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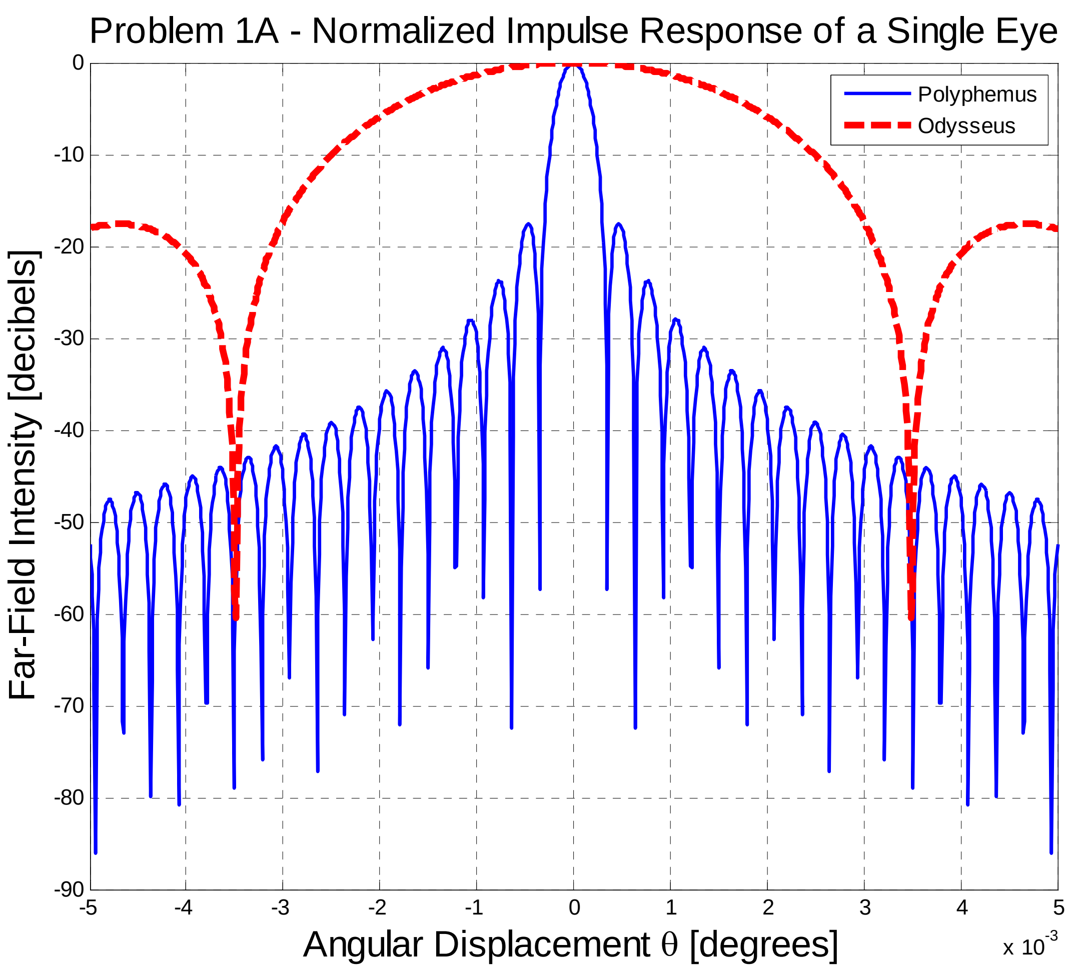
EE 262 – 2D Imaging

Problem Set VIII – The Optical Transfer Function

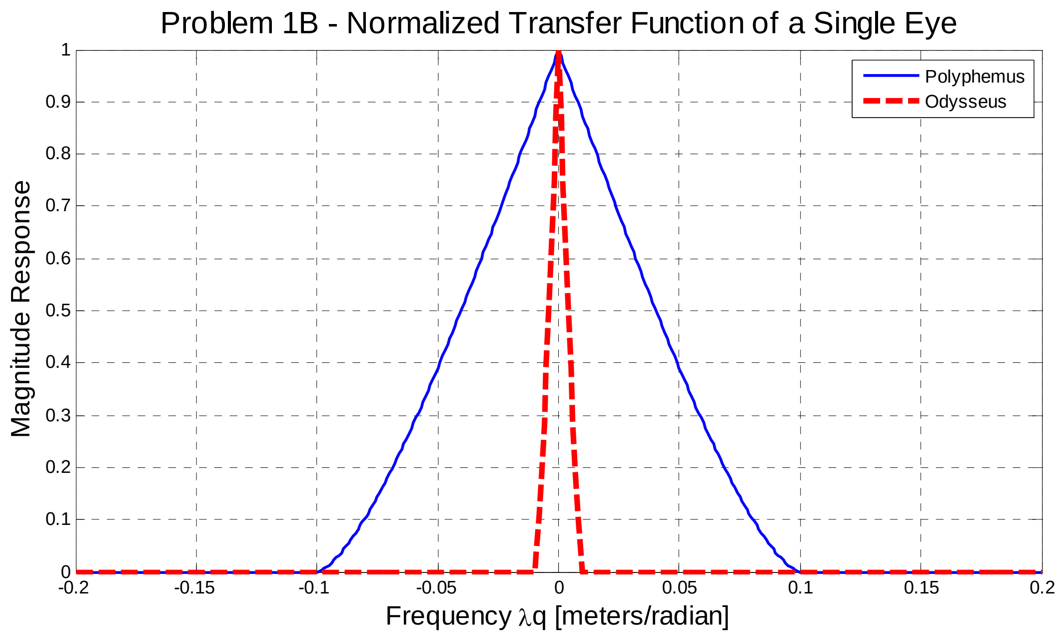
Problem #1 – Circular Vision

(a.) Because the eye is circular in diameter, both Polyphemus and Odysseus possess *circ* pupils, where Polyphemus' aperture is centered and Odysseus' aperture is shifted. Because the pupil functions are circular, the incoherent point spread functions have the shape of squared jincs:



