Christopher Tsai
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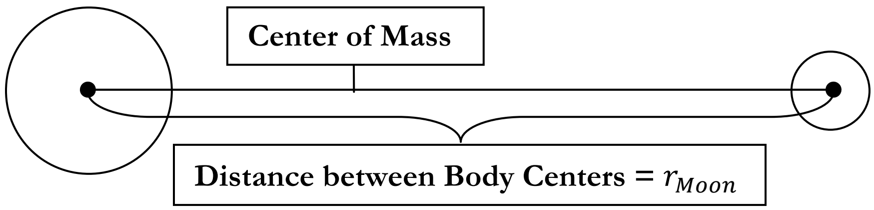
EE 106 – Planetary Exploration

Problem Set II
Orbits and Gravity
Earth, Moon, and Sky

Problem 1 – Planet-Moon Barycenters

The *barycenter* of a two-body system is the center of mass of the two bodies:



Defining the origin at the planet's center so that the moon's radial coordinate is r_{Moon}, we can compute the barycenter's distance from the center of Earth by calculating the center of mass as a mass-weighted distance:

$$M_{Earth} \cdot (r_{CM} - r_{Earth}) = M_{Moon} \cdot (r_{CM} - r_{Moon})$$

$$M_{Earth} \cdot (r_{CM} - r_{Earth}) = M_{Moon} \cdot (r_{Moon} - r_{CM})$$

$$(M_{Earth} + M_{Moon}) \cdot r_{CM} = M_{Earth} \cdot r_{Earth} + M_{Moon} \cdot r_{Moon}$$

$$r_{CM} = \frac{M_{Earth}}{(M_{Earth} + M_{Moon})} \cdot (0\ m) + \frac{M_{Moon}}{(M_{Earth} + M_{Moon})} \cdot r_{Moon}$$

$$r_{CM} = \frac{M_{Moon}}{M_{Earth} + M_{Moon}} \cdot r_{Moon}$$

$$r_{CM} \approx \frac{(735 \times 10^{20}\ kg)}{(5.977 \times 10^{24}\ kg) + (735 \times 10^{20}\ kg)} \cdot (384 \times 10^{6}\ m)$$

$$\boxed{\boldsymbol{r_{CM} \approx 4.664738 \times 10^{6}} \textbf{ m from the center of Earth}}$$

The equatorial radius of Earth is about 6.378 $\times$ 10^6 m, so $r_{CM} \approx 4.664738 \times 10^{6}\ m < r_{Earth}$.

Thus, the barycenter of Earth and the Moon lies **inside** the Earth, within one radius.

We repeat this analysis for Pluto and its moon Charon:

$$r_{CM} = \frac{M_{Charon}}{M_{Pluto} + M_{Charon}} \cdot r_{Charon}$$

$$r_{CM} \approx \frac{(19 \times 10^{20}\ kg)}{(0.0025 \cdot 5.977 \times 10^{24}\ kg) + (19 \times 10^{20}\ kg)} \cdot (19.7 \times 10^{6}\ m)$$

$$\boxed{\boldsymbol{r_{CM} \approx 2.222354 \times 10^{6}} \text{ m from the center of Pluto}}$$

The equatorial radius of Pluto is about 1.15×10^6 m, so $r_{CM} \approx 2.222354 \times 10^6\ m > r_{Pluto}$.

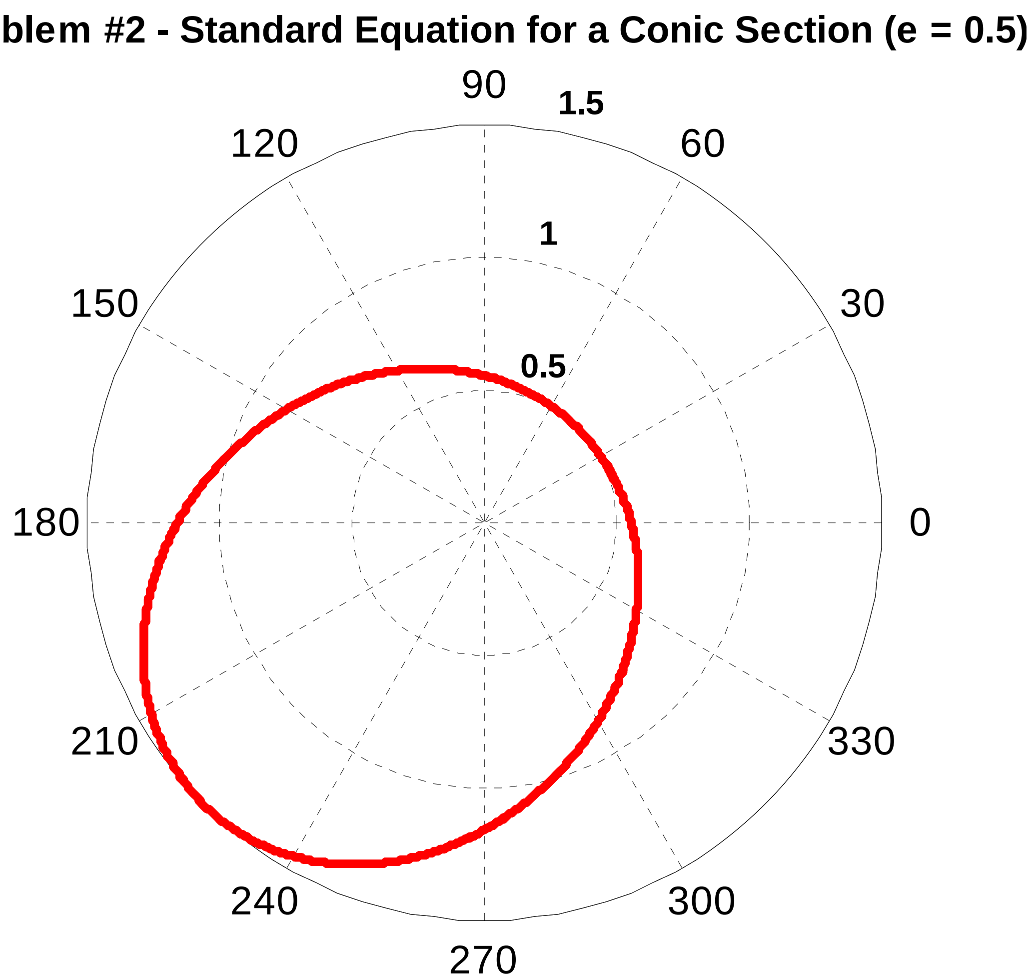
Thus, the barycenter of Pluto and Charon lies **outside** Pluto, but much closer to Pluto than Charon.

Problem 2 – Polar Equation for a Conic Section

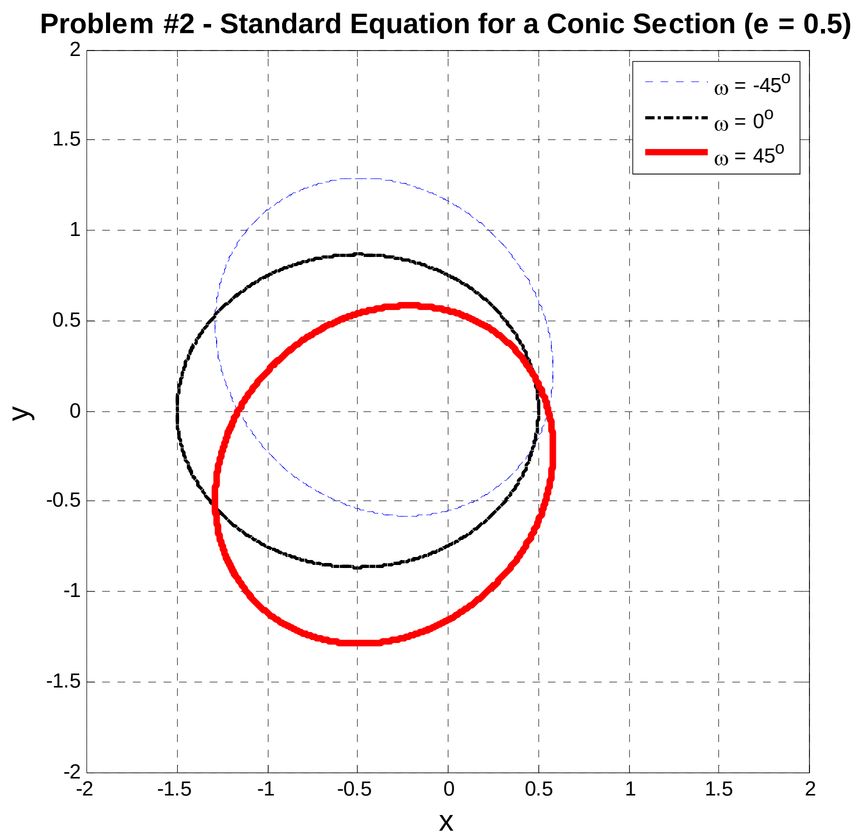
The standard polar equation for a conic section is

$$r = \frac{p}{1 + e \cdot cos(\theta - \omega)}$$

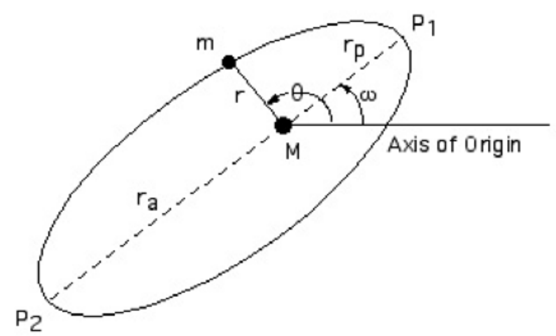
…where e is the eccentricity and p is a constant. For $e = 0.5$, $a = 1$, $\omega = \frac{\pi}{4}$, and $p = 1$, the conic section appears to be an **ellipse**:



As we vary the angular displacement ω in the cosine argument, the ellipse rotates about the origin:



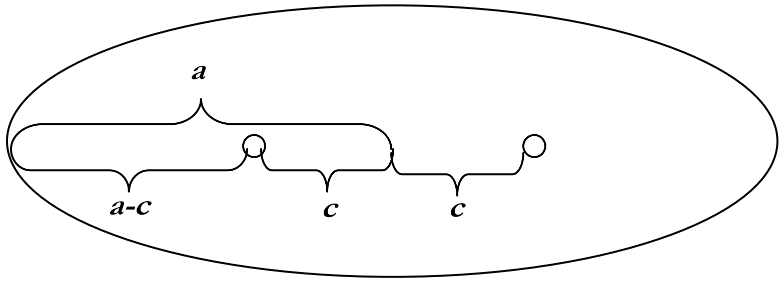
The major axis of the ellipse is always tilted at inclination ω with respect to the positive *x*-axis:



Thus, the value of **ω influences the orientation** of the ellipse's major axis with respect to the horizontal, and varying its value will rotate or tilt the ellipse and its major and minor axes.

Because $e = \frac{c}{a}$, the focal distance $c = ae = (1)(0.5) = 0.5$. The distance between the two foci is twice this focal distance, as pictured on the following page:

The distance between the ellipse's two foci is $\boxed{d = 2c = 2ae = 2(1)(0.5) = \mathbf{1}}$.



Problem 3 – Eccentricity of Earth's Heliocentric Orbit

The orbital eccentricity of the Earth's orbit around the Sun is approximately 0.017, according to Appendix 7. Noting that the semimajor axis of Earth's elliptical orbit around the Sun is approximately 1 AU ≈ 1.496 $\times$ 10^{11}m, the semiminor axis of Earth's elliptical orbit is therefore

$$b = \sqrt{a^2 - c^2} = a\sqrt{1 - \left(\frac{c}{a}\right)^2} = a\sqrt{1 - e^2}$$

$$b \approx a\sqrt{1 - (0.017)^2} \approx 0.9998555 \cdot a$$

Because $\boxed{\boldsymbol{a \approx 1\,AU \approx 1.496 \times 10^{11}\,m}}$, $\boxed{\boldsymbol{b \approx 0.9998555\,AU \approx 1.4957838 \times 10^{11}\,m}}$.

The axes differ by only $\frac{\mathrm{a-b}}{\mathrm{b}} \approx \frac{1\text{ AU} - 0.9998555\text{ AU}}{0.9998555\text{ AU}} \approx 0.0001445 \approx 0.01445\%$. This percentage is low enough for us to consider Earth's orbit circular; the eccentricity's proximity to zero makes the elliptical orbit virtually circular.

The focal distance of the Sun from the center of Earth's elliptical orbit is $c = ae \approx 0.017\ AU$. Thus, the Earth's distances from the Sun at perihelion and aphelion are

$$\boxed{\boldsymbol{r_{perihelion} = a - c \approx 0.983\,AU \approx 1.470568 \times 10^{11}\,m}}$$

$$\boxed{\boldsymbol{r_{aphelion} = a + c \approx 1.017\,AU \approx 1.521432 \times 10^{11}\,m}}$$

The Earth is only $\frac{r_{aphelion} - r_{perihelion}}{r_{average}} \approx \frac{1.017\ AU - 0.983\ AU}{1\ AU} \approx 0.034 \approx 3.4\%$ closer to the Sun at perihelion than at aphelion.

Perihelion occurs in January, during the Southern Hemisphere summer. Because the Earth is closer to the Sun at perihelion, the Earth must travel faster during this time to sweep out the same area that it would sweep in the same time interval at aphelion; by Kepler's Second Law, the Earth must spend less time around perihelion than it does near aphelion, meaning that the Southern Hemisphere experiences a slightly shorter summer (during Earth's perihelion) than the Northern Hemisphere (during Earth's aphelion). However, the Earth's closer proximity to the Sun during the Southern Hemisphere summer partially compensates for the summer's shorter duration, in that the Sun will be slightly closer to Earth's surface during the shorter summer, resupplying some of the "lost" heat from the slight decrease in summer days by shining more intense light. Nevertheless, neither the difference in distance to the Sun nor the duration discrepancy will produce noticeable effect on humans; the elliptical orbit is so close to circular ($e \approx 0$) that macroscopic variables like temperature and tan darkness exhibit almost no dependence on the slightly smaller distance or slightly shorter summer. The Southern Hemisphere summer may be slightly hotter and shorter, but not to any significant degree.

Problem 4 – Titius' Sequence

In 1766, a man named Daniel Titius wrote the number sequence:

0, 3, 6, 12, 24, 48, 96, 192, 384, 768, 1536, …

Adding four to each number in the sequence, he obtained the sequence:

4, 7, 10, 16, 28, 52, 100, 196, 388, 772, 1540, …

Dividing the sum by ten, Daniel Titius yielded the sequence:

0.4, 0.7, 1, 1.6, 2.8, 5.2, 10, 19.6, 38.8, 77.2, 154, …

Tabulated beside the semimajor axes [AU] of the planetary orbits around the Sun, we see

Planet	**Semimajor Axis [AU]**	**Titius' Sequence**
Mercury	0.3871	0.4
Venus	0.7233	0.7
Earth	1.0000	1
Mars	1.5237	1.6
Ceres	2.7671	2.8
Jupiter	5.2028	5.2
Saturn	9.538	10
Uranus	19.191	19.6
Neptune	30.061	38.8
Pluto	39.529	77.2

From the similarity in value of the eight terms in Titius' sequence, it would appear that the semimajor axes of large planetary bodies' orbits around the Sun follow the same sequence that Titius fabricated mathematically, with the orbital semimajor axis of the planet closest to the Sun (Mercury) described by the first term in the sequence (0.4 AU). In fact, even before astronomers knew of Ceres, Titius' sequence accurately predicted the existence of this massive asteroid orbiting the Sun with a semimajor axis of approximately 2.8 AU. Thus, the mathematical sequence seemed to reveal the size of planetary orbits mystically, with no rhyme or reason, while also validating the adoption of the earthcentrically defined astronomical unit (AU) as a sort of natural measurement of orbit distances.

However, as the semimajor axis of Neptune differs considerably from the ninth term in Titius' sequence, we see that the entire hypothesis was nothing more than mere coincidence. All in all, this sequence reaffirms the notion that we can fit *any* set of data with a simple mathematical model if the sequence is small enough – least-squares or trial-and-error provide us with the means. Unfortunately, Titius' sequence is nothing more than that: a mathematical model that fits exceptionally well for a set of eight data points but fails to extrapolate outside the domain of interest.

Problem 2.19

Let M_{planet} be the mass of the orbiting planet whose orbit has a semimajor axis of a_{planet}= 4 AU.

Let $m_{asteroid}$ be the mass of the asteroid whose orbit has a semimajor axis of $a_{asteroid}$= 10 AU.

Let M_{sun} be the mass of the Sun, about which our planet and asteroid orbit.

Not knowing the mass of either the planet or the asteroid, we must assume that the Sun is much more massive than both orbiting bodies: $M_{sun} \gg M_{planet}$ and $M_{sun} \gg m_{asteroid}$. In that case, Newton's Law of Gravity reduces to

$$\frac{P^2}{a^3} = \frac{4\pi^2}{G \cdot (M_{sun} + m)} \approx \frac{4\pi^2}{G \cdot M_{sun}}$$

$$P \approx \sqrt{\frac{4\pi^2 a^3}{G \cdot M_{sun}}}$$

Recalling that one astronomical unit (AU) amounts to 1.496 $\times$ 10^{11} m, we calculate periods:

$$P_{planet} \approx \sqrt{\frac{4\pi^2 a_{planet}^3}{G \cdot M_{sun}}} \approx \sqrt{\frac{4\pi^2(4 \cdot 1.496 \times 10^{11}\ m)^3}{\left(6.672 \times 10^{-11}\ N\ \frac{m^2}{kg^2}\right) \cdot (1.989 \times 10^{30}\ kg)}} \approx 8\ years$$

$$P_{asteroid} \approx \sqrt{\frac{4\pi^2 a_{asteroid}^3}{G \cdot M_{sun}}} \approx \sqrt{\frac{4\pi^2(10 \cdot 1.496 \times 10^{11}\ m)^3}{\left(6.672 \times 10^{-11}\ N\ \frac{m^2}{kg^2}\right) \cdot (1.989 \times 10^{30}\ kg)}} \approx 31.625\ years$$

We can check these results using Kepler's AU-years formulation, again assuming a massive Sun:

$$\boxed{\boldsymbol{P_{planet}} \approx \sqrt{a_{planet}^3} \approx \sqrt{(4\ AU)^3} \approx \textbf{8 years}}$$

$$\boxed{\boldsymbol{P_{asteroid}} \approx \sqrt{a_{asteroid}^3} \approx \sqrt{(10\ AU)^3} \approx \textbf{31.623 years}}$$

Problem 2.22

Newton showed that the periods and distances in Kepler's Third Law depend on the masses of the objects. At 1 astronomical unit (AU) from the Sun, Earth revolves around the Sun with a period of approximately $P_{earth} \approx \sqrt{\frac{a_{earth}^3}{M_{sun}+M_{earth}}} \approx \sqrt{\frac{(1\,AU)^3}{(1\,sun\,mass)+\left(\frac{5.977\times10^{24}kg}{1.989\times10^{30}kg}\,sun\,mass\right)}} \approx 1$ year.

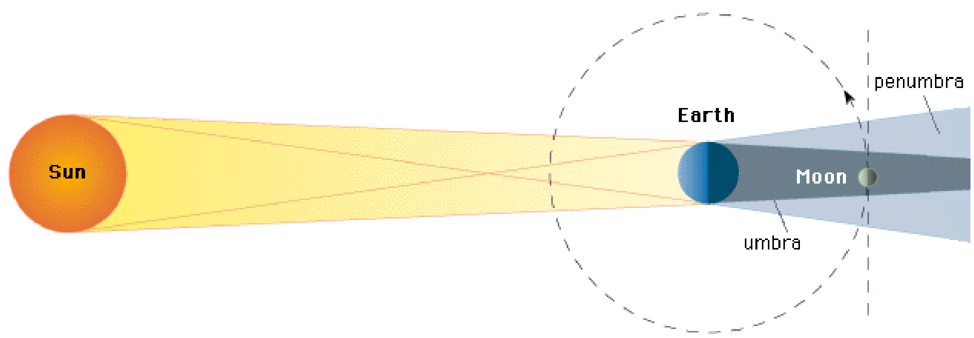
However, if the Sun had twice its true mass instead, we must double the radicand's denominator:

$$P_{earth} \approx \sqrt{\frac{a_{earth}^3}{M_{sun}+M_{earth}}} \approx \sqrt{\frac{(1\,AU)^3}{(2\,sun\,mass)+\left(\frac{5.977\times10^{24}kg}{1.989\times10^{30}kg}\,sun\,mass\right)}} \approx 0.7071 \text{ year} \approx \boxed{258.27 \text{ days.}}$$

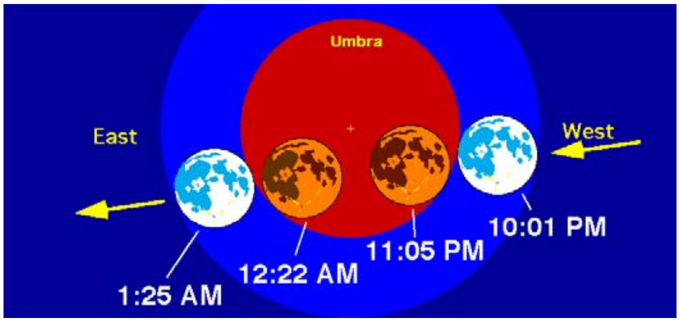
We can verify this result with Newton's Law of Gravitation, assuming that $M_{sun} \gg M_{earth}$:

$$P_{earth} \approx \sqrt{\frac{4\pi^2 a_{earth}^3}{G\cdot 2M_{sun}}} \approx \sqrt{\frac{4\pi^2(1.496\times10^{11}\,m)^3}{\left(6.672\times10^{-11}\,N\,\frac{m^2}{kg^2}\right)\cdot(2\cdot1.989\times10^{30}\,kg)}} \approx 22{,}316{,}026 \text{ seconds} \approx \boxed{258.287 \text{ days.}}$$

Problem 3.17

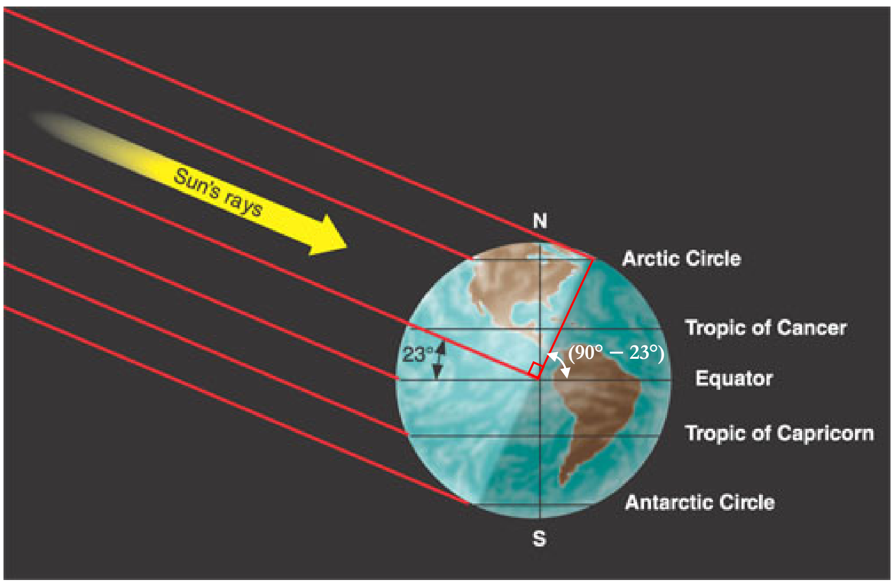


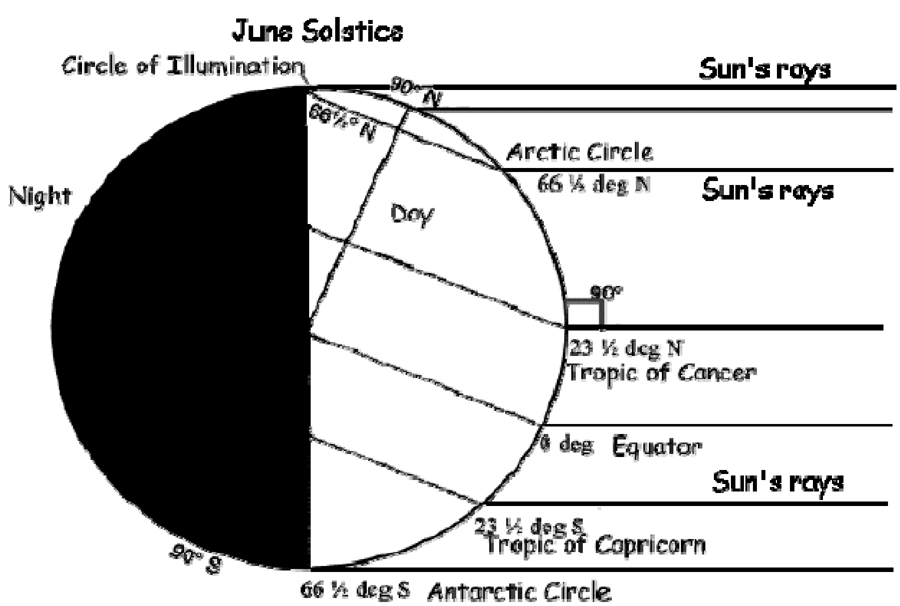
During a lunar eclipse, the Moon enters the shadow of the Earth from the **west** side, since, from a terrestrial perspective, the Moon travels *eastward* through the sky relative to the Sun and stars:



Problem 3.20

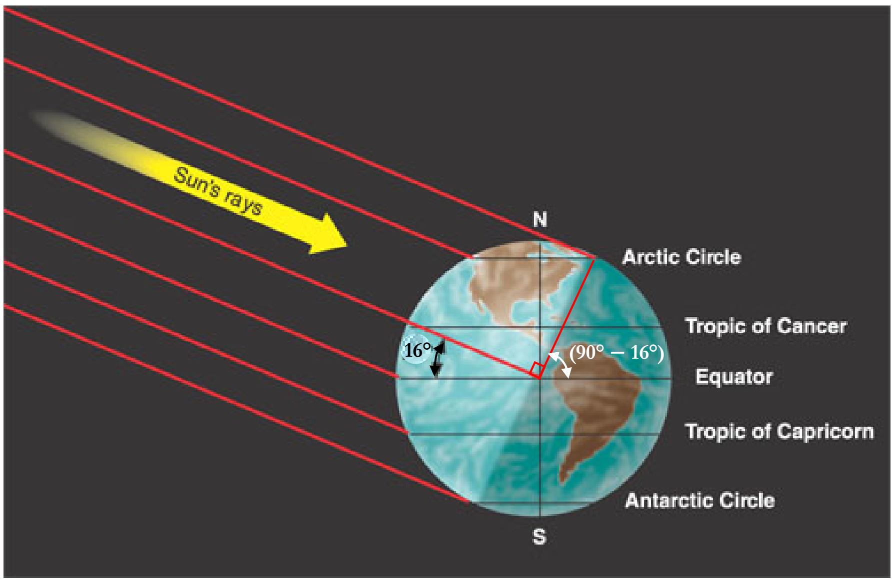
When the Earth's tilt is 23°, the Sun passes through the zenith at noon on the first summer day (≈June 22) for all locations at 23° N latitude, denoting the Tropic of Cancer at 23° N latitude. Meanwhile, during this summer solstice, the North Pole and all locations within 23° latitude of the North Pole receive an uninterrupted 24-hour-long day of sunlight, thus joining all locations at or north of 67° latitude as part of the Arctic Circle:





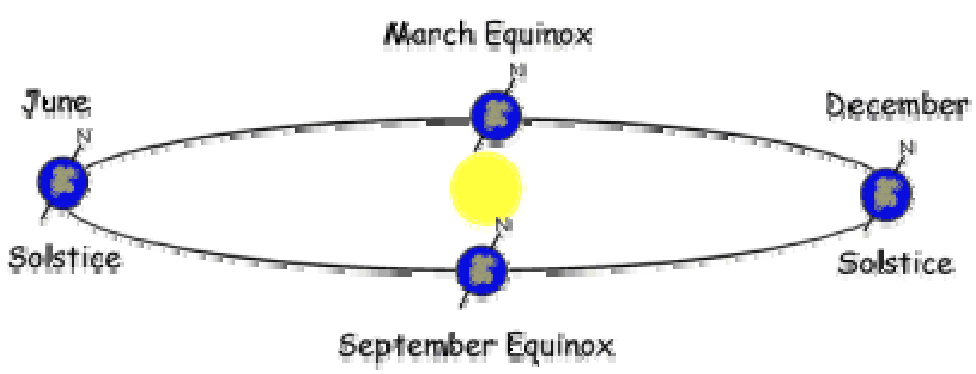
For this actual tilt of 23°, the Tropic of Cancer and the Arctic Circle are (67° - 23°) = 44° apart.

If the tilt of the Earth's axis of rotation were only 16°, then the Sun would pass through the zenith at noon on the first summer day for locations at 16° N latitude, shifting the Tropic of Cancer to 16° N latitude. Meanwhile, the uninterrupted 24-hour-long day of sunlight would affect only locations at or above 74° latitude, moving the Arctic Circle northward to 74° N latitude. The line of tangency always occurs at the complement latitude of the zenith normal, as pictured below:



Thus, the difference in latitude between the Arctic Circle and the Tropic of Cancer would increase to a total of (74° - 16°) = **58°**.

With this milder tilt to Earth's axis, Earth's seasons would be less pronounced. All four seasons would still exist, but the Sun's rays would give less preference to either hemisphere during Earth's revolution about the Sun:



In other words, the weaker axial tilt would temper the differences between seasons. For example, during the Northern Hemisphere's summer, the Northern Hemisphere would lean less vigorously toward the Sun, and the Sun's time above horizon would also decrease from its typical 15-hour duration; conversely, during the Northern Hemisphere's winter, the Northern Hemisphere would lean less vigorously away from the Sun, and the number of daylight hours would not be dip as low as 9 hours as they currently do during winter. Meanwhile, the angle at which the sunlight strikes the surface would generally oscillate with a lower standard deviation, varying less wildly between summer and winter and staying closer to a moderate value. All in all, the extremities of the summer and winter seasons would move closer to the mean, precipitating more moderate fluctuation in temperature and relatively constant (11-13) daylight hours per day throughout the year.

If the strength of seasons fluctuated as a sinusoid, then the decreased axial tilt would reduce the amplitude, as depicted below: