

# An Algorithm for Amplitude-Constrained Input Design for System Identification

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**Abstract**—We propose an algorithm for design of optimal inputs for system identification when amplitude constraints on the input and output are imposed. In contrast to input design with signal power constraints, this problem is non-convex and non-smooth. We propose an iterative solution: in the first step, a convex optimization problem is solved for input design under power constraints. In subsequent steps, the constraints considered are the  $p$ -norms of the input and output signals,  $p$  increases for each iteration step. This is an adaptation of the classical Pólya algorithm for function approximation, which has previously been used for the related problem of signal crest-factor optimization. Although the difficulty of the problem prevents a proof of optimality, the performance of the algorithm is discussed with reference to a simple example.

## I. INTRODUCTION

System identification is the process of computing a mathematical model of a real-world system based on experimental input-output data. In many cases, though not all, the input to the real-world system can be chosen with some freedom. Ideally, the input should be chosen to maximize model quality, whilst minimizing experiment time or disturbance to the system's operation.

Over several decades, a large body of literature has developed on the topic of *optimal experiment design* (e.g. [1], [2]). Computationally speaking, the most powerful techniques are based on the recognition that, for a linear system, the inverse of the error covariance matrix is an affine function of the input power spectrum. One can also choose constraints that are affine functions of the input spectrum, and furthermore the input spectrum can be affinely parametrized in various ways. In this way, optimal input design can be characterized as a convex optimization problem, for which efficient computational tools are readily available [3].

In this framework it is natural to consider power constraints on the input; frequency-by-frequency amplitude constraints can also be imposed. Unfortunately, in many practical cases, the real constraint is the peak amplitude of the input or input of the system. The peak value of a signal on the particular realization of its power spectrum: not just the

power of each frequency but phases as well. The relationship between phases and peak amplitude is highly non-linear and non-smooth, making optimization under amplitude constraints computationally challenging. Nevertheless, practical applications render the problem worth considering.

A closely-related task is that of multi-sine crest-factor optimization: a signal with a particular discrete power spectrum is desired, and the realization, i.e. phases, should be chosen so that the peak amplitude is minimized [4]. The “crest factor” or “peak factor” is the ratio of the peak amplitude to the power. Again, this problem is highly non-convex and non-smooth, and no general solution is known. However, an iterative algorithm related to the classical Pólya algorithm of function approximation was proposed [5] and showed impressive results.

In this paper, we propose adapting the approach of [5] to the problem of amplitude-constrained input design, whilst taking full advantage of the computational advantages of power-constrained input design. Roughly speaking, our approach is: perform a power-constrained optimization, then iteratively move towards the peak-amplitude-constrained problem by considering signal  $p$ -norms as  $p$  goes from 2 to infinity. We consider design of a periodic signal expressed as a multisine.

## A. Motivating Application

Some medical diagnostic tasks are expressible as system identification problems. In particular, the author was drawn to the present problem through work in identification of the cerebrospinal fluid system [6], [7], [8], [9]. In this procedure, fluid is pumped through a needle into the patient's intracranial cavity with a pre-defined flow pattern and pressure recordings are taken with another needle in order to detect abnormalities in the absorption and fluid dynamics, which can be an indicator of a disease known as hydrocephalus. For patient comfort and efficient use of hospital resources, it is important to minimize the time spent performing the procedure (times can be upwards of an hour). However, it is critical that safe pressure levels in the brain cavity are not exceeded. Hence we have an optimal input design problem with signal amplitude constraints.

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B. Paper Organization

In the next section we define some notation used throughout the paper. In Section 3 we give the mathematical problem statement. In Section 4 we discuss preliminary matters relating to system identification, experiment design, and crest-factor optimization. In Section 5 we give the algorithm which is the main contribution of this paper. In Section 6 we discuss the results of the algorithm on some simple examples. In Section 7 we make some brief conclusions and discuss future work.

II. NOTATION

For a periodic discrete-time signal of period  $N$ , we make use of the following standard norms:

- The peak value over the period:

$$\|x\|_\infty = \max_{t=0,1,\dots,N} |x(t)|$$

- The  $p$ -norm, for  $p \geq 2$ :

$$\|x\|_p = \left( \frac{1}{N} \sum_{t=0}^N |x(t)|^p \right)^{1/p}$$

which we refer to as “power” for the case  $p = 2$ .

