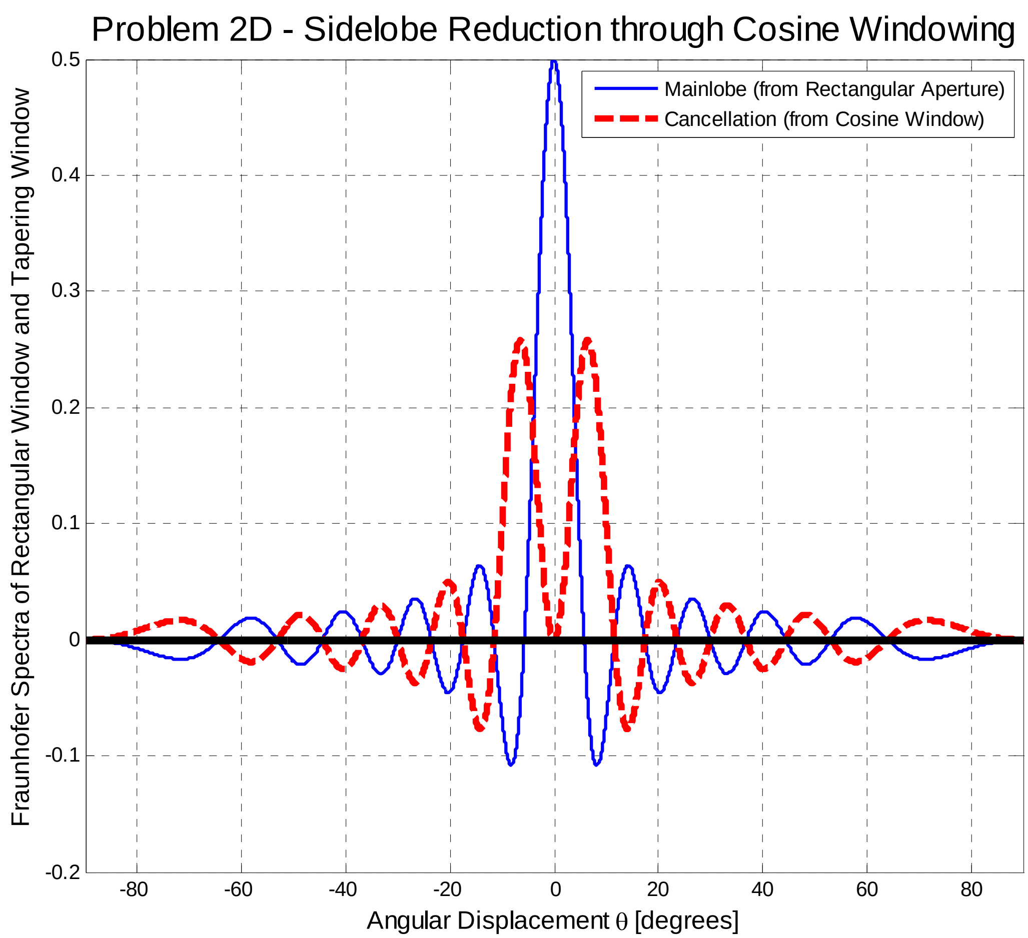


Unraveling the angular axis into a straight line, we obtain the following intensity plot [in dB]:



As the power pattern divulges, the taper has increased the peak-to-sidelobe ratio from 13.26 dB to $\boxed{\textbf{31.467313 dB}}$. The central peak is now approximately 1400 times more intense than the first sidelobe, and the polar plot now bears fewer of the secondary side fringes seen previously in the original polar pattern.

(d.) Furthermore, we can trace our sidelobe cancellation to its aboriginal form in the transform of the cosine in the convolution theorem:



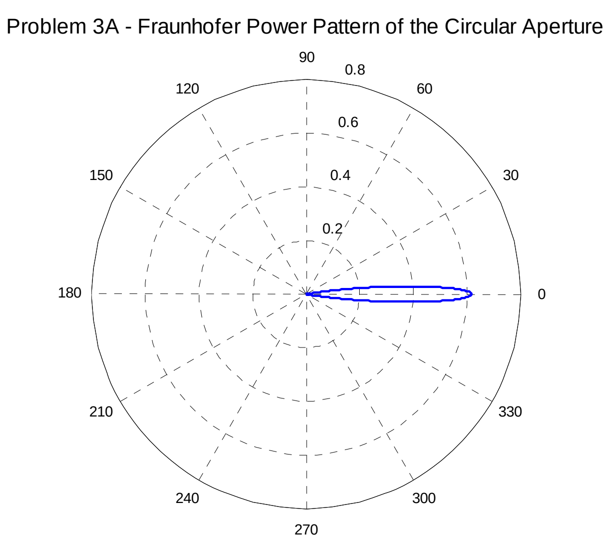
The original rectangular aperture contributes a sinc-like amplitude centered in the figure. The cosine window, on the other hand, generates two adjacent sinc patterns offset just enough so that their sidelobes weaken the sidelobes of the central sinc through additive cancellation (shown in red). Because the cosine window's sinc pair is shifted just one lobe away from the central pattern, the sidelobe signs oppose the original sinc's sidelobe signs, allowing the cancellation to occur.

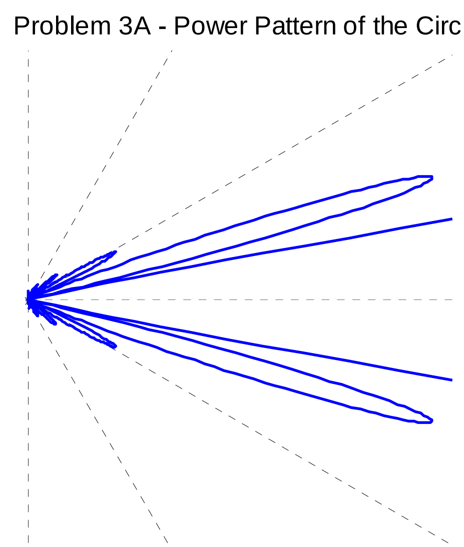
Problem #3 – Circular Antenna Fraunhofer Pattern

A circular antenna with one meter aperture $\Pi(r)$ with ideal cutoff generates a jinc-like Fraunhofer pattern, whose square we observe as intensity.

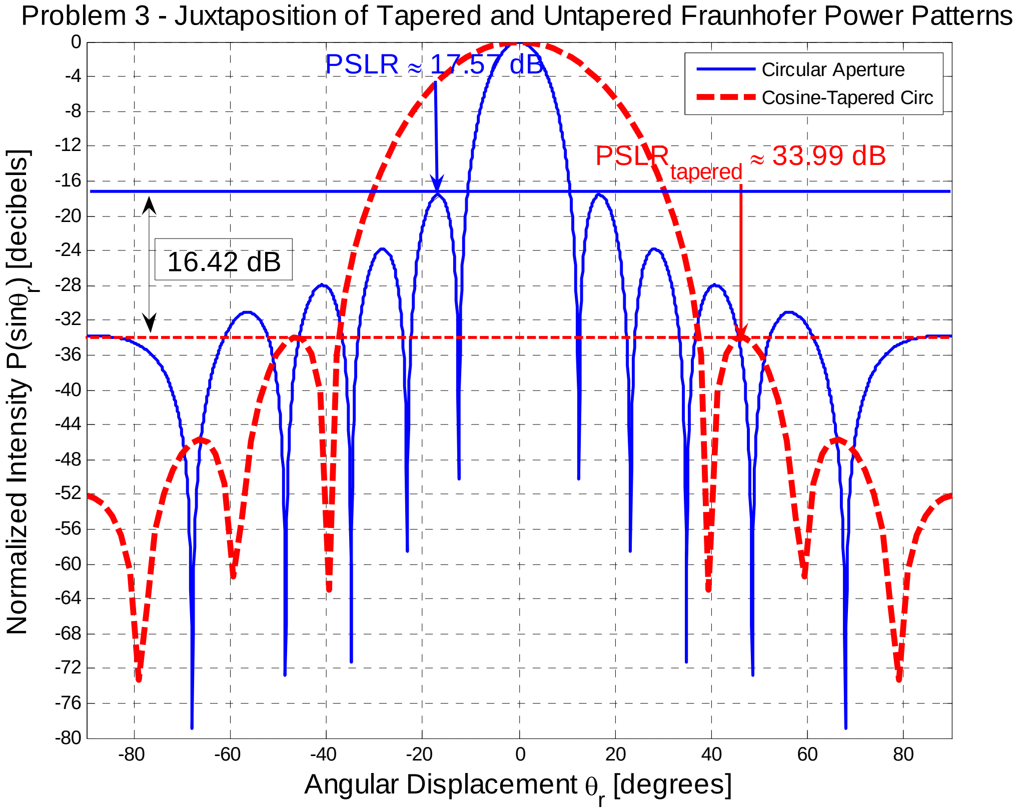
$$I \propto \left| \int_{-\infty}^{\infty} p(x) e^{-j2\pi r \frac{\sin\theta_r}{\lambda}} r dr \right|^2 = jinc^2\left(\frac{\sin\theta_r}{\lambda}\right)$$

Analogous to the sinc-like power pattern of the one-dimensional rectangular window, the jinc-like power pattern of the two-dimensional circ suffers from strong sidelobes, the curse of a sharp transition. We plot a one-dimensional cross-section below:

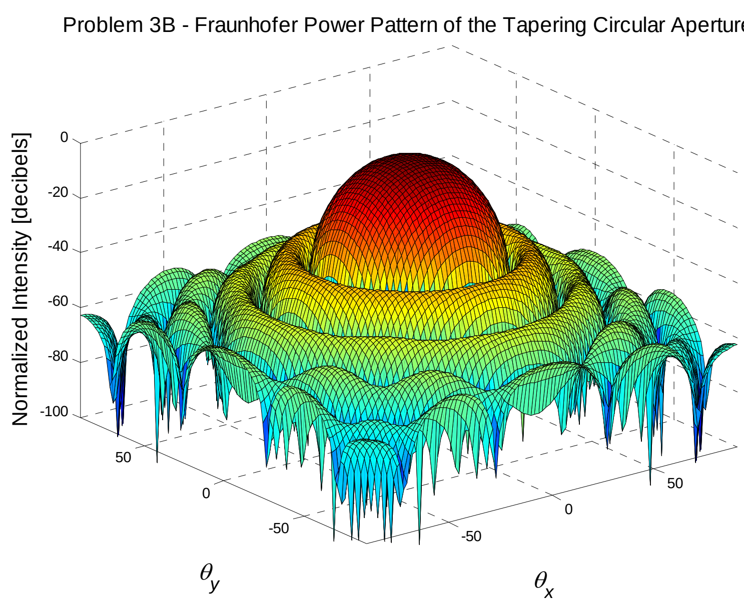




(a.) The peak-to-sidelobe ratio of this circular antenna pattern initially appears to be around $\boxed{\textbf{17.57 dB}}$, but we can hone its performance through tapering yet again employing the same techniques we tapped for the rectangular aperture with the radially cosinusoidal window $\frac{1+\cos 2\pi r}{2}$. In order to facilitate sidelobe height comparison, we graph only a central slice along the two-dimensional jinc pattern:

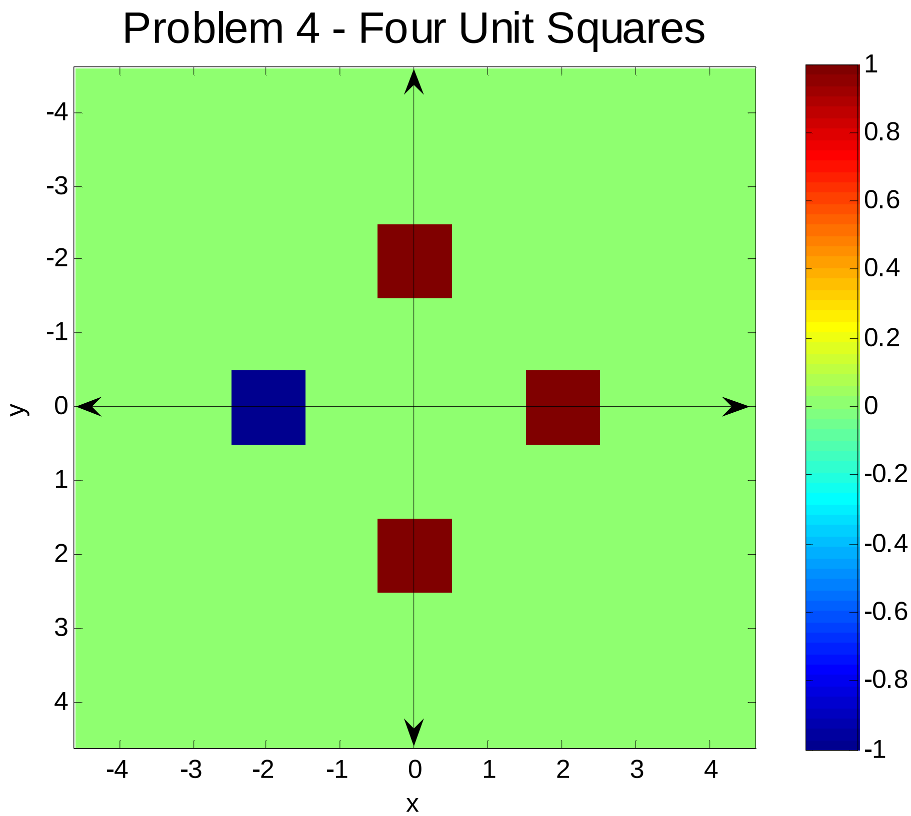


(b.) According to the power pattern cross-section plotted above, the PSLR improves to approximately **33.99 dB** following cosine function tapering. Following normalization, our tapering operation suppresses the first sidelobe by approximately **16.42 dB**! We illustrate the richness of the two-dimensional pattern from which we extracted a central slice:

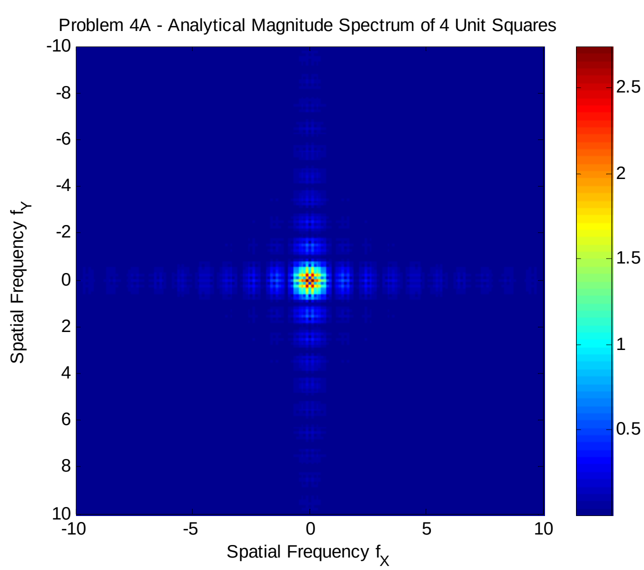


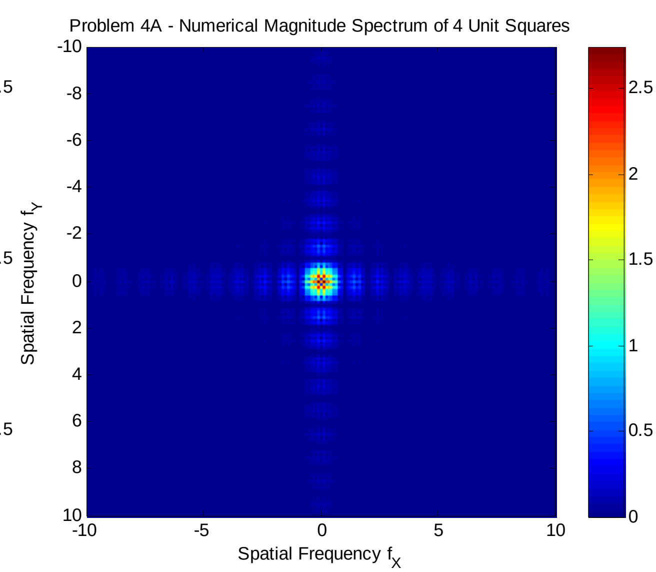
Problem #5 – Four Unit Squares

We transform the following spatial distribution of four unit squares, one of which is negative signed:

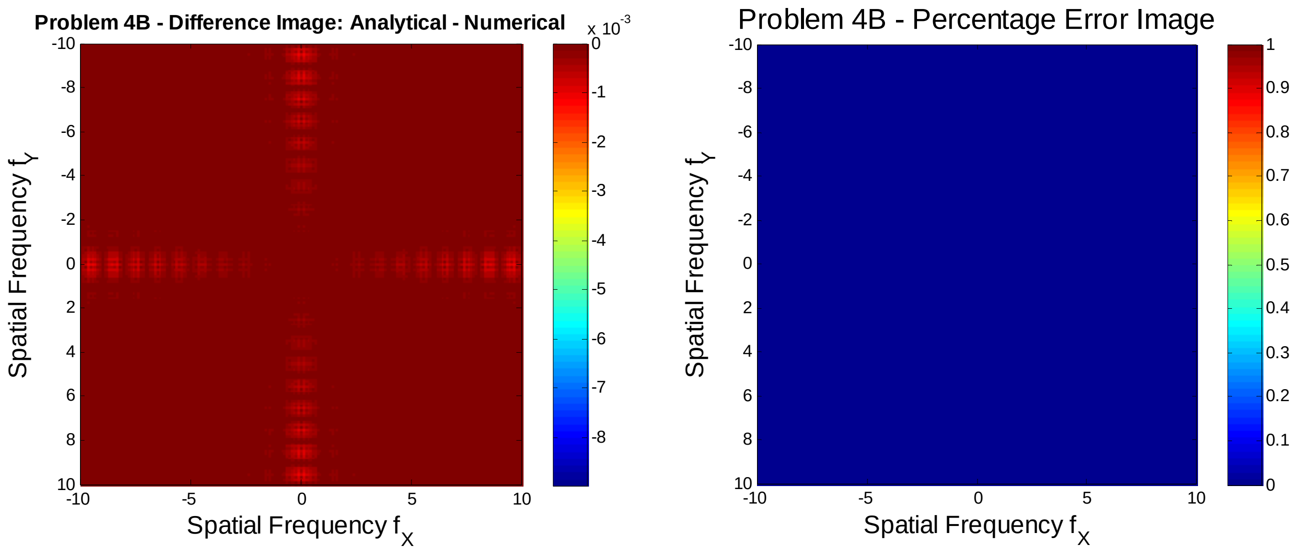


(a.) Our analytical solution $F(u,v) = 2\,sinc(u)\,sinc(v)\,[\cos(4\pi v) - j\sin(4\pi u)]$, when sampled discreetly, nearly perfectly matches the fast-Fourier-transformed numerical solution pointwise:





(b.) Finally, we compute the pointwise difference between the two methods by subtracting the numerical solution from the analytical one:



This difference image, strictly negative, reveals that the numerical solution almost always overestimates the true analytic solution, which comes as no surprise since the discrete Fourier transform packs all the energy of the analytical solution outside the bounds of the image within the limits imposed by the finite nature of the DFT and FFT. In other words, because our analytical image is the mere evaluation – or sampling – of a band-unlimited function, not all its energy is present within the bounds of the image; meanwhile, the discretized numerical transform, by definition, accounts for the entire spectrum in a finite number of coefficients. The most severe error occurs along the two principal axes, where the spectrum changes most rapidly. As the percentage error image reveals, however, even these discrepancies are minor when compared to the actual transform values; indeed, the FFT close approximates the sampled spectrum.