Wavelets

Marcus Brubaker (mbrubake@cs.toronto.edu)

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Completeness of the Basis

In lecture on Wednesday it was mentioned that wavelet coefficients could be viewed as the coordinates of the image in $\mathbb{R}^{2^N}$ where $\phi_0^0, \psi_0^0, \psi_0^1, \ldots$ were the basis. In other words, every image can be represented as a linear combination of the images in the wavelet basis. To see this we need simply show that $\phi_0^0, \psi_0^0, \psi_0^1, \ldots$ are linearly independant or, alternately, that W has full rank. This was already effectively done in class when it was shown that WW^T is diagonal with non-zero entries. (Why?)

But to get a better sense of this we will compute the coefficients for an arbitrary 1D image. Let

$$I = \begin{bmatrix} 9 & 7 & 3 & 5 \end{bmatrix}$$

and

$$W = \begin{bmatrix} \phi_0^0 \\ \psi_0^0 \\ \psi_0^1 \\ \psi_1^1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} & -\frac{1}{4} \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

where the basis vectors are the rows of W. We want to be able to compute coefficients I^0, D_0^0, D_0^1, D_1^1 such that

$$I = I^0\phi_0^0 + D_0^0\psi_0^0 + D_0^1\psi_0^1 + D_1^1\psi_1^1$$

To compute the coordinates, you compute the dot-product between each row and the image. Thus the wavelet coefficients for our example image are:

$$\begin{aligned} I^0 &= I \cdot \phi_0^0 \\ &= 6 \\ \\ D_0^0 &= I \cdot \psi_0^0 \\ &= 2 \\ \\ D_0^1 &= I \cdot \psi_0^1 \\ &= 1 \\ \\ D_1^1 &= I \cdot \psi_1^1 \\ &= -1 \end{aligned}$$

Optimality of Wavelet Compression

The compression algorithm presented in class boiled down to sorting the wavelet coefficients by magnitude and keeping the k largest. We will show that, in terms of squared error, this is the best that can be done for a given compression level k.

Let $u_1, \ldots, u_m$ be an orthonormal basis for $\mathbb{R}^m$ and $c_1, \ldots, c_m$ be the coordinates of image I such that

$$I = \sum_{i=1}^{m} c_i u_i$$

Now let $\sigma(i)$ be some ordering of the numbers $1, \ldots, m$ of which we will keep the first $\tilde{m}$ to compress the image. The reconstructed image is then

$$\tilde{I} = \sum_{i=1}^{\tilde{m}} c_{\sigma(i)} u_{\sigma(i)}$$

Finally the reconstruction error when using these $\tilde{m}$ coefficients is

$$\begin{aligned}
\left\| I - \tilde{I} \right\|^2 &= \left\| \sum_{i=1}^{m} c_i u_i - \sum_{i=1}^{\tilde{m}} c_{\sigma(i)} u_{\sigma(i)} \right\|^2 \\
&= \left\| \sum_{i=\tilde{m}+1}^{m} c_{\sigma(i)} u_{\sigma(i)} \right\|^2 \\
&= \sum_{i=\tilde{m}+1}^{m} \sum_{j=\tilde{m}+1}^{m} c_{\sigma(i)} c_{\sigma(j)} \left(u_{\sigma(i)} \cdot u_{\sigma(j)} \right) \\
&= \sum_{i=\tilde{m}+1}^{m} \left(c_{\sigma(i)} \right)^2
\end{aligned}$$

where the last step is because the u_i's are orthonormal and thus $u_i \cdot u_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$. Thus, the sigma which minimizes the squared error is the one which minimizes the sum of the squares of the excluded coefficients.

Notice that the u_i's could be any orthonormal basis and thus picking the largest coefficients is the optimal thing to do for any orthonormal basis.

2D Wavelet Transforms

The 2D Haar wavelet basis can be computed in two ways. In one way it can be computed directly by applying the 2D Haar wavelet to the image. Alternately it can be computed as a sequence of 1D Haar wavelet transforms. For reference, a basis for a 2x1 (1D) image

$$
\begin{aligned}
b_1 &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \\
b_2 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

To see that these are equivalent lets look at a 2D basis for a 2x2 image.

$$
\begin{aligned}
\phi_0^0 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
\psi_0^0 &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\
\psi_1^0 &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\
\psi_2^0 &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}
\end{aligned}
$$

Notice though that

$$
\begin{aligned}
\phi_0^0 &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= b_1 b_1^T
\end{aligned}
$$

$$
\begin{aligned}
\psi_0^0 &= \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \\
&= b_1 b_2^T
\end{aligned}
$$

$$
\begin{aligned}
\psi_1^0 &= \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \\
&= b_2 b_1^T
\end{aligned}
$$

$$
\begin{aligned}
\psi_2^0 &= \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \\
&= b_2 b_2^T
\end{aligned}
$$

In other words, the Haar wavelet transform can be viewed as a seperable convolution.