Combinatorial Geometry

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Previous Lecture

- Algorithm for matching A in B:
  - Take any pair $a,a' \in A$, let $r=||a-a'||$
  - Find all pairs $b,b' \in B$ such that $||b-b'||=r$
  - For all such pairs
    - Compute t that transforms (a,a') into (b,b')
    - Check if $t(A) \subseteq B$

Combinatorial Question

- Given a set A of n points in the plane, what is the maximum number of $p,p'\in A$ such that $||p-p'||=1$?
  - Erdos'46: $O(n^{3/2})$
  - Jozsa, Szemeredi'73: $o(n^{3/2})$
  - Beck, Spencer'84: $O(n^{1.44\ldots})$
  - Spencer, Szemeredi, Trotter'84: $O(n^{4/3})$
  - Szekely'96: $O(n^{4/3})$, proof in 4 slides

Crossing Number

- Crossing number of a graph G: smallest k such that G can be drawn on the plane with at most k edges crossing
- Interested in bounds of the form $k \geq f(n,e)$

Simple Bound

- We know that $k \geq e - 3n$
- Proof:
  - Assume G with $k<e-3n$
  - Then there is a graph with $k-1$ crossings and $e-1$ edges
  - .....
  - There is a graph with 0 crossings and $e-k$ edges
  - But $e-k \leq 3n$ – a contradiction

Bounds

- The earlier lower bound is pretty weak. E.g., it lower bounds k by at most e
- Complete graph has crossing number $\Omega(n^4)$
- Need to “amplify” the bound

Probabilistic Amplification

- Given G, construct G' by random sampling
- Each node is included in G' with probability $p=4\ n/e$ (assume $e \geq 4n$)
- The expected parameters n',e',k' of G' are:
  - $E[n']=pn$
  - $E[e']=p^2e$
  - $E[k'] \leq p^4k$

Proof

- We know that $k' + 3n' - e' \geq 0$
- Thus $E[k' + 3n' - e'] = E[k'] + 3E[n'] - E[e'] \geq 0$
- We get:

$$p^4 k + 3pn - p^2 e \geq 0$$

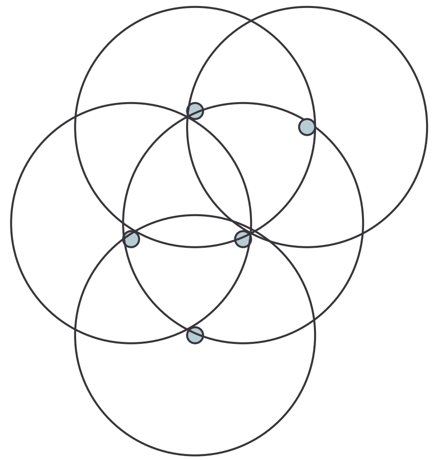
$$p^3 k \geq pe - 3n = 4n - 3n = n$$

$$k \geq e^3/(4^3 n^2) = \Omega(e^3/n^2)$$

$$e = O(\,(kn^2)^{1/3}\,)$$ [Leighton'83]

[Ajtai,Chvatal,Newborn,Szemeredi'82]

Number of Unit Distances



- Nodes = points = n
- Multi-edges defined by arcs ≥ #unit distances
- Keep one out of ≤ 4 edges, so we get a graph
- # crossings ≤ $2\ n^2$

QED

Other Bounds

- Number of incidences between n lines and n points = $O(n^{4/3})$
- Given n points, the number of distinct distances between them is at least $\Omega(n^{4/5})$

Improved Algorithm

- Take the pair $a,a' \in A$ with the lowest multiplicity in B; let $r=||a-a'||$
- Find all pairs $b,b' \in B$ such that $||b-b'||=r$
- For all such pairs
  - Compute t that transforms (a,a') into (b,b')
  - Check if $t(A) \subseteq B$

Analysis

- For a distance t, let $m_A(t)$ be the multiplicity of t in A
- $\sum_t m_B(t) \leq n^2$
- There are at least $n^{4/5}$ different t's such that $m_A(t) \geq 1$
- So, if there is a match, there must exist t such that $m_A(t) \geq 1$ and $m_B(t) \leq n^{6/5}$
- Algorithm has running time $O(n^{11/5})$

Higher Dimensions

- What is the number of unit distances between n points in R^4 ?
- At least $n^2/4$:
  - Let $A=\{(x,y,z,u): x^2+y^2=1, z=u=0\}$
  - Let $B=\{(x,y,z,u): z^2+u^2=1, x=y=0\}$
  - For any $a\in A$, $b\in B$, we have $||a-b||^2=1$
  - Take $n/2$ points from A and $n/2$ points from B