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EE 364 – Convex Optimization

CVX Problem Set VI – Approximation and Fitting

Problem #1 – Minimax Rational Fit to the Exponential

We fit a rational function of the following form to the exponential e^t:

$$f(t) = \frac{a_0 + a_1 t + a_2 t^2}{1 + b_1 t + b_2 t^2}$$

The optimal fitting function results from the following coefficients:

$$a_0 \approx 1.0099$$

$$a_1 \approx 0.6121$$

$$a_2 \approx 0.1134$$

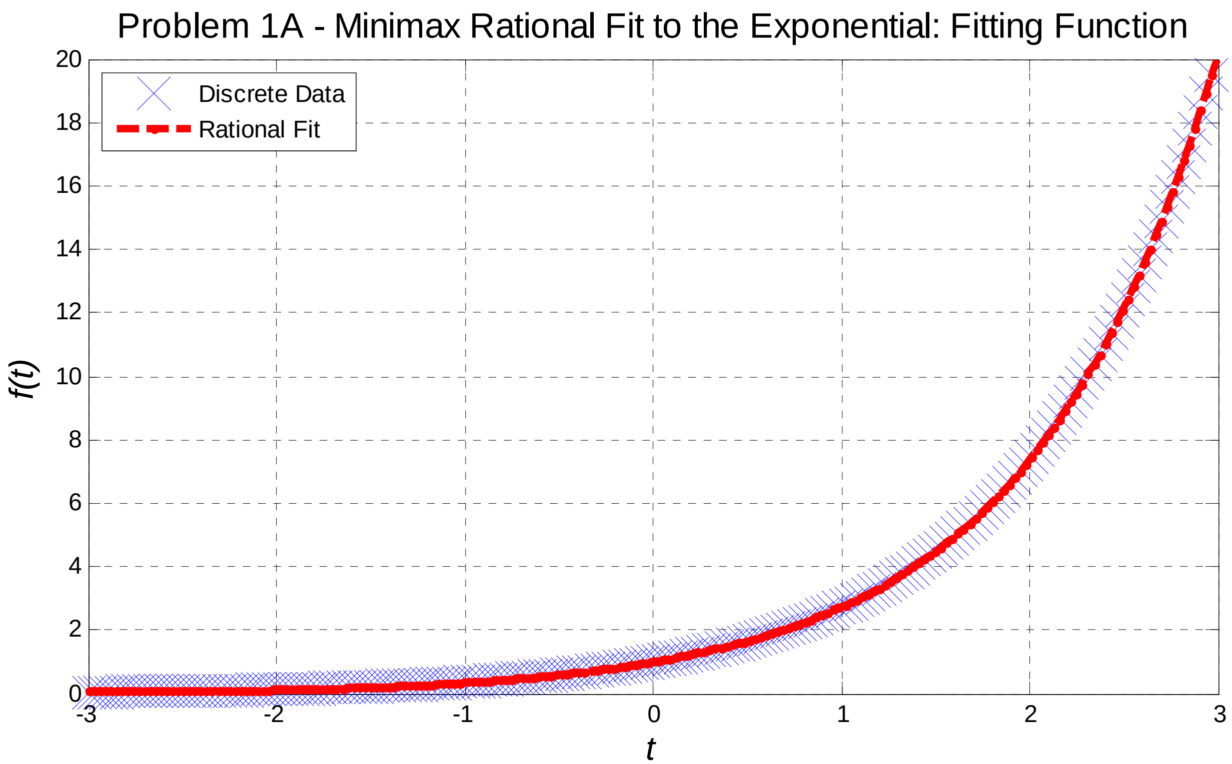
$$b_0 \approx -0.4146$$

$$b_1 \approx 0.0485$$

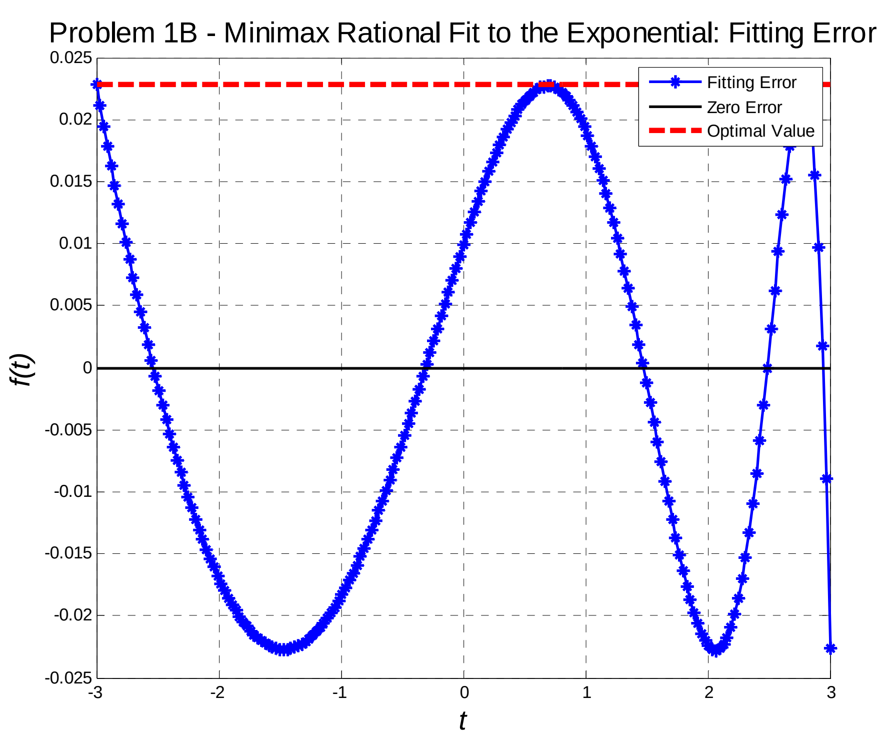
yielding an optimal minimax residual of

$$\min\left\{\max_{i=1,\cdots,k}\left|\frac{a_0 + a_1 t_i + a_2 {t_i}^2}{1 + b_1 t_i + b_2 {t_i}^2} - y_i\right|\right\} \approx 0.0228$$

After 17 iterations, our bisection algorithm converges to the following "best fitting" rational function for the exponential:



To even a trained human eye, the rational function fits the exponential without noticeable error along any interval; for all intents and purposes, our 0.001-accuracy fit is perfect, and the minuscule error plot corroborates our success:



Problem #2 – Maximum Likelihood Prediction of Team Ability

From the data provided in `team_data`, we predict the following team ability:

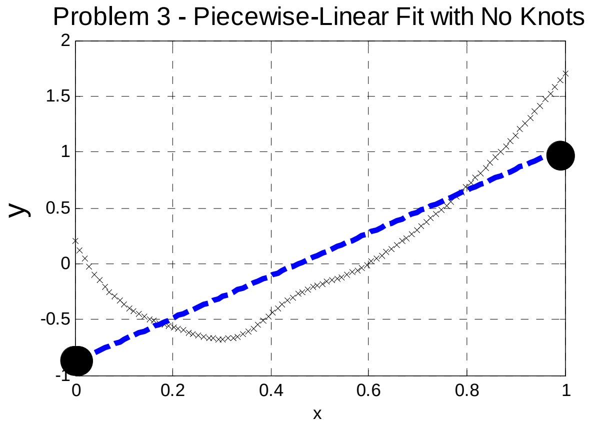
$$\hat{a} \approx \begin{bmatrix} \mathbf{1.0000} \\ \mathbf{0.0000} \\ \mathbf{0.6829} \\ \mathbf{0.3696} \\ \mathbf{0.7946} \\ \mathbf{0.5779} \\ \mathbf{0.3795} \\ \mathbf{0.0895} \\ \mathbf{0.6736} \\ \mathbf{0.5779} \end{bmatrix}$$

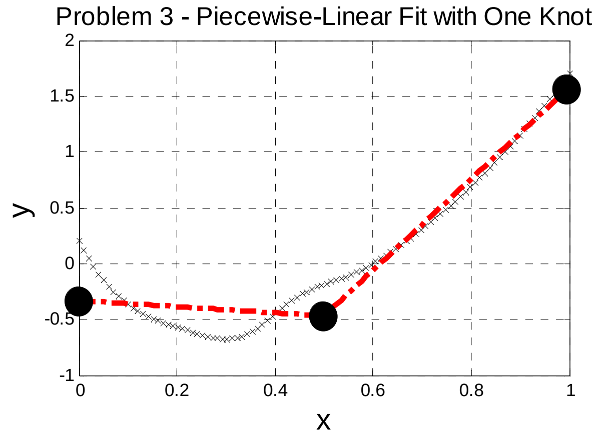
Maximum Likelihood Predicted Results	Last Year's (Training) Results	This Year's (Test) Results
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
1	**1**	**1**
-1	**-1**	**-1**
-1	**-1**	**1**
-1	**-1**	**-1**
-1	**-1**	**-1**
-1	**-1**	**-1**
-1	**-1**	**1**
-1	**-1**	**-1**
-1	**-1**	**-1**
1	**1**	**1**
-1	**-1**	**-1**

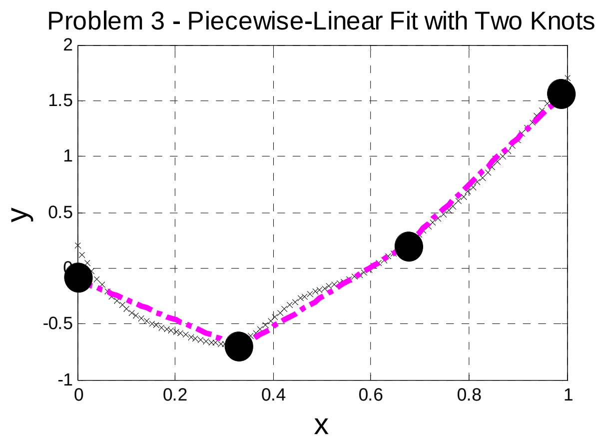
1	-1	1
1	1	1
1	1	1
1	1	-1
1	1	1
-1	-1	-1
-1	-1	-1
-1	1	-1
1	1	1
-1	-1	-1
-1	-1	-1
1	1	-1
1	1	1
1	1	1
1	-1	1
1	1	1
1	1	1
1	1	1
-1	-1	1
1	-1	1
1	1	1
-1	1	-1
-1	-1	1
-1	-1	-1
-1	-1	-1
1	1	1

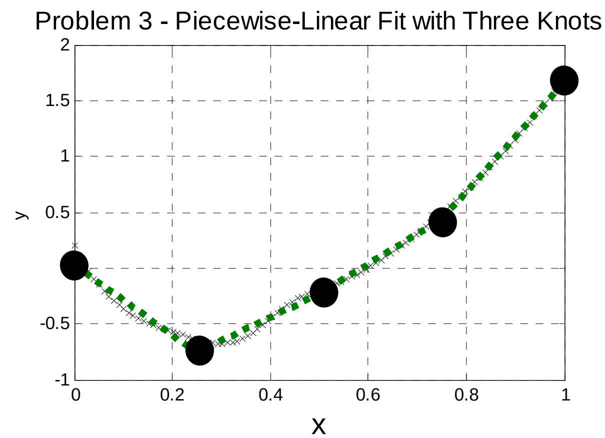
Using the maximum likelihood estimate $\hat{a}$ derived in Part (b.), we correctly predict 39 of this year's 45 games (86.67% correct). Simply assuming that last year's victor prevails again this year leads to a correct prediction in only 34 of this year's 45 games (75.56% correct). Many outcomes are easily predictable and therefore readily replicated!