III. PROBLEM STATEMENT

We consider single-input single-output linear time-invariant systems of the form<sup>1</sup>:

$$y = G(z, \theta)u + H(z, \theta)e \tag{1}$$

where  $e$  is a Gaussian white noise with variance  $\sigma_e^2$ , and  $u$  is a control signal which may be chosen by the designer, and  $G$  and  $H$  are rational transfer functions depending on some parameter vector  $\theta$ . The aim is to estimate  $\theta$  from experimental records of  $u$  and  $y$ . It is assumed that the model set contains the true system, i.e. there exists a  $\theta_0$  such that relation 1 holds with  $\theta = \theta_0$ .

Let  $P_\theta$  be the expected covariance of  $\theta$  given a prediction-error estimation. Then we consider the following optimization problem: to search for a signal  $u(t)$  such that

$$\begin{aligned} &P_\theta \rightarrow \min, \\ \text{subject to } &\|u\|_\infty \leq u_{\max}, \\ &\|y\|_\infty \leq y_{\max}, \end{aligned}$$

where some scalar function of  $P_\theta$

IV. PRELIMINARIES

A. Input Design

Optimal input design for system identification has a substantial history beginning in the 1970’s, as a marriage of the theories of linear systems and statistical experiment design (see, e.g., [1], [2]). There has recently been renewed interest in optimal input design as it has become clear that many problems can be solved efficiently by methods of convex optimization (see, e.g., [3], [10]).

<sup>1</sup>We will often drop the  $z$  argument for  $G$  and  $H$  for the sake of brevity

The usual formulation of an optimal input design is as follows: minimize some scalar function of the parameter estimate covariance  $P_\theta$ , subject to some conditions on the input signal, and possibly the system output signal. A dual approach is to minimize in some sense the size of the input signal, whilst satisfying constraints on the model quality [10], although it has been shown that these approaches are essentially equivalent [11]. In this paper this paper we will only consider the first approach, however our method can clearly be adapted to the dual approach.

In the prediction-error framework [2], the inverse of the covariance matrix of the parameter estimate approximated asymptotically by

$$P^{-1} \cong M = \frac{N}{2\pi} \int_{-\pi}^{\pi} \mathcal{F}(e^{j\omega}, \theta_0) \Phi(\omega) \mathcal{F}^*(e^{j\omega}, \theta_0) d\omega$$

with  $\mathcal{F}$  representing the sensitivity of the prediction error to changes in the parameter vector, close to  $\theta_0$ :

$$\mathcal{F}(z, \theta_0) = H^{-1}(z, \theta_0) \left[ \frac{\partial G(z, \theta_0)}{\partial \theta} \quad \frac{\partial H(z, \theta_0)}{\partial \theta} \right]$$

and the joint input-disturbance spectrum

$$\Phi(\omega) = \begin{bmatrix} \Phi_u(\omega) & \Phi_{ue}(\omega) \\ \Phi_{ue}(\omega)^* & \sigma_e^2 \end{bmatrix}.$$

In this paper we will consider only open-loop experiments, for which  $\Phi_{ue} = 0$ . These expressions are, in general, only true asymptotically.

Several reasonable optimization criteria exist for the matrix  $M$  such as the determinant, trace, and the smallest eigenvalue. In this paper, we will consider in particular the objective function

$$J = \det(M)^{1/n} \tag{2}$$

where  $n$  is the dimension of  $M$ . This function is concave in  $M$  and since  $M$  is affine in the input power spectrum  $\Phi_u$  which can be freely chosen, the optimization problem is to maximize a concave function of the decision variables.

At present the problem is infinite dimensional, however. To render it finite-dimensional, a finite affine parametrization of  $\Phi_u$  must be chosen. If a periodic input signal is desired, then a fourier series truncated at a certain reasonable frequency presents a natural choice. Aperiodic signals can be affinely parametrized by either finite dimensional or partial covariance forms [3] though we do not consider this further here.

In this paper we restrict our attention to inputs of the form:

$$u(t) = \sum_{i=1}^{n_f} A_i \cos(\omega_i t + \phi_i)$$

The frequencies  $\omega_i$  are assumed predefined, and integrally related. The amplitudes  $A_i$  and phases  $\phi_i$  are the design variables.

By Parseval’s relation, the power of the signal  $u(t)$  can be written in terms of the amplitudes alone:

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{t=1}^N u(t)^2 = \sum_{i=1}^{n_f} \frac{A_i^2}{2}$$

if  $N$  is an integer number of periods of the fundamental frequency of the multisine. Hence the power of a multisine signal is affine in the input spectrum.

*B. Crest-Factor Optimization*

In many applications, such as communications, radar, and sonar, one would like to minimize the peak-to-peak amplitude of a signal having a certain pre-defined power spectrum [4]. This is known as crest-factor optimization, or sometimes peak-factor optimization.

The crest factor of a signal  $u(t)$  is defined as

$$CF := \frac{\|u(t)\|_\infty}{\|u(t)\|_2}$$

i.e. the ratio of its peak value to its power.

Often this problem is considered in the case where  $u(t)$  is a multisine [4], [12], [13], [5] with given amplitudes  $A_i$ . The power  $\|u(t)\|_2$  does not depend on the phases  $\phi_i$ , and so the crest-factor optimization problem is equivalent to choosing the phases so as to minimize  $\|u(t)\|_\infty$ .

Unfortunately, this problem is highly nonlinear, non-convex and non-smooth. There is no known algorithm guaranteeing a global minimum, except in the simplest cases where a search over all possible phases is feasible.

The first widely applicable algorithm was given by Schroeder [4]. Schroeder's formula is straightforward to compute, and is based on the intuition of considering a signal that switches between its component frequencies, rather than sums them, and deriving approximate phase angles from this.

*C. Pólya Algorithm for Crest-Factor Optimization*

The Pólya algorithm is a classical method for computing  $\infty$ -norm optimal function approximations. The idea is to consider  $p$ -norm optimizations, and iterate with increasing values  $p$  until a sufficient approximation is found, based on the fact that

$$\lim_{p \rightarrow \infty} \|x\|_p = \|x\|_\infty.$$

If one can find the optimal  $p$ -norm approximation at each step, then under certain conditions the algorithm is guaranteed to converge to the  $\infty$ -norm optimal approximation [14].

Guillaume and co-workers applied this algorithm to the problem of crest-factor optimization [5]. The phases given by Schroeder's algorithm were chosen as a starting point, and then  $\|u(t)\|_p$  was optimized for increasing values of  $p$ . Since each step is a nonlinear optimization, typically a good initial guess is required. Using the Pólya algorithm, each step in the optimization uses the output of the previous step as its initial condition.

Unfortunately, due to the highly complex way in which phases effect the shape of a signal, optimal solutions for the intermediate  $p$ -norm stages cannot be guaranteed, and hence the algorithm is not certain to converge to a global minimum. Despite this, comparisons with the Schroeder phases and previous algorithms for crest-factor optimization showed it to give impressive results. To the author's knowledge it is the best presently-known algorithm for crest-factor optimization.

V. THE ALGORITHM

The algorithm we propose in the present paper can be considered as an extension of the crest factor optimization algorithm of [5].

However, rather than the Pólya algorithm being applied to the objective function, in our algorithm it is applied to the constraint equations. For every iteration, the objective function remains that given in (2). So we consider a sequence of constrained optimization problems, the output of each becoming the initial guess for the next nonlinear optimization problem.

**Step 1: Convex Optimization**

Optimize power spectrum subject to power constraints:

$$\lambda^* = \arg \max_{\lambda} \det(P^{-1}(\lambda))^{1/n},$$

subject to

$$\begin{aligned} \sum_i \lambda_i &\leq 2u_{\max}^2, \\ \sum_i |G(e^{j\omega_i})|^2 \lambda_i &\leq 2y_{\max}^2, \\ \lambda_i^* &\geq 0 \quad \forall i. \end{aligned}$$

The initial amplitudes are chosen as  $A_{1,i} = \sqrt{\lambda_i}$ . Initial phases are chosen according to Schroeder's formula [4]:

$$\phi_{1,j}^* = -2\pi \sum_{l=1}^{j-1} (j-l)\lambda_l.$$

**Step  $k = 2, 3, 4... : \text{Iterate Nonlinear Optimization}$**

Define the following constraint functions:

$$\begin{aligned} f_{u,p}(A, \phi) &:= \|u(A, \phi)/u_{\max}\|_p^p, \\ f_{y,p}(A, \phi) &:= \|y(A, \phi)/y_{\max}\|_p^p, \end{aligned}$$

with  $p = 2^k$ . Take output of the previous step, rescaled to fit the constraints:

$$\begin{aligned} \hat{A}_k &= \frac{A_{k-1}^*}{\max[f_{u,p}(A_{k-1}^*, \phi_{k-1}^*), f_{y,p}(A_{k-1}^*, \phi_{k-1}^*)]}, \\ \hat{\phi}_k &= \phi_{k-1}^*. \end{aligned}$$

Perform the following nonlinear optimization with  $\hat{A}_k, \hat{\phi}_k$  as an initial guess:

$$A_k^*, \phi_k^* = \arg \max_{A, \phi} \det(P^{-1}(A))^{1/n}$$

subject to

$$\begin{aligned} f_{u,p}(A, \phi) &\leq 1, \\ f_{y,p}(A, \phi) &\leq 1. \end{aligned}$$

Iterate until an acceptable design is obtained. ■

### A. Derivatives

An advantage of the formulation we have chosen is that analytical derivatives of the objective and constraint functions can be derived, which can speed up nonlinear optimization algorithms.

The derivative of the objective function is given by:

$$\frac{\partial J}{\partial A_i} = \frac{1}{n} (\det(M))^{(1/n-1)} \det(M) \text{Tr} \left( M^{-1} \frac{\partial M}{\partial A_i} \right) \quad (3)$$

where

$$M = P^{-1}(A), \quad \frac{\partial M}{\partial A_i} = 2A_i \mathcal{F} \text{diag}(e_i) \mathcal{F}^*, \quad (4)$$

where  $e_i$  be a vector of length  $n_f$  which is all zeros, except for its  $i^{\text{th}}$  term, which is 1.

Since the information matrix does not depend on phases, we have

$$\frac{\partial J}{\partial \phi_i} = 0.$$

To compute the derivatives of the constraints, we note that since  $p$  is a multiple of 2, can write the constraint functions as:

$$f_u = \frac{1}{N} \eta_u^T \eta_u, \quad f_y = \frac{1}{N} \eta_y^T \eta_y.$$

with

$$\begin{aligned} \eta_u(t) &:= u(t)^q / u_{\max}^q, \\ \eta_y(t) &:= y(t)^q / y_{\max}^q, \end{aligned}$$

and  $q = p/2$ . Then we have

$$\begin{aligned} \frac{\partial f_u}{\partial A_i} &= \frac{2}{N} \eta_u^T \frac{\partial \eta_u}{\partial A_i}, \quad \frac{\partial f_u}{\partial \phi_i} = \frac{2}{N} \eta_u^T \frac{\partial \eta_u}{\partial \phi_i}, \\ \frac{\partial f_y}{\partial A_i} &= \frac{2}{N} \eta_y^T \frac{\partial \eta_y}{\partial A_i}, \quad \frac{\partial f_y}{\partial \phi_i} = \frac{2}{N} \eta_y^T \frac{\partial \eta_y}{\partial \phi_i}, \end{aligned}$$

with

$$\begin{aligned} \frac{\partial \eta_u}{\partial A_i}(t) &= qu^{q-1} \cos(\omega_i t + \phi_i), \\ \frac{\partial \eta_u}{\partial \phi_i}(t) &= -qu^{q-1} A_i \sin(\omega_i t + \phi_i), \\ \frac{\partial \eta_u}{\partial A_i}(t) &= qy^{q-1} |G(e^{j\omega})| \cos[\omega_i t + \phi_i + \angle G(e^{j\omega})], \\ \frac{\partial \eta_u}{\partial \phi_i}(t) &= -qy^{q-1} |G(e^{j\omega})| A_i \sin[\omega_i t + \phi_i + \angle G(e^{j\omega})]. \end{aligned}$$

### B. Modification to Improve Convergence

It may occur that the power-constraint optimal result of the initial convex optimization has many amplitudes  $A_i$  equal to zero. In particular, it is well known that the optimal input for an LTI system with power constraints on the input alone is the sum of a finite number of sinusoids (see, e.g., [15]).

If  $A_i = 0$ , then it is clear from Equations (3) and (4) that the derivative of the objective function with respect to  $A_i$  is zero. As such, a numerical optimization algorithm may not search along this dimension, despite the fact that the objective function – which is to be maximized – in fact increases with increasing  $A_i$ .

This being the case, we have found that with some numerical optimization packages more reliable convergence can be obtained if, instead of setting

$$\hat{A}_k = \frac{A_{k-1}^*}{\max[f_{u,p}(A_{k-1}^*, \phi_{k-1}^*), f_{y,p}(A_{k-1}^*, \phi_{k-1}^*)]},$$

one sets

$$\hat{A}_k = \frac{A_{k-1}^* + \epsilon \mathbf{1}}{\max[f_{u,p}(A_{k-1}^*, \phi_{k-1}^*), f_{y,p}(A_{k-1}^*, \phi_{k-1}^*)]},$$

where  $\epsilon$  is a small positive number and  $\mathbf{1}$  is a vector of all ones.

## VI. ILLUSTRATIVE EXAMPLE

To illustrate the algorithm, we consider a optimizing the input for a simple output-error structure:

$$y = G(\theta)u + e$$

in which  $\theta$  is a scalar parameter and

$$G(\theta) = \frac{1 - \theta}{z - \theta}$$

corresponding to a first-order low-pass filter with DC-gain of 1 and a bandwidth of  $-\log \theta$ . We set  $\theta_0 = \exp(-0.1) \approx 0.905$  for a bandwidth of 0.1 rad/s. For this example, we set  $\epsilon = 10^{-2}$ .

The convex optimization was performed using CVX [16], [17] and the nonlinear optimizations in the remaining stages were performed using the MATLAB optimization toolbox.

For this system

$$\mathcal{F}(z, \theta) = \frac{-z + 1}{z^2 - 2\theta + \theta^2}$$

In the case of a scalar parameter  $\theta$ , the objective function reduces to

$$J = \det(P_\theta^{-1})^{1/n} = \frac{1}{\sigma_\theta^2}$$

where  $\sigma_\theta^2$  is the asymptotic variance of the estimate of  $\theta$ .

The input is to be optimized subject to the constraint

$$\|u\|_\infty \leq 1.$$

The input  $u(t)$  was parametrized as a multisine having frequencies 0.01*k* rad/s, where  $k = 1, 2, \dots, 30$ .

It is well known that the optimal input for such a system under input power constraints is a pure sinewave with a frequency equal to the bandwidth of the system [1]. Indeed, this is what is found as a result of the first step in the algorithm. The final step computed, corresponding to  $p = 60$ , appears closer to a square wave, see Figure 1. Clearly as a square wave allows the signal to have more power at the fundamental frequency than a sinewave of the same amplitude, and furthermore the additional frequencies contribute further information. The spectra of the initial and final stages are plotted in Figure 2.

To evaluate the gain in information each iteration's computed input signal was rescaled so that it satisfies  $\|u\|_\infty \leq 1$ , and the resulting asymptotic error covariance  $\sigma_\theta^2$  was computed. This is plotted in Figure 3. Note that, between the

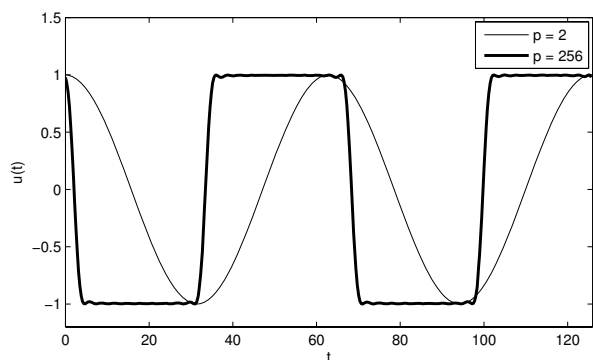


Fig. 1. Optimal input signals for  $\|u\|_p < 1$  with  $p = 2$  and  $p = 256$ .

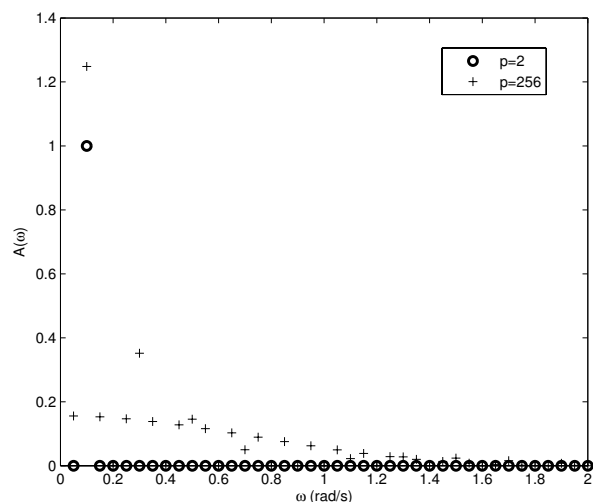


Fig. 2. Optimal input power spectra for  $p = 2$  (o) and  $p = 156$  (+).

first and last iteration, a 38% reduction in error covariance has been achieved.

A simpler approach to amplitude-constrained input design might be the following:

- 1) calculate a multisine input power spectrum using convex optimization, assuming power constraints;
- 2) choose the sinewave phases using a crest-factor optimization routine, such as [4], [5].
- 3) rescale the input so that the amplitude constraints are satisfied.

However, it is clear from Figure 2 that such an approach would not be successful for this example. The result of stage 1 would have a single sinewave, and hence stage 2 would be superfluous, and from the results in Figure 3 the error covariance would be significantly greater than what is possible.

The essential difference in our strategy is that at each stage, a coupled optimization is performed on both the amplitudes and phases.

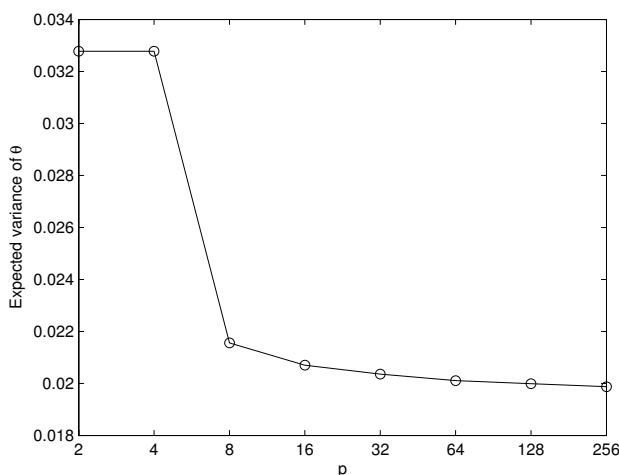


Fig. 3. Expected covariance of the estimate of  $\theta$  for optimizations with different  $p$ .

### VII. CONCLUSIONS AND FUTURE WORK

In this paper we have approached the problem of designing an optimal input signal for system identification, with amplitude constraints on the input and output signals. The algorithm we have proposed is modelled on the Pólya algorithm of function approximation, in which an  $\infty$ -norm is approached by successive  $p$ -norm approximations.

Due to the highly nonlinear, nonconvex, and nonsmooth nature of the problem, one cannot prove that the algorithm will always converge to the global optimum. Thus its performance must be established over application to a series of real problems. In this paper, we have proposed the algorithm, and shown its performance with a simple example. In this case, it seems to have converged close to a true optimum. Other examples will be considered in the future.

It is clear from the example that results obtained by our algorithm are better than could be obtained by separately considering power-constrained optimization and crest-factor optimization.

As mentioned in the introduction, the motivation for studying this problem was an application in neurological diagnosis that can be expressed as a system identification problem [6]. As such, immediate future work will be on adapting this algorithm to that problem. In particular: the effect of feedback in the system will need to be investigated.

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