Thus, along the principal axes, we see that the wider Cyclops pupil yields a sharper, narrower impulse response main lobe; Odysseus' main lobe, pictured in red, spreads over a larger angular displacement, indicating that he sees a wider range of view but can discern details much less coarsely than Polyphemus, who sees a narrower visual field to much higher accuracy. This behavior makes sense; Odysseus has much smaller, narrower eyes and hence a wider transfer function by duality.

(b.) We plot the transfer function along a principal axis for both Odysseus and Polyphemus:



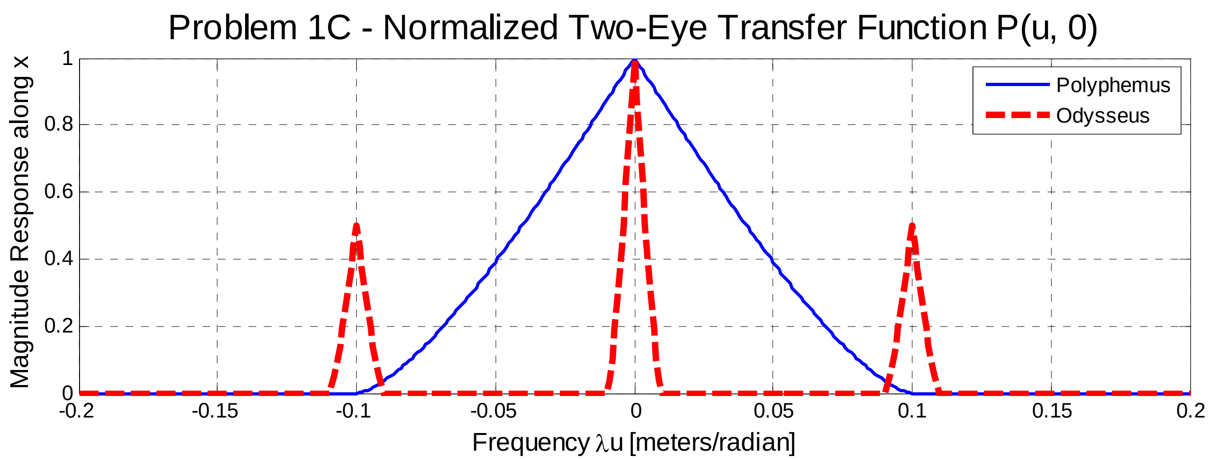
The transmission region is circular, so the optical transfer function – the autocorrelation of two circular regions, or the ratio of overlapping area to total area – is a *chat* function:

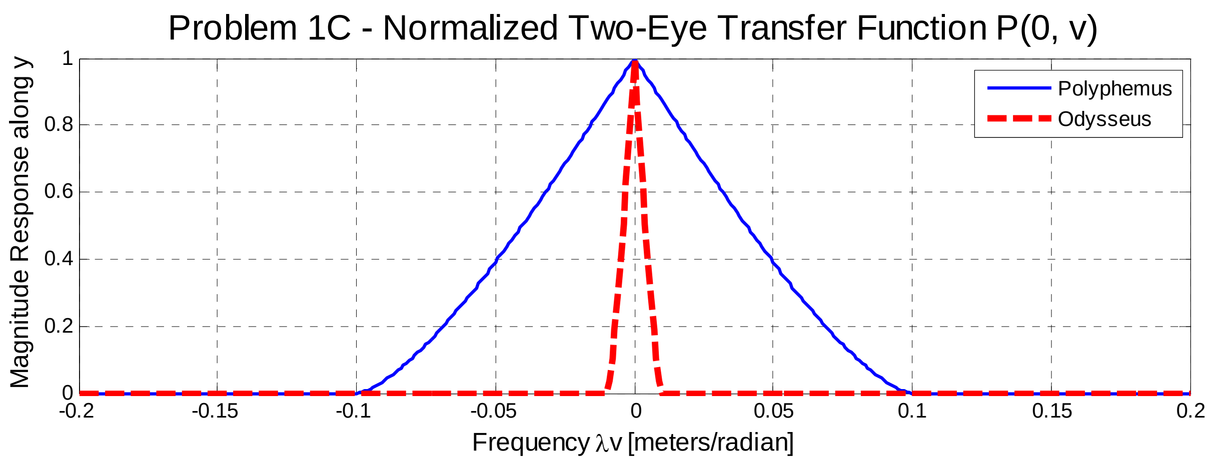
$$chat(q) \triangleq \frac{1}{2}\left(\cos^{-1} x - x\sqrt{1-x^2}\right)$$

$$\mathcal{H}(q) = D^2\, chat\left(\frac{q}{D}\right)\Pi\left(\frac{q}{D}\right)$$

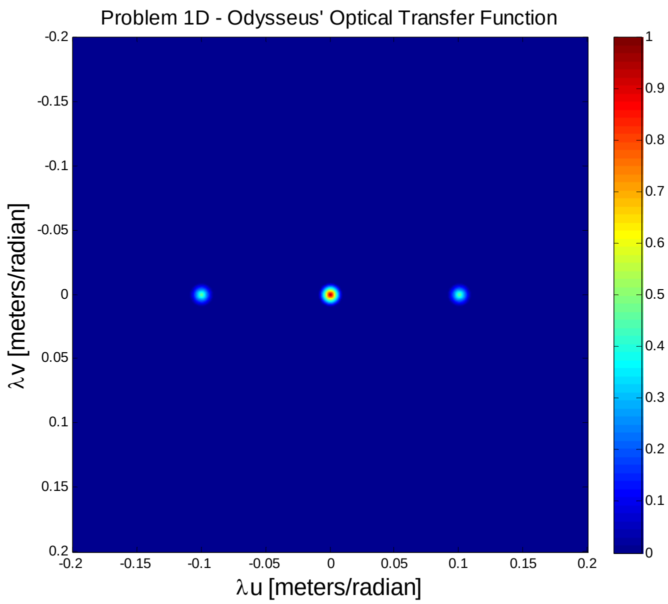
Unlike the incoherent point spread functions, Odysseus' optical transfer function is narrower than Polyphemus. In effect, Odysseus has much smaller, narrower pupils, so his eyes experience a shorter range of angles; as a result, his eyes receive fewer rays from strange, oblique angles, resulting also in a more limited spatial frequency range. On the other hand, Polyphemus' lone eye collects rays from a multitude of different directions, so his angular range is increased, thereby increasing the scope of his transfer function. To view this mathematically, note that the function $\mathcal{H}(q) = D^2\, chat\left(\frac{q}{D}\right)\Pi\left(\frac{q}{D}\right)$ increases in width when D increases, whereas the function contracts for lower values of the pupil diameter, representing the decreased bending of rays received.

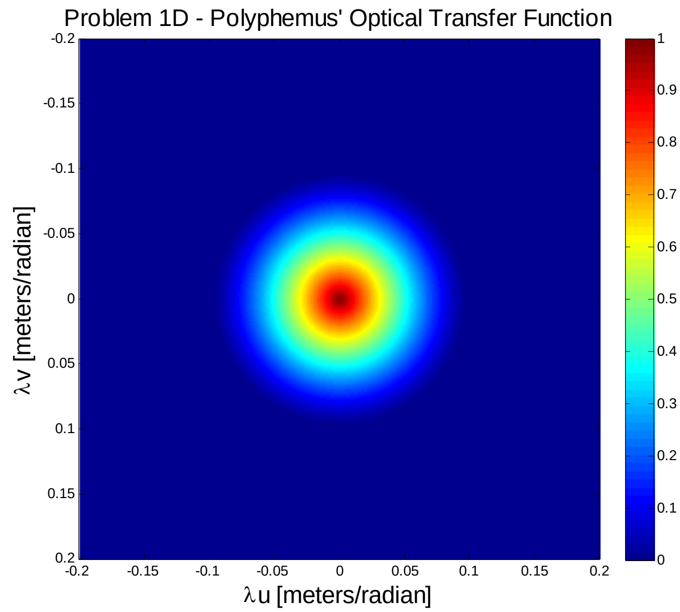
(c.) Suppose Odysseus can combine the signals from each of his two eyes interferometrically, so that his transfer function along the axis containing both eyes also experiences a combined response:





In two dimensions, Odysseus' optimal transfer function is a pair of circs (two eyes), so his transfer function contains a high central peak representing initial overlap and two islands to represent the overlap present when one eye's response slides past the other eye's response.





(d.) Despite their high frequency sensitivity, Odysseus' eyes cannot perceive intermediate frequencies due to the narrowness of their optical transfer function. In order to obtain the frequency sensitivity of Polyphemus, Odysseus could rotate his head, turning around in full circle to cover the entire circumference at $d = 0.1$ meter per radian away.

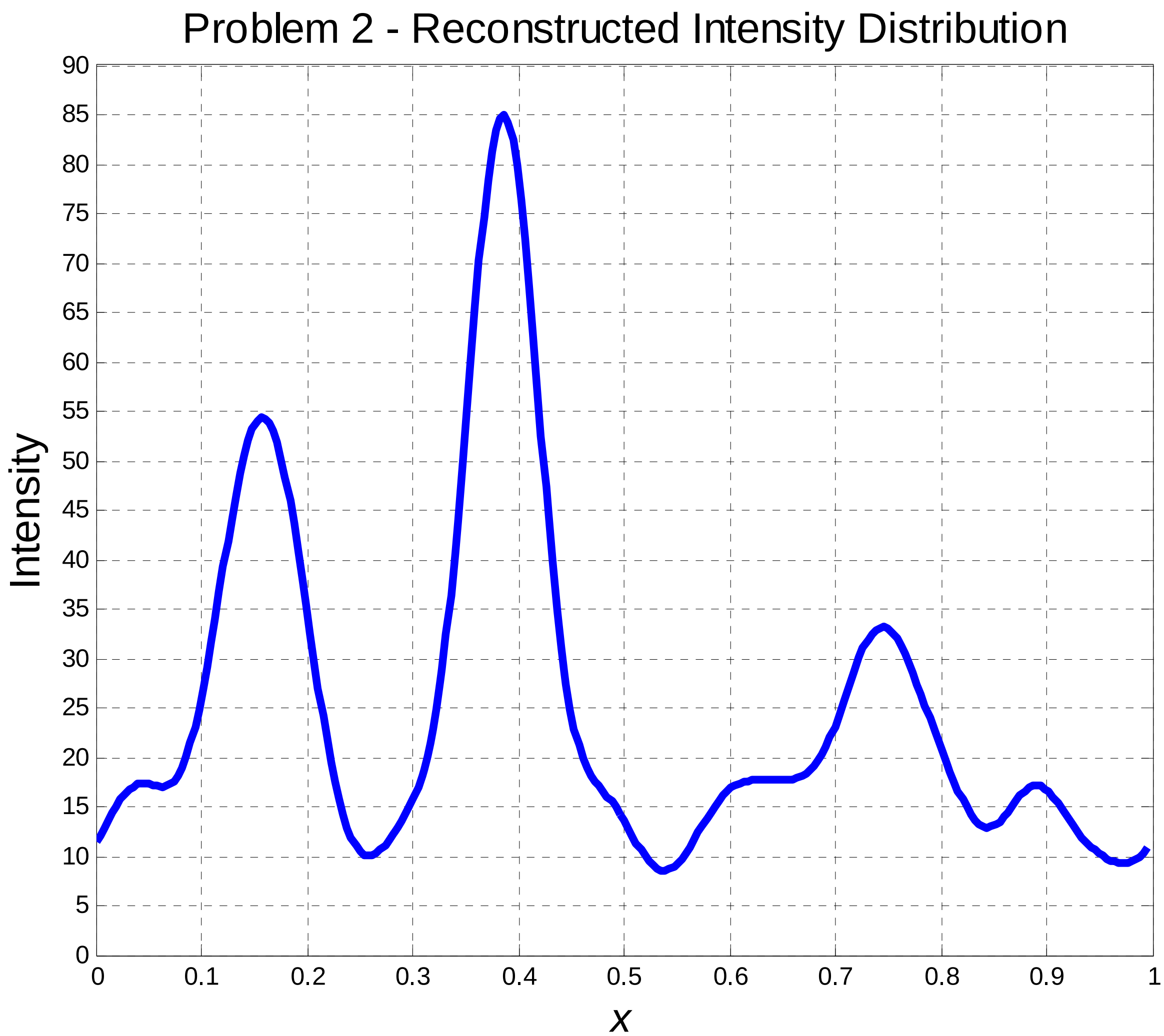
However, rotation cannot completely solve Odysseus' medium frequency sensitivity; only lenses and mirrors that alter or filter the high frequencies can allow Odysseus to see in the intermediate range between the low-frequency spectrum and the high-frequency spectrum consummated through rotation.

Alternatively, Odysseus could elect to obtain several different looks.

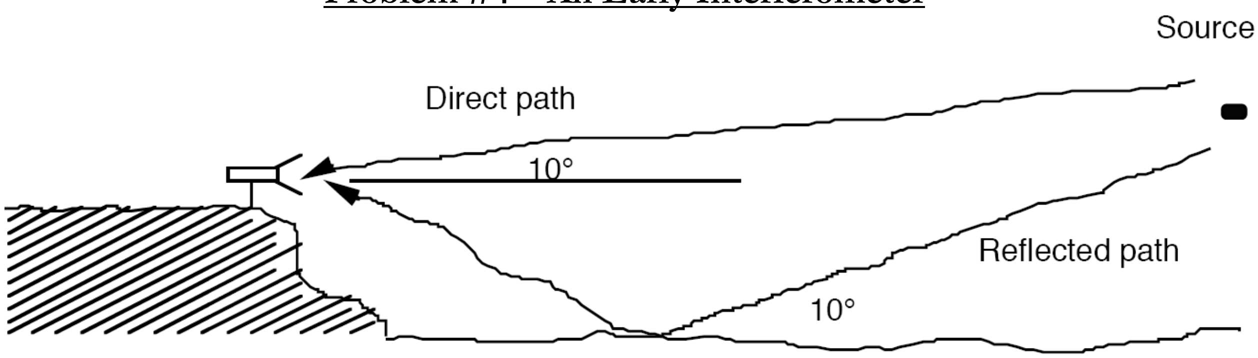
Problem #2 – Fourier Components for Interferometry

Parsing the raw data into physical quantities and building an array of complex values, we synthesize the Fourier components one by one through summation of complex exponentials. After we fully synthesize the Fourier spectrum, we compute the intensity by computing the absolute value:

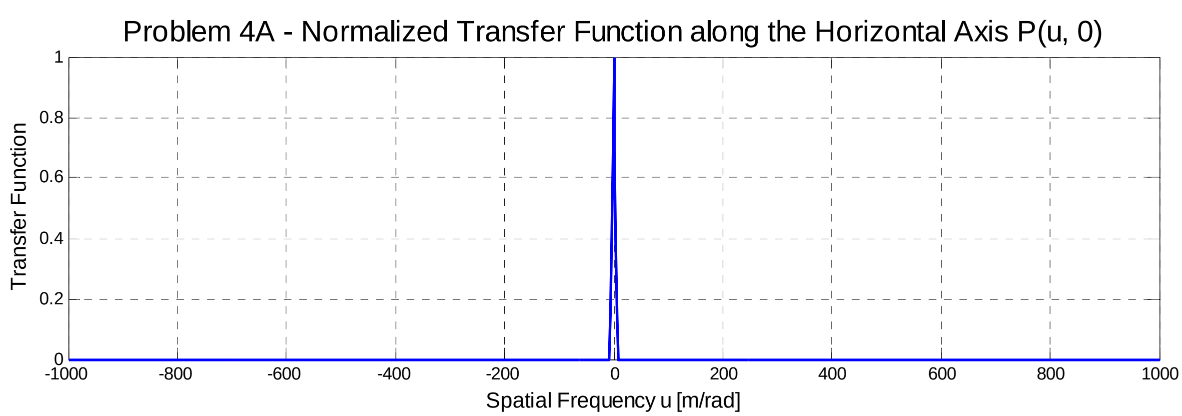
$$I(x) = \left| \sum_{i=-14}^{14} c_k e^{-j2\pi kx} \right|$$

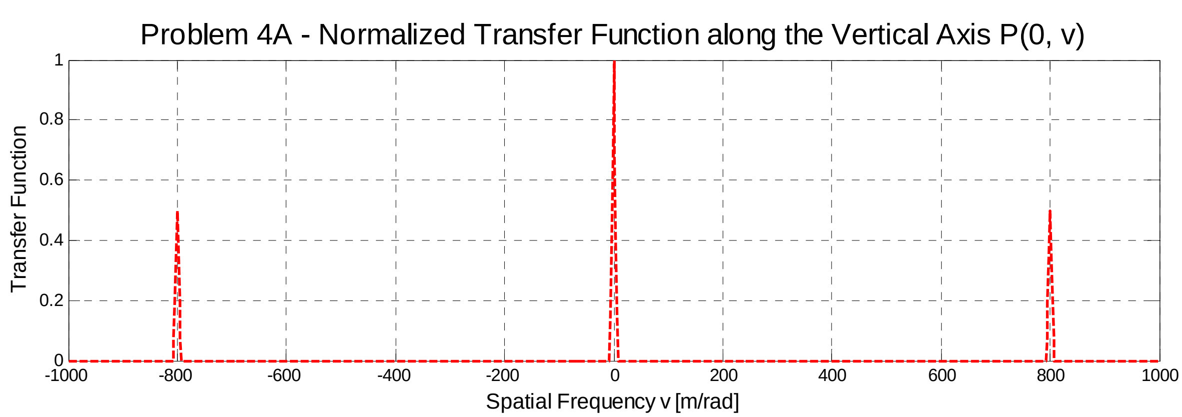


Problem #4 – An Early Interferometer



The interferometer resembles a pair of equidistant point sources when we neglect the attenuating effect of the lower beam's reflection. If we assume a reflection coefficient of unity, then the receiver will exhume two identically separated copies of the source. If we further assume that the aperture is a round 8-meter-diameter circ function, then the antenna will canvass two symmetrically displaced circular pupils, much as Odysseus interferometrically accrued the images from his two symmetrically spaced eyes. Thus, the interferometer transfer function will assume the form of the ubiquitous chat function, with two autocorrelated circular regions eliciting an arccosinusoidal variation:





(b.) Because the transfer function of the interferometer vanishes between 10 cycles/radian and 390 cycles/radian, any frequency content with spatial terms of

$$e^{-j\frac{2\pi}{\lambda}\sin\theta_y \cdot \lambda q}$$

with $10\,\frac{\text{cycles}}{\text{rad}} < \text{q} < 390\,\frac{\text{cycles}}{\text{rad}}$ would be invisible to the interferometer. In essence, this region of zero transfer function is like a blind spot in the spectrum, much like the intermediate frequencies in Odysseus' eyes' transfer function were blind spots to his vision. One invisible distribution is

$$f(\theta_x, \theta_y) = \cos\left(\frac{2\pi}{\lambda} \times \lambda \left[100\,\frac{cycles}{radian}\right] \times \sin\theta_y\right)$$

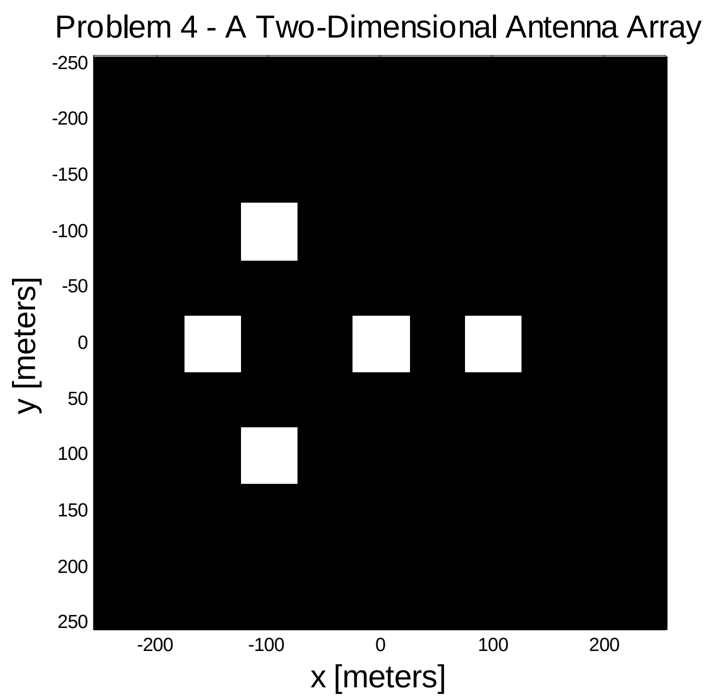
(c.) Random waves on the ocean surface would serve as smaller facets or roughness that would tilt the surface at various haphazard angles, effectively scattering the reflected waves and attenuating their contribution to the far-field pattern, which will no longer be symmetric as in the smooth specular reflector. In order to infer the sea state, or the RMS roughness of the ocean surface, we can estimate the mean surface tilt through repeated measurement of the local change in angle Δ, which would appear in our power pattern as

$$4D^2 jinc^2\left(\frac{D\sqrt{\theta_x^2 + \theta_y^2}}{\lambda}\right) cos^2\left(\frac{2\pi}{\lambda}\sin[\theta_y - \Delta]\right)$$

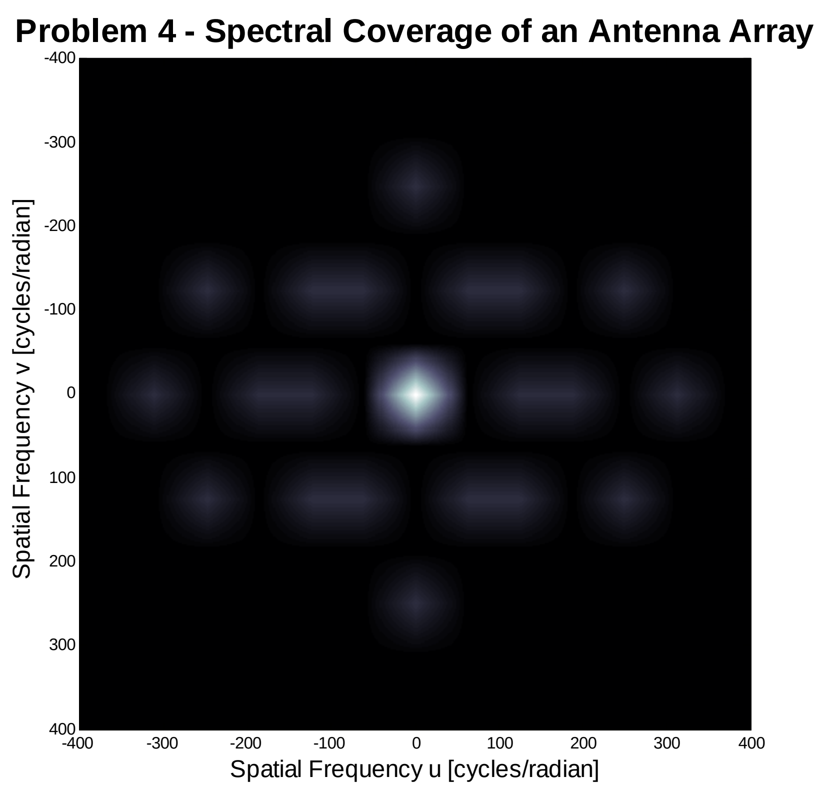
Detecting this change Δ is tantamount to plumbing the local tilt.

Problem #4 – Spectral Coverage of an Antenna Array

We seek the spectral coverage of the following quasisymmetric square antenna array:

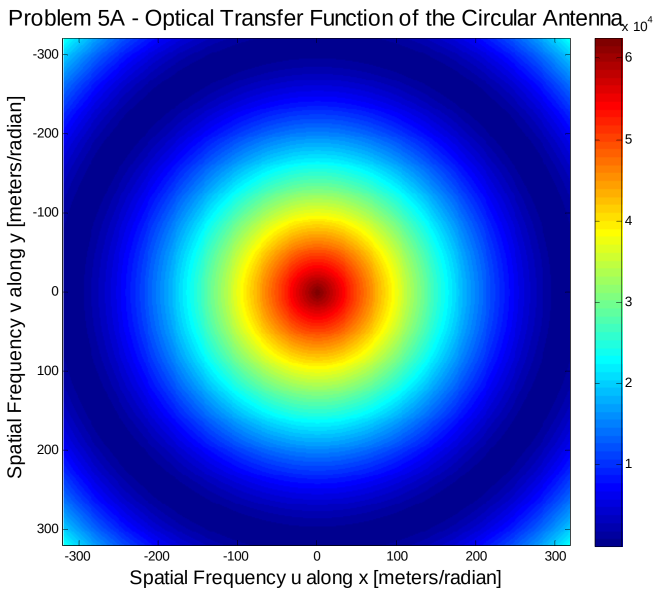


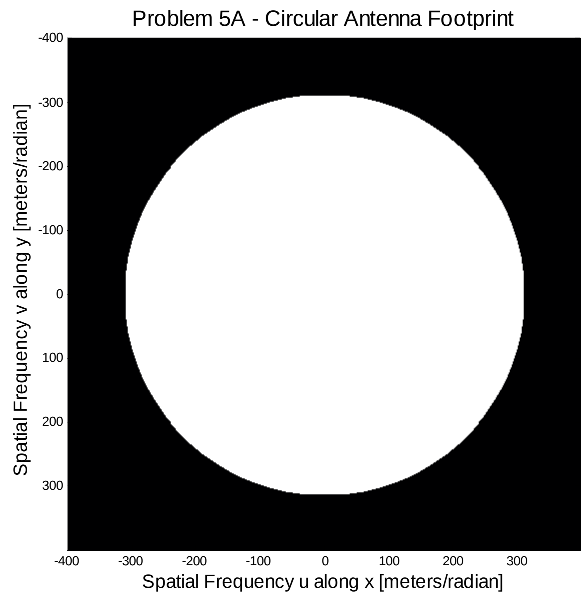
We transform each shifted rect function as a complex exponentially modulated sinc function, and sum:



Problem #5 – Imaging the Sky

(a.) Assuming that we acquire our vista through a radio telescope of diameter 250 m…

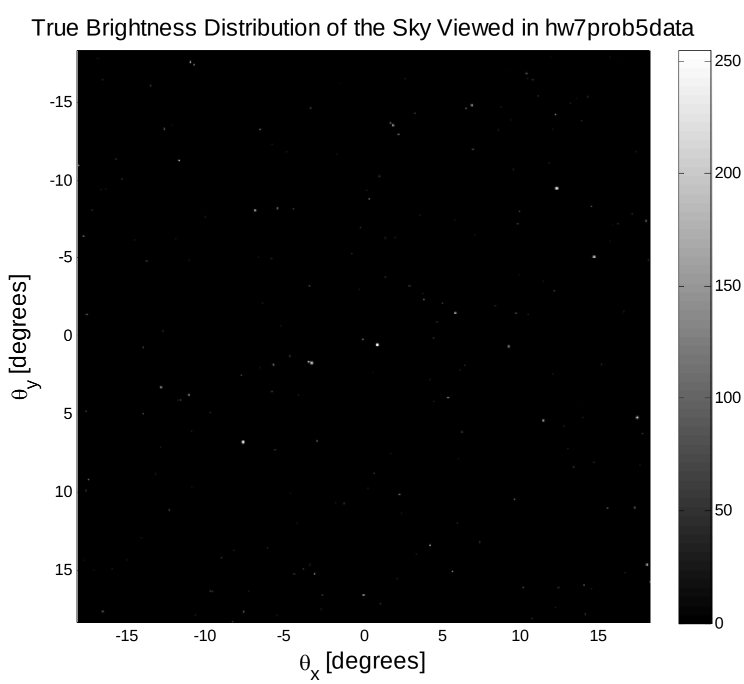


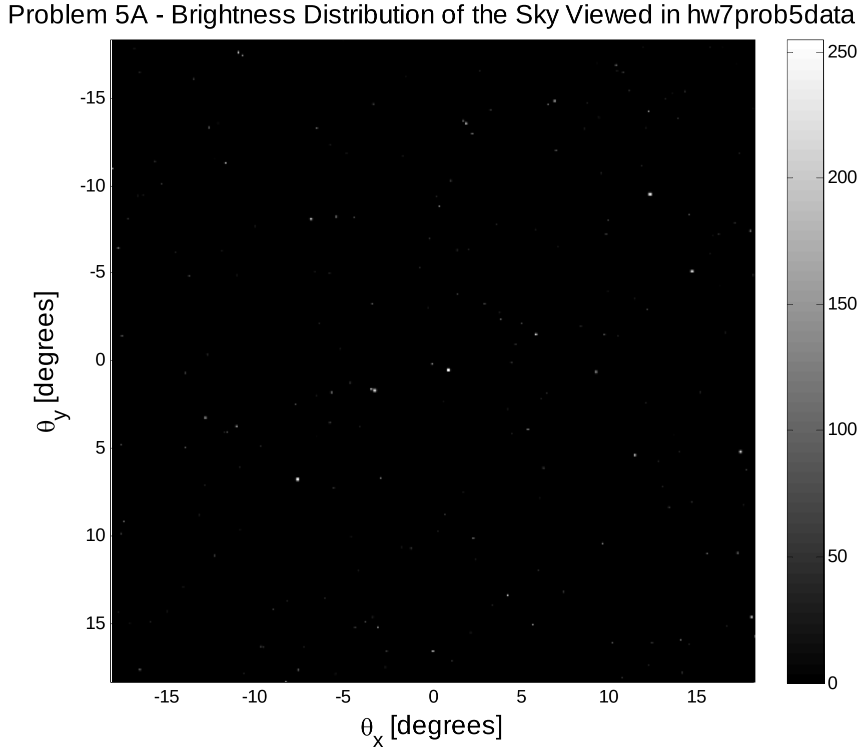


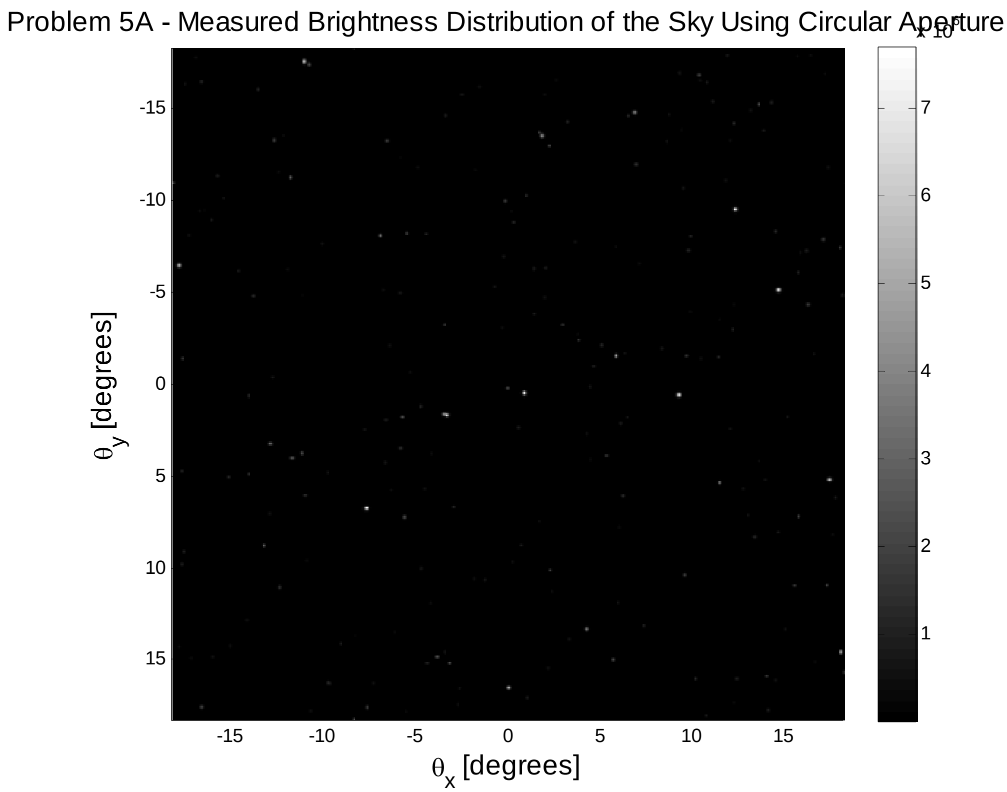
The total area of our antenna is simply the area of a circle with radius 125 m.

$$A = \pi R^2 = \pi(125\ m)^2 = 15{,}625\pi \approx \mathbf{49,087.38521\ m^2}$$

Tapping this telescope's circular transfer function we can image the sky by calculating the source distribution of radio energy. We juxtapose our acquired distribution with the true distribution:

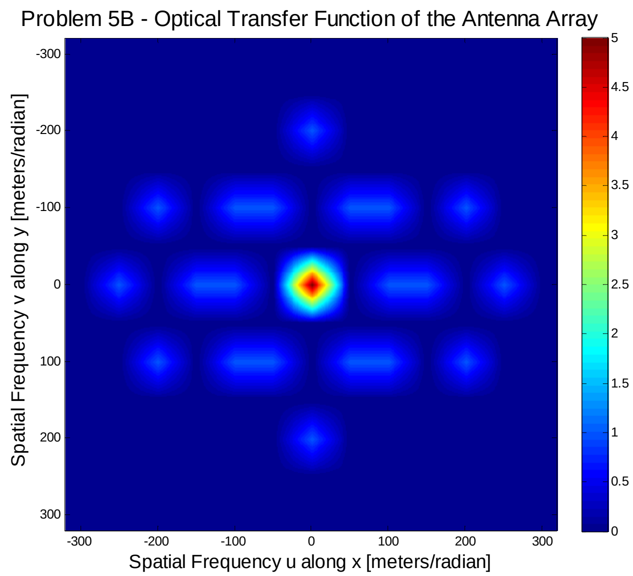


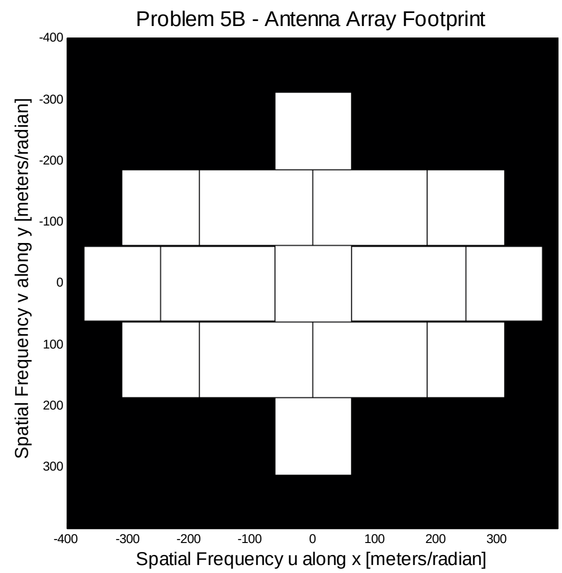




Besides the dwindling of some of the weaker stars and a dimming of stars, the measured distribution and the true brightness distribution displayed in the data closely match!

(b.) Once again considering the provided data file as the source distribution, we can create a synthetic interferometer image under the array with optical transfer function:





We now possess five square apertures, so the total area of the antenna is

$$A = 5s^2 = 5(50\ m)^2 = \mathbf{12{,}500\ m^2}$$

This antenna produces a much blurrier image due to the extended optical transfer function: