Supplemental Proofs for "Mixed-Order Compositing for 3D Paintings"

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We show that the mixed-order compositing function C constructed in the paper satisfies all of our desired properties.

1. *Stroke order:* If fragments i and $i+1$ have the same depth, then compositing them in stroke order leaves $S(z)$ unchanged. Therefore, the replacement colors c'' for all other fragments remain the same. Let $(c_x, \alpha_x) = (c_i, \alpha_i) \oplus (c_{i+1}, \alpha_{i+1})$ be the result of compositing these fragment in stroke order. Because replacement colors are only a function of depth, we have $(c'_x, \alpha'_x) = (c'_i, \alpha'_i) = (c'_{i+1}, \alpha'_{i+1})$. Because the replacement colors are the same, up to the premultiplied alpha factor, $(c''_x, \alpha_x) = (c''_i, \alpha_i) \oplus (c''_{i+1}, \alpha_{i+1})$, and therefore the final composite result is unchanged.

2. *Depth order:* If $z_{i+1} > z_i + d$, then for $z \le z_i + d/2$, $S(z)$ only depends on fragments up to i and for $z > z_i + d/2$, $S(z)$ only depends on fragments $i+1$ and after. Because $\gamma \le 1$, the replacement color for all fragments up to i does not depend on $S(z)$ for any $z > z_i + d/2$. Similarly, the replacement color for fragments $i+1$ and after does not depend on $S(z)$ for any $z \le z_i + d/2$. So replacement colors for fragments up to i only depend on the parameters of fragments up to i, and similarly for fragments $i+1$ and after. Therefore these two groups can be mixed-order composited separately and composited in depth order without affecting the outcome.

3. *Zero alpha:* A fragment with $\alpha_i = 0$ has no effect on $S(z)$ and does not contribute to the final composite and may therefore be removed without changing the result.

4. *Continuity:* The continuity of C in colors and alphas is clear: the over operator is continuous in color and α and which fragments are composited in what order only depends on the depths. With respect to the depths, we prove that if z_i changes by a small ϵ, while all other z's are held constant, then the change in the value of C is bounded by a function that approaches zero as $\epsilon \to 0$ and that does not depend on other variables. This is sufficient to prove that C is continuous in all depths simultaneously. The argument is technical, but the idea is simple: we bound the change in each step of the computation of C individually.

   Without loss of generality, we analyze what changes when z_i changes to $\hat{z}_i = z_i + \epsilon$, where $\epsilon > 0$. Also assume that $\epsilon^{2/3} < \gamma d/4$ and $\epsilon < 1$. First of all, $S(z) \neq \hat{S}(z)$ only at $z \in [z_i - d/2, z_i + \epsilon - d/2] \cup [z_i + d/2, z_i + \epsilon + d/2]$, so $\frac{1}{\gamma d} \int_{-\infty}^{\infty} |S(z) - \hat{S}(z)|_\infty \le \frac{2\epsilon}{\gamma d}$. Therefore, for $j \neq i$, $|(c'_j, \alpha'_j) - (\hat{c}'_j, \hat{\alpha}'_j)|_\infty \le \frac{2\epsilon}{\gamma d}$. For fragment i,

$$\begin{aligned}
|(c'_i, \alpha'_i) - (\hat{c}'_i, \hat{\alpha}'_i)|_\infty &= \frac{1}{\gamma d} \left| \int_{z_i - \gamma d/2}^{z_i + \gamma d/2} S(z)\, dz - \int_{\hat{z}_i - \gamma d/2}^{\hat{z}_i + \gamma d/2} \hat{S}(z)\, dz \right|_\infty \leq \\
&\leq \frac{1}{\gamma d} \left| \int_{z_i - \gamma d/2}^{z_i + \gamma d/2} S(z)\, dz - \int_{\hat{z}_i - \gamma d/2}^{\hat{z}_i + \gamma d/2} S(z)\, dz \right|_\infty + \\
&\qquad + \frac{1}{\gamma d} \left| \int_{\hat{z}_i - \gamma d/2}^{\hat{z}_i + \gamma d/2} S(z)\, dz - \int_{\hat{z}_i - \gamma d/2}^{\hat{z}_i + \gamma d/2} \hat{S}(z)\, dz \right|_\infty \leq \\
&\leq \frac{1}{\gamma d} \left| \int_{z_i + \gamma d/2}^{\hat{z}_i + \gamma d/2} S(z)\, dz - \int_{z_i - \gamma d/2}^{\hat{z}_i - \gamma d/2} S(z)\, dz \right|_\infty + \frac{2\epsilon}{\gamma d} \leq \frac{4\epsilon}{\gamma d}
\end{aligned}$$

For each fragment, we have bounded the change of (c', α') by $\frac{4\epsilon}{\gamma d}$, but we need a bound on $|c'' - \hat{c}''|_\infty$. For fragment $j \in \{1, \ldots, n\}$, we split our analysis into two cases, depending on its alpha: if $\alpha_j > 2\epsilon^{1/3}$, then:

$$\begin{aligned}
\left|c''_j - \hat{c}''_j\right|_\infty &= \left| \frac{c'_j \alpha_j}{\alpha'_j} - \frac{\hat{c}'_j \alpha_j}{\hat{\alpha}'_j} \right|_\infty \leq \left| \frac{c'_j \alpha_j}{\alpha'_j} - \frac{\hat{c}'_j \alpha_j}{\alpha'_j} \right|_\infty + \left| \frac{\hat{c}'_j \alpha_j}{\alpha'_j} - \frac{\hat{c}'_j \alpha_j}{\hat{\alpha}'_j} \right|_\infty \leq \\
&\leq \frac{4\epsilon}{\gamma d} + \hat{c}'_j \alpha_j \left| \frac{1}{\alpha'_j} - \frac{1}{\hat{\alpha}'_j} \right| = \frac{4\epsilon}{\gamma d} + \hat{c}'_j \alpha_j \left| \frac{\hat{\alpha}'_j - \alpha'_j}{\hat{\alpha}'_j \alpha'_j} \right| \leq \\
&\leq \frac{4\epsilon}{\gamma d} + \left| \frac{\hat{\alpha}'_j - \alpha'_j}{(2\epsilon^{1/3} - 4\epsilon/\gamma d) 2\epsilon^{1/3}} \right| \leq \frac{4\epsilon}{\gamma d} + \left| \frac{4\epsilon/\gamma d}{4\epsilon^{2/3} - 8\epsilon^{4/3}/\gamma d} \right| = \\
&= \frac{4\epsilon}{\gamma d} + \left| \frac{\epsilon}{\epsilon^{2/3}(\gamma d - 2\epsilon^{2/3})} \right| \leq \frac{4\epsilon}{\gamma d} + \frac{2\epsilon}{\gamma d \epsilon^{2/3}} \leq \frac{4\epsilon + 2\epsilon^{1/3}}{\gamma d} \leq \frac{6\epsilon^{1/3}}{\gamma d},
\end{aligned}$$

where at the end we have used the fact that $2\epsilon^{2/3} < \gamma d/2$ and $\epsilon < 1$. If $\alpha_j \leq 2\epsilon^{1/3}$, then $|c''_j|_\infty \leq 2\epsilon^{1/3}$ and $|\hat{c}''_j|_\infty \leq 2\epsilon^{1/3}$ because premultiplied-alpha color components cannot be greater than the α. Therefore $|c''_j - \hat{c}''_j|_\infty \leq 4\epsilon^{1/3}$. In either case, we have just shown that $|c''_j - \hat{c}''_j|_\infty \leq 4\epsilon^{1/3} + \frac{6\epsilon^{1/3}}{\gamma d}$. The over operator with premultiplied alpha returns a linear combination of colors with each coefficient less than or equal to one. Therefore:

$$|(c''_1, \alpha_1) \oplus \cdots \oplus (c''_n, \alpha_n) - (\hat{c}''_1, \alpha_1) \oplus \cdots \oplus (\hat{c}''_n, \alpha_n)|_\infty \leq 4n\epsilon^{1/3} + \frac{6n\epsilon^{1/3}}{\gamma d}$$

The remaining concern is that the depth order may have changed. This is not a problem because fragments close in depth have similar replacement colors. Note that because all elements of $S(z)$ are between 0 and 1, its convolution with a box of width γd is Lipschitz with constant $\frac{1}{\gamma d}$. Let us bound the change from swapping the order in which two fragments are composited in the final stage:

$$\begin{aligned}
\left|(\hat{c}''_i, \alpha_i) \oplus (\hat{c}''_{i+1}, \alpha_{i+1}) - (\hat{c}''_{i+1}, \alpha_{i+1}) \oplus (\hat{c}''_i, \alpha_i)\right|_\infty &= \left|(\hat{c}''_i + (1 - \alpha_i)\hat{c}''_{i+1} - \hat{c}''_{i+1} - (1 - \alpha_{i+1})\hat{c}''_i, 0)\right|_\infty = \\
&= \left|\hat{c}''_i \alpha_{i+1} - \hat{c}''_{i+1} \alpha_i\right|_\infty = \\
&= \left| \alpha_i \alpha_{i+1} \left(\frac{\hat{c}'_i}{\hat{\alpha}'_i} - \frac{\hat{c}'_{i+1}}{\hat{\alpha}'_{i+1}} \right) \right|_\infty = \\
&= \frac{\alpha_i \alpha_{i+1}}{\hat{\alpha}'_i \hat{\alpha}'_{i+1}} \left| \hat{c}'_i \hat{\alpha}'_{i+1} - \hat{c}'_{i+1} \hat{\alpha}'_i \right|_\infty \leq \\
&\leq \left| (\hat{c}'_i - \hat{c}'_{i+1}) \hat{\alpha}'_{i+1} + \hat{c}'_{i+1} (\hat{\alpha}'_{i+1} - \hat{\alpha}'_i) \right|_\infty \leq \\
&\leq \left| \hat{c}'_i - \hat{c}'_{i+1} \right|_\infty + \left| \hat{\alpha}'_{i+1} - \hat{\alpha}'_i \right| \leq \frac{2}{\gamma d} |\hat{z}_i - \hat{z}_{i+1}|,
\end{aligned}$$

where the last inequality follows from the Lipschitz condition on the convolution of S with the box. Therefore, swapping the order in which final fragments are composited over a distance of at most ϵ changes the result by at most $\frac{2\epsilon}{\gamma d}$. Overall, using the triangle inequality, changing from z_i to $\hat{z}_i$ changes the final result in the L_∞ norm by at most $4n\epsilon^{1/3} + \frac{6n\epsilon^{1/3}+2\epsilon}{\gamma d}$. This is not tight, of course: we conjecture that C is actually Lipschitz in each variable with a constant that does not depend on n.