Problem #3 – Piecewise Linear Fitting









For a piecewise linear fit with **no knots**,

$$\alpha_1 \approx 1.9110$$

$$\beta_1 \approx -0.8725$$

$$\min \sum_{i=1}^{m} [f(x_i) - y_i]^2 \approx 12.7407$$

For a piecewise linear fit with **one knot**,

$$\alpha_1 \approx -0.2708$$

$$\alpha_2 \approx 4.0928$$

$$\beta_1 \approx -0.3325$$

$$\beta_2 \approx -2.5143$$

$$\min \sum_{i=1}^{m} [f(x_i) - y_i]^2 \approx 2.6243$$

For a piecewise linear fit with **two knots**,

$$\alpha_1 \approx -1.8094$$

$$\alpha_2 \approx 2.6785$$

$$\alpha_3 \approx 4.2059$$

$$\beta_1 \approx -0.1022$$

$$\beta_2 \approx -1.5982$$

$$\beta_3 \approx -2.6165$$

$$\min \sum_{i=1}^{m} [f(x_i) - y_i]^2 \approx 0.5897$$

For a piecewise linear fit with **three knots**,

$$\alpha_1 \approx -3.1558$$

$$\alpha_2 \approx 2.1155$$

$$\alpha_3 \approx 2.6762$$

$$\alpha_4 \approx 4.8993$$

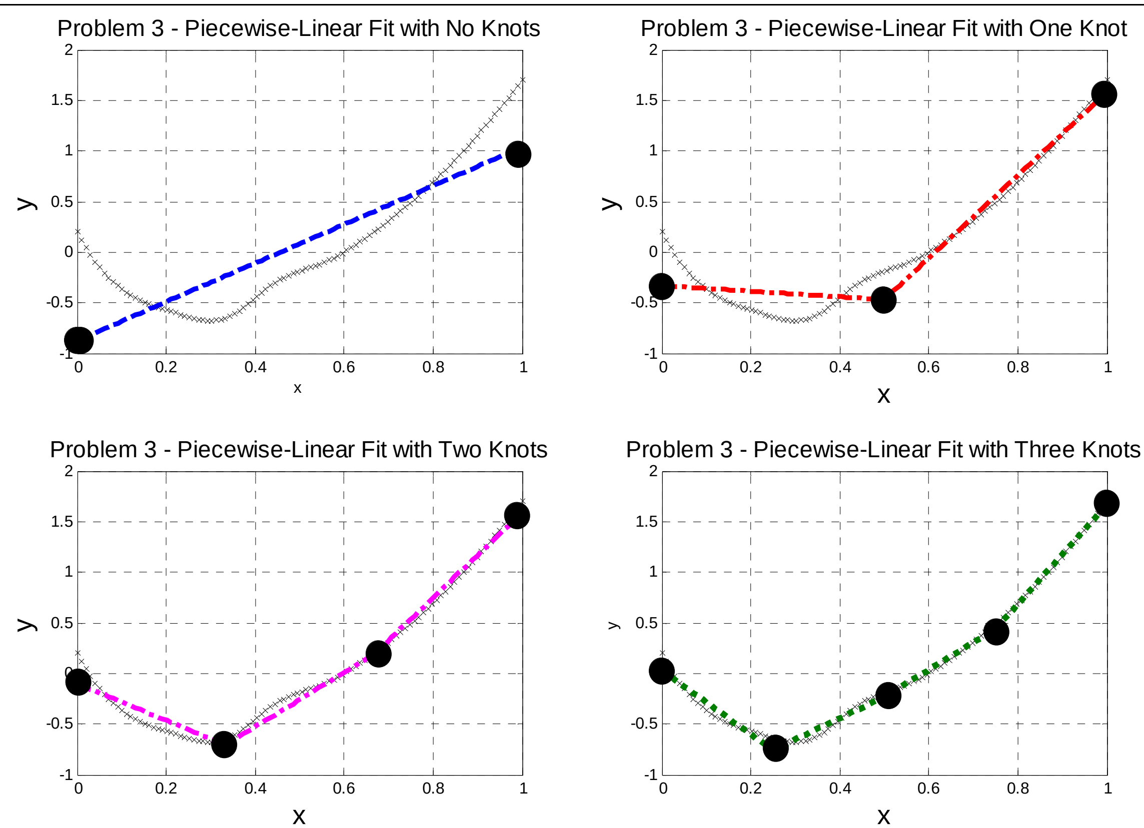
$$\beta_1 \approx 0.0309$$

$$\beta_2 \approx -1.2869$$

$$\beta_3 \approx -1.5673$$

$$\beta_4 \approx -3.2345$$

$$\min \sum_{i=1}^{m} [f(x_i) - y_i]^2 \approx 0.219$$



Problem #4 – Robust Least-Squares with Interval Coefficient Matrix

We first consider the nominal problem:

$$\text{minimize } \|\bar{A}x - b\|_2$$

This problem admits the least-squares solution

$$x_{ls} \approx \begin{bmatrix} -53.160004241256 \\ +51.600004138684 \\ -107.700008577305 \end{bmatrix}$$

The minimal least-squares residual norm is $\|\bar{A}x_{ls} - b\|_2 \approx 7.5895$.

The worst-case least-squares residual norm is $\||\bar{A}x_{ls} - b| + R|x_{ls}|\|_2 \approx 26.7012$.

However, we can improve our stability by solving the robust least-squares problem:

$$\text{minimize } \||\bar{A}x_{ls} - b| + R|x_{ls}|\|_2$$

This problem admits the robust least-squares solution

$$x_{rls} \approx \begin{bmatrix} -0.28105375806103 \\ +0.00000009231658 \\ -0.75850674531460 \end{bmatrix}$$

The minimal least-squares residual norm is $\|\bar{A}x_{rls} - b\|_2 \approx 17.7106$.

The worst-case least-squares residual norm is $\||\bar{A}x_{rls} - b| + R|x_{ls}|\|_2 \approx 17.7940$.

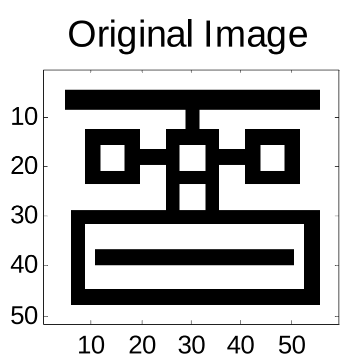
Problem #5 – Total Variation Image Interpolation

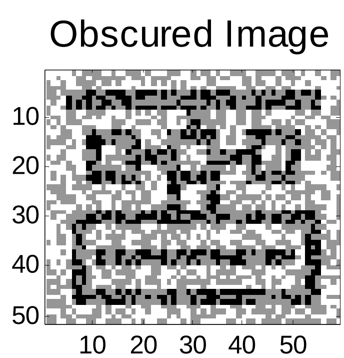
We solve the following convex optimization problem for optimal ℓ_2 variation:

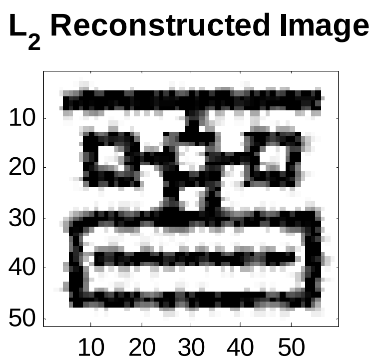
```
cvx_begin;
    variable Ul2(m,n);
    Ux = Ul2(2:end, :) - Ul2(1:(end-1), :);
    Uy = Ul2(:, 2:end) - Ul2(:, 1:(end-1));
    minimize (norm(Ux, 'fro') + norm(Uy, 'fro'))
    subject to
        Ul2(Known) == Uorig(Known);
        Ul2 >= 0;
cvx_end;

% Optimal value (cvx_optval): +7422.94
```

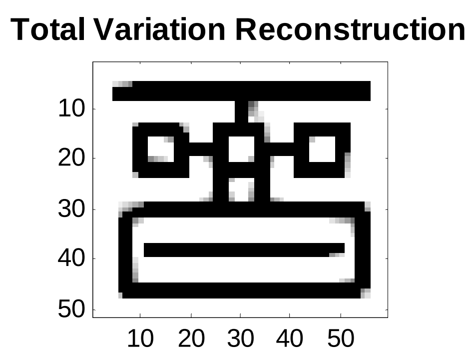
Similarly, we replace the ℓ_2-norm with the ℓ_1-norm for optimal total variation:

```
cvx_begin;
    variable Utv(m,n);
    Ux = Utv(2:end, :) - Utv(1:(end-1), :);
    Uy = Utv(:, 2:end) - Utv(:, 1:(end-1));
    minimize (norm([Ux(:) ; Uy(:)], 1))
    subject to
        Utv(Known) == Uorig(Known);
        Utv >= 0;
cvx_end;

% Optimal value (cvx_optval): +167790
```









Problem #6 – Relaxed and Discrete A-Optimal Experiment Design

We first solve the relaxed A-optimal experiment design problem:

$$minimize\ \frac{1}{m}\ Trace\left(\sum_{i=1}^{p} \lambda_i v_i v_i^T\right)^{-1}$$

$$subject\ to\ \sum_{i=1}^{p} \lambda_i = 1, \qquad \lambda \succcurlyeq 0$$

We obtain the optimal $\lambda^\star$ vector:	leading to the following discretized $\hat{m}$ vector:
0.00000000099085	0
0.13813261762990	4
0.06336198279685	2
0.12740605141608	4
0.03794162807680	1
0.13035163490341	4
0.09117813911511	3
0.00000000036918	0
0.12483692256096	4
0.00000000163787	0
0.00000000196060	0
0.00000000044819	0
0.00000000253303	0
0.00000000056890	0
0.11333654573959	3
0.00000000033047	0
0.00715728720602	0
0.00000000366153	0
0.00000000026620	0
0.16629717764358	5

The optimal value of our relaxed problem is $\frac{1}{m}\ Trace\left(\sum_{i=1}^{p} \lambda_i^\star v_i v_i^T\right)^{-1} \approx 0.24808807792122$.

The suboptimal point for the discrete problem is $\frac{1}{m}\ Trace\left(\sum_{i=1}^{p} \hat{m}_i v_i v_i^T\right)^{-1} \approx 0.2483\ 1384936181$.

Thus, the gap between our upper bound and lower bound is approximately **0.00022577144**.