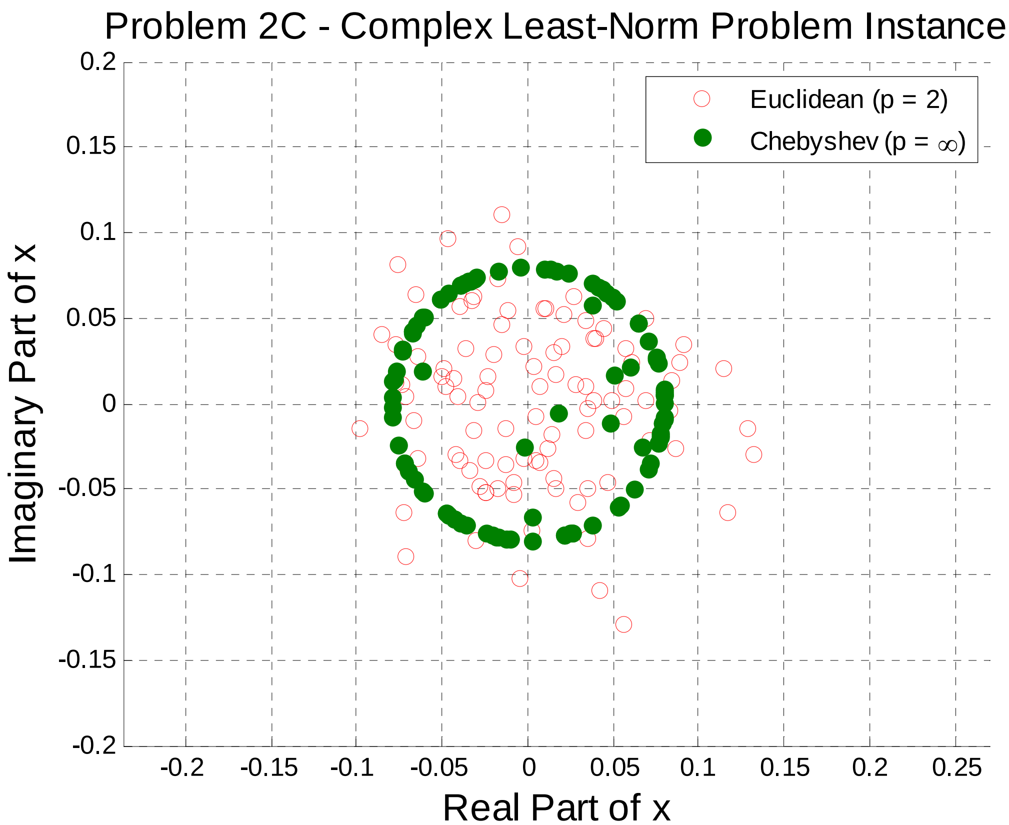
CVX Problem Set IV – Convex Optimization Problems

Problem #2 – Complex Least-Norm Problem



In minimizing the Euclidean norm, we obtain a cluster of points close to but scattered at random distances from the origin, reflecting our desire to minimize distance (complex modulus) from the Argand origin subject to random constraints on each individual Euclidean distance.

The Chebyshev norm, on the other hand, strives to minimize the maximum modulus, which yields a circular locus of points since the cost function is computed as a complex modulus, equivalent to (radial) distance in the complex plane. By minimizing this complex modulus subject to our constraint, almost all of the points would naturally fall at the boundary dictated by the minimized maximum magnitude of the complex number. After all, the optimal solution is often a compromise, with each complex element bearing approximately equal weight from a complex norm standpoint.

Problem #3 – Numerical Perturbation Analysis Example

$$\boldsymbol{p^\star \approx 8.2222}$$

$$\boldsymbol{x^\star = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \approx \begin{bmatrix} -2.3333 \\ 0.1667 \end{bmatrix}}$$

$$\boldsymbol{\lambda^\star = \begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \approx \begin{bmatrix} 1.5787 \\ 3.6890 \\ 0.1131 \end{bmatrix}}$$

The following Karush-Kuhn-Tucker conditions hold:

$$\lambda^\star \succcurlyeq 0$$

$$\left(\begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 5 & 76 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & 2 \\ 1 & -4 \\ 5 & 76 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} -2 \\ -3 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$2x_1^\star - x_2^\star - 1 + \lambda_1^\star + \lambda_2^\star + 5\lambda_3^\star = 0$$

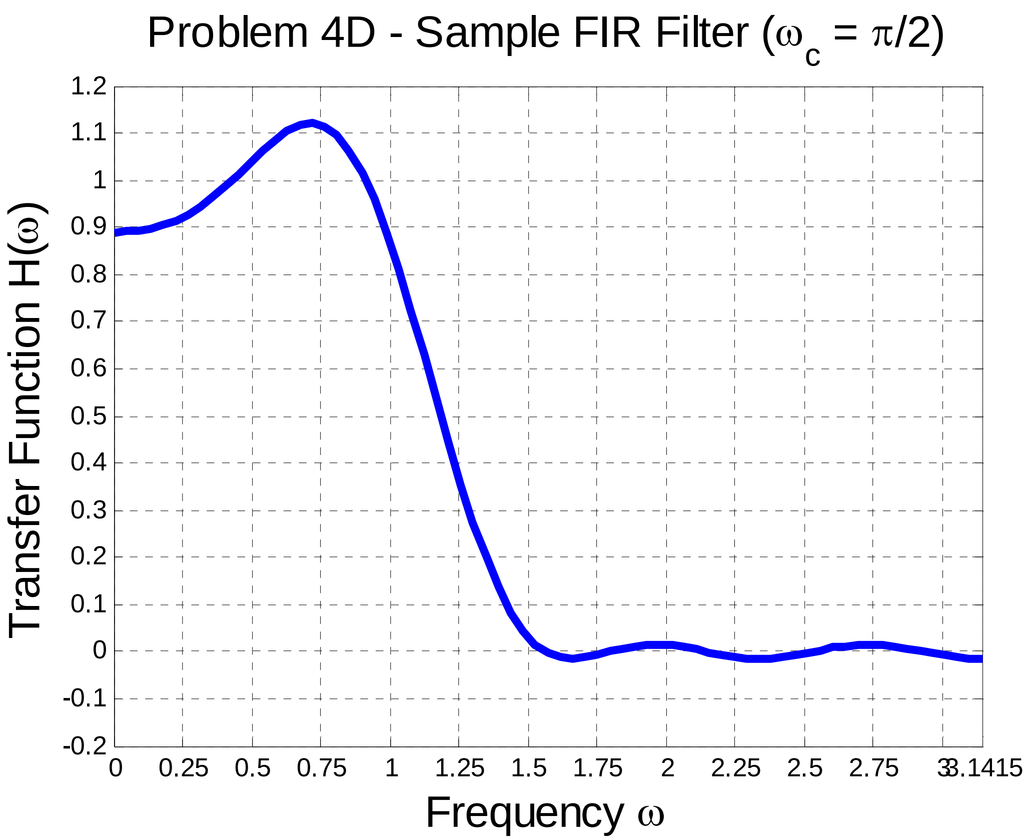
$$-x_1^\star + 4x_2^\star - 1 + 2\lambda_1^\star - 4\lambda_2^\star + 76\lambda_3^\star = 0$$

$\boldsymbol{\delta_1}$	$\boldsymbol{\delta_2}$	$\boldsymbol{p^\star_{pred}}$	$\boldsymbol{p^\star_{exact}}$
$\mathbf{0}$	0	**8.2222**	**8.2222**
$\mathbf{0}$	-0.1	**8.5911**	**8.7064**
$\mathbf{0}$	$+0.1$	**7.8533**	**7.9800**
-0.1	0	**8.3801**	**8.5650**
-0.1	-0.1	**8.7490**	**8.8156**
-0.1	$+0.1$	**8.0112**	**8.3189**
$+0.1$	0	**8.0644**	**8.2222**
$+0.1$	-0.1	**8.4332**	**8.7064**
$+0.1$	$+0.1$	**7.6955**	**7.7515**

In each case, the predicted optimal value underestimates the exact optimal value, as one might expect from a convex function; linear estimates are global underestimators: $\boxed{\boldsymbol{p^\star_{pred} \le p^\star_{exact}}}$.

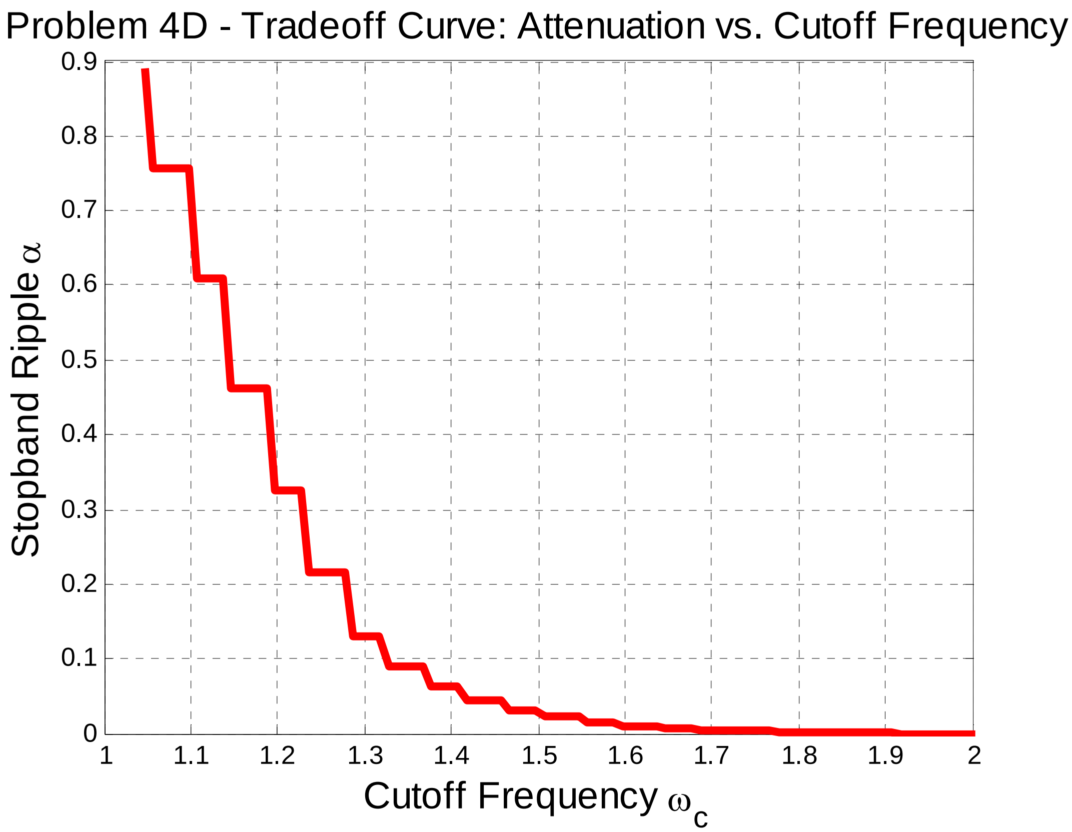
Problem #4 – FIR Filter Design

First, we design a sample filter and optimize the stopband attenuation with a fixed cutoff frequency, just to test our convex optimization prowess:



The filter meets the specifications delineated in the sample diagram. Indeed, the maximum passband ripple oscillates between 0.89 and 1.12, while the filter enters stopband at approximately 1.57, or $\frac{\pi}{2}$ as designed. Finally, we note a stopband attenuation of 0.0154, which is reasonably small.

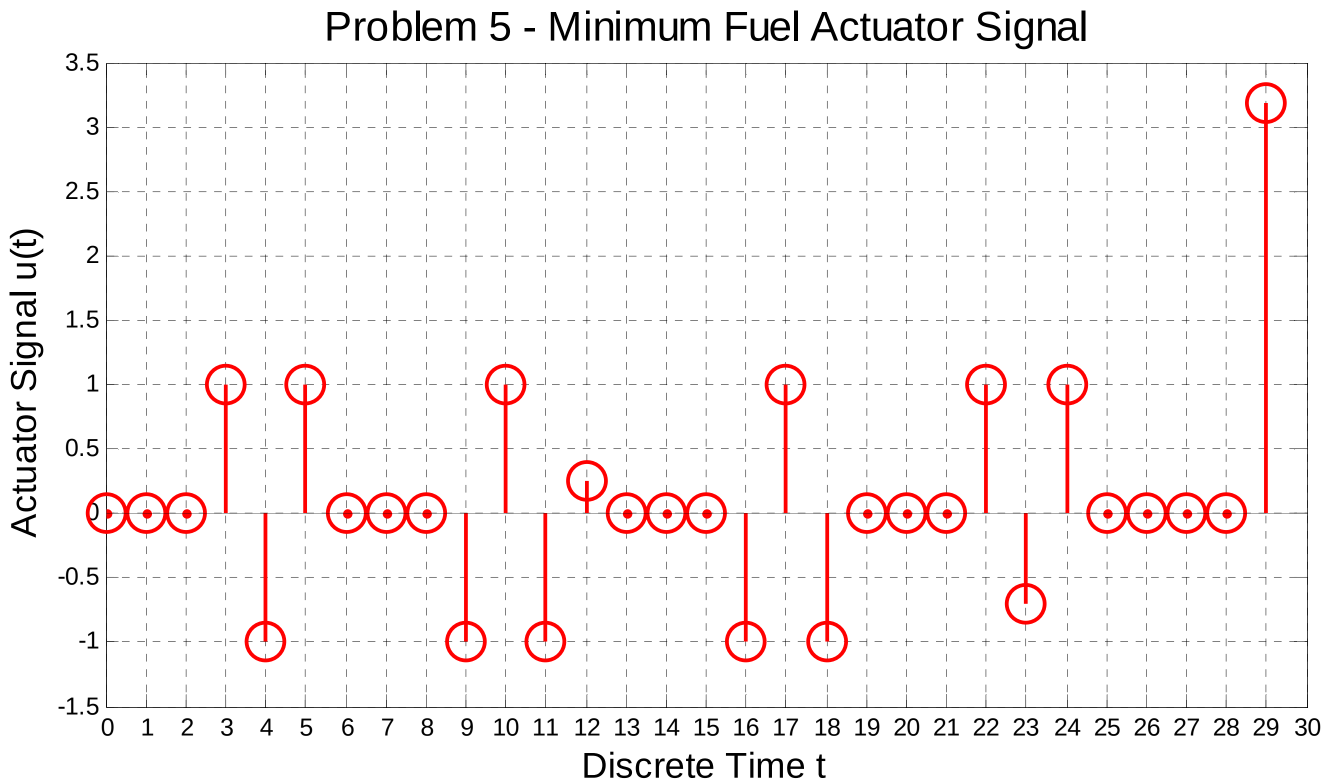
However, as we tighten the cutoff frequency, decreasing the transition bandwidth, we also increase the amount of ripple in the stopband. For example, we observe the stopband attenuation (ripple amplitude) monotonically decreases as we increase the cutoff frequency; in other words, we can exchange filter sharpness for stopband ringing:



Generally, the ripple decreases as we increasing the cutoff frequency; as we loosen the restraints on our transition band by stretching the transition we are willing to tolerate, we reduce the ringing and strengthen the attenuation of our stopband. However, although the general shape appears concave upward, the set of achievable specifications is not convex, since the flat regions along the curve are momentarily concave. Instead, the function appears quasiconvex.

The flat portions in the curve represent constant attenuation over a small range of cutoff frequency choices; in other words, we can twiddle the transition bandwidth without altering the stopband ripple because the ringing that occurs can fit a certain number of cosinusoidal cycles before inevitably increasing the ripple amplitude. Instead of rippling *more* for every fraction of bandwidth we remove from the transition band, the energy simply redistributes itself across the stopband until the bandwidth decreases to the point at which the transition is so sharp that the sidelobes must increase in amplitude to accommodate more energy.

Problem #5 – Minimum Fuel Optimal Control



We remark that the actuator signal is, for the most part, turned off for several consecutive cycles at a time; for example, for the first 27 seconds, five clusters of three-second dormancy alternate with four clusters of oscillating bursts. The actuator signal appears to switch between three seconds of inactivity and three seconds of oscillatory input. The final input is especially large. With this exception, the general distribution of fuel use is well balanced, as one often sees in an optimal solution.

The minimal total amount of fuel consumed is approximately **$p^\star \approx$ 17.3236 units of fuel